$$f(x) = ax^3 + bx^2 + cx + d$$

Algebra 2

AND TRIGONOMETRY

$$\sin(x_1 + x_2) = \sin x_1 \cos x$$

revised edition

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Algebra 2
and Trigonometry

REVISED EDITION

Mary P. Dolciani
William Wooton
Edwin F. Beckenbach
Sidney Sharron

EDITORIAL ADVISER
Robert H. Sorgenfrey

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Contents

Symbols

Symbol		PAGE
$\{\ \}$	the set whose members are	2
$\in$	is a member of, is an element of	2
$\subset$	is a subset of	2
$\emptyset$	empty set, null set	2
$=$	equals, is equal to	2
$/$	is not, used to negate other symbols	2
$\therefore$	therefore	3
$\mathcal{R}$	{the real numbers}	4
$\ldots$	continues without end	5
$-a$	additive inverse of a	7
$\frac{1}{a}$	multiplicative inverse of a	7
x^n	nth power of x	31
$m\angle A$	measure of angle A	40
$<$	is less than	45
$>$	is greater than	45
$\leq$	is less than or equal to	50
$\geq$	is greater than or equal to	50
$\cap$	intersection	51
$\cup$	union	51
$\lvert a\rvert$	absolute value of a	58
(x, y)	the ordered pair with first coordinate x and second coordinate y	67
$\rightarrow$	assigns one value to another value	68
$f(x)$	f of x	68
$\overset{\circ\!\rightarrow}{PR}$	half-line P, R	77
a_n	a-sub-n	80
$P(x, y)$	point P with coordinates (x, y)	80
$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	determinant	116
$\overline{PQ}$	segment P, Q	127
$\overleftrightarrow{AB}$	line A, B	146
b^{-n}	reciprocal of b^n	173
$m(\overline{AB})$	measure of segment A, B	189
$\sum$	summation sign	221
$\lim\limits_{n\to\infty}$	limit as n increases without bound	241
$\sqrt[n]{b}$	the nth root of b	260
$\pm$	plus or minus	261
$0.0\overline{45}$	the repeating decimal 0.0454545 . . .	267
$\approx$	equals approximately	269
i	a square root of -1	296
$\mathcal{C}$	the set of all complex numbers	299
$\bar{z}$	conjugate of z	300
$d(P_1P_2)$	the distance from P_1 to P_2	342
R^{-1}	inverse relation	388
g^{-1}	inverse function	389
$\log_b a$	logarithm of a to the base b	391
$A \times B$	the Cartesian product of set A and set B	422
${}_nP_r$	the number of permutations of n different elements taken r at a time (Also, $P(n, r)$ or P^n_r)	425
${}_nC_r$	the number of combinations of n elements taken r at a time (Also, $\binom{n}{r}$, $C(n, r)$ or C^n_r)	431
$P(A)$	probability of the event A	444
$\bar{A}$	complement of A	445
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	matrix	461
$A_{m\times n}$	matrix A with dimensions m and n	462
det A	the determinant of matrix A	482
$2'10''$	2 minutes 10 seconds	505
$m^\circ(\alpha)$	measure in degrees of angle α	505
$m(\alpha)$	measure of α	505
α^R	α radians	506
$m^R(\alpha)$	measure in radians of angle α	506
$\sin \alpha$	sine of α	511
$\cos \alpha$	cosine of α	511
$\tan \alpha$	tangent of α	530
$\cot \alpha$	cotangent of α	530
$\sec \alpha$	secant of α	530
$\csc \alpha$	cosecant of α	530
(r, θ)	polar coordinates	593
$\overrightarrow{AB}$	vector A, B	605
$\lVert\overrightarrow{AB}\rVert$	norm (length) of vector A, B	605
$\mathbf{v}$	the vector $\mathbf{v}$	605
metric symbols		635

These natural steam springs in northern California are used to generate electricity.

1
Review of Essentials

Basic Properties

OBJECTIVES for Sections 1-1 through 1-4:
1. *Use set notation.*
2. *Solve simple open sentences, given the replacement set.*
3. *Graph sets of real numbers on a number line.*
4. *Apply the basic properties of real numbers.*

1-1 Sets and Symbols

Can you read the statement

$$5 \in \{1, 3, 5, 7\}?$$

You can if you recall from earlier mathematics courses that the symbol $\in$ is read "is a member of" or "is an element of" and that $\{1, 3, 5, 7\}$ is read "the set whose members are 1, 3, 5, and 7."

Can you read the statement

$$\{1\} \subset \{1, 3, 5, 7\}?$$

You can if you recall that the symbol $\subset$ is read "is a subset of." These and other familiar set symbols are reviewed in the table on the following page.

Symbolism	How Read	Meaning
$\{\ \}$	the set whose members are	Denotes a collection or set.
$\in$	is a member of or is an element of	Is in the collection or set.
$A \subset B$	A is a subset of B	Every member of set A is also a member of set B. Of course, for every set A, $A \subset A$.
$\emptyset$	the null set or the empty set	The set which contains no elements. The empty set is considered a subset of every set.
$A = B$	A is equal to B	A and B denote the same set.
$/$	is not	Used in conjunction with $\in$, $=$, or $\subset$ to denote negation. $5 \notin A$ means "5 is not an element of A."

Oral Exercises

Tell whether the statement is true or false.

1. $\{2, 4, 6\} \subset \{\text{the even integers}\}$
2. $\{3, 5, 7\} \in \{\text{the odd integers}\}$
3. $9 \in \{\text{the odd integers}\}$
4. $\emptyset \not\subset \{2, 4, 6\}$
5. $9 \notin \{3, 5, 7\}$
6. $\{2, 4, 6\} \neq \{2, 4, 6\}$
7. $\{3, 5, 7\} \subset \{3, 5, 7\}$
8. $\emptyset \subset \{3, 5, 7\}$

Written Exercises

Replace each $\underline{?}$ with one of the symbols $\in$, $\subset$, $=$, or negations of these to make a true statement. There may be more than one correct answer.

A

1. $4 \underline{?} \{\text{the even integers}\}$
2. $\{6, 28\} \underline{?} \{\text{the even integers}\}$
3. $\frac{1}{4} \underline{?} \{0.5, 0.25, 0.125\}$
4. $\{3, 5, 7\} \underline{?} \{3, 5\}$
5. $\frac{1}{8} \underline{?} 0.125$
6. $\{3, 4\} \underline{?} \{2, 4, 6, 8\}$
7. $\dfrac{3 + 5}{2} \underline{?} \dfrac{7 + 9}{8}$
8. $\frac{1}{4} - \frac{1}{5} \underline{?} 0.05$

B

9. $\sqrt{3^2 + 4^2} \underline{?} 7$
10. $\emptyset \underline{?} \{0\}$
11. $\{\text{even prime numbers}\} \underline{?} \{2\}$
12. $\{\text{powers of } 3\} \underline{?} \{\text{multiples of } 3\}$
13. $\{\text{multiples of } 3\} \underline{?} \{\text{the odd integers}\}$
14. $\{\text{multiples of 2 and } 3\} \underline{?} \{\text{multiples of } 6\}$

C 15. {numbers of the form $n^2 - (n - 1)^2$, where n is a positive integer} ? {the positive odd integers}

16. {perfect squares of integers} ? {positive integers with an odd number of whole number factors}

1-2 Solving Open Sentences

Meaningful groups of symbols involving $\in$, $\subset$, and $=$ are called **sentences.** Sentences, such as

$$5 + 4 = 9 \qquad \text{and} \qquad 3 \notin \{1, 2, 3\},$$

which can be classified as true or false are called **statements.** On the other hand, a sentence such as

$$x - 3 \neq 2$$

is neither true nor false unless you know the object referred to by the symbol x. In this usage, x is called a variable. A **variable** is a symbol which may represent any one of the members of a specified set, called the **replacement set** or **domain** of the variable. The members of the domain are called the **values** of the variable. A variable with just one value is called a **constant.**

Sentences containing variables are called **open sentences.** The set that consists of the values of the variable for which an open sentence is true is called the **solution set** or **truth set** of the open sentence over the domain of the variable. Each member of the solution set is said to **satisfy** and to be a **solution** or **root** of the open sentence. To solve an open sentence over a given domain means to find its solution set over this domain.

EXAMPLE Solve $x - 3 \neq 2$, if $x \in \{3, 4, 5, 6\}$.

SOLUTION Replacing x in the given sentence with each of its values in turn, you have

$3 - 3 \neq 2$	True	$5 - 3 \neq 2$	False
$4 - 3 \neq 2$	True	$6 - 3 \neq 2$	True

$\therefore$ (read "therefore") the solution set is $\{3, 4, 6\}$. Answer.

Oral Exercises

Tell whether the given open sentence is true or false for the given value of the variable.

1. $x - 5 = 7$; 2
2. $2y - 3 = 5$; 4
3. $3z \neq 12$; 5
4. $9 - 2a = a$; 3
5. $7b - 2 \neq 5$; 1
6. $3u + u = 5u$; 0
7. $r \notin$ {prime numbers}; 11
8. $2v \in$ {even integers}; 3
9. $t \in \emptyset$; 0

Written Exercises

Specify the solution set of the given open sentence over $\{1, 2, 3, 4, 5\}$. If the set is $\emptyset$, so state.

A

1. $x + 5 = 0$
2. $y + 3 = 7$
3. $2z - 1 = 1$
4. $3t = 15$
5. $4v - 3 = 5$
6. $3x \neq 9$
7. $u + u = 6$
8. $2y + 9 = 13$
9. $3v = 0$
10. $4w + w = 5w$
11. $3t - t = t + 3$
12. $4d + 3d = 5d$

B

13. $(x - 1)(x - 4) = 0$
14. $2n + 1 \in \{\text{odd integers}\}$
15. $p^2 + 1 \in \{\text{multiples of } 4\}$
16. $2q - 1 \notin \{\text{prime numbers}\}$
17. $2y \in \{\text{prime numbers}\}$
18. $z^2 \in \{\text{multiples of } 9\}$

C

19. $|x - 3| = x - 3$
20. $(-1)^x = -1$

1-3 Sets of Numbers

The diagram below pictures the **graph of the set** $\{-\frac{3}{2}, \frac{1}{2}, \sqrt{3}\}$ on a **number line.**

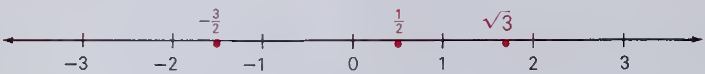
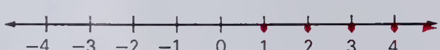


Figure 1

On a number line, the point paired with a number is called the **graph** of the number, while the number paired with a point is the **coordinate** of the point.

The set of all the positive numbers, the negative numbers, and zero is called the set $\mathcal{R}$ of real numbers. A basic assumption is that for each real number there corresponds a point on the number line, and, conversely, for each point on the number line there corresponds a real number. Thus, there is a one-to-one correspondence between the members of $\mathcal{R}$ and the points on a geometric line. Below are shown the graphs of several familiar subsets of $\mathcal{R}$.

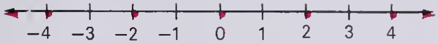
1. $\{\text{the natural numbers}\} = \{\text{the positive integers}\} = \{1, 2, 3, \ldots\}$



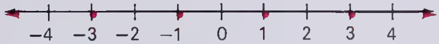
2. $\{\text{the whole numbers}\} = \{0, 1, 2, 3, \ldots\}$

3. $\{\text{the integers}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

4. {the even integers} = $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$



5. {the odd integers} = $\{\ldots, -3, -1, 1, 3, \ldots\}$



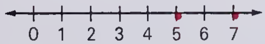
Notice that in the examples above each set is specifed by an *incomplete roster,* or list. The three dots indicate that the pattern shown in the list continues in one or both directions without end. The heavy arrowhead on the accompanying diagram indicates that the graph of the set similarly continues without end.

Oral Exercises

Describe each graph as a subset of A or B listed below.

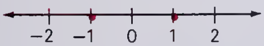
$A = \{-16, -8, -4, -2, -1, 1, 2, 4, 6, 8, 12\}$
$B = \{-18, -6, -\sqrt{3}, 0, 2, 2.3, 3.1, 5, 7\}$

EXAMPLE

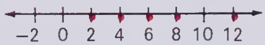


SOLUTION {all the odd integers in B}

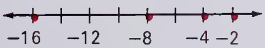
1.



2.



3.



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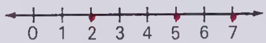

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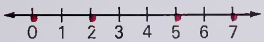

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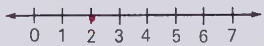


Written Exercises

Graph each set. Select a suitable unit of measure.

A

1. $\{-3, 2, 0, 4\}$
2. $\{-1, -2, 3\}$
3. $\{-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, 0\}$
4. $\{-15, -30, 30, 45\}$
5. $\{1.5, 2, -2.5\}$
6. $\{\frac{1}{3}, \frac{1}{4}, -\frac{1}{2}, 0\}$

Let $B = \{-8, -7, -5.5, -1, 0, 2, 4, 5\frac{1}{4}, 7, 9\}$. Graph the subset of B that contains

7. all the positive integers in B
8. all the whole numbers in B
9. all the positive odd integers in B
10. all the numbers in B that are not integers

Let $A = \{-7, -4, -2, 0, 3, 5\}$ and $B = \{-4, -1, 0, 1, 6\}$. Graph each set.

B 11. {the integers in A that are also in B}

12. {the integers in A or in B or in both sets}

13. {the integers in B but not in A}

14. {the even integers in A or in B or in both}

15. {the odd integers in A but not in B}

C 16. {negative odd integers in A or in B}

17. {negative even integers in A that are also in B}

18. {odd whole numbers in A or in B}

19. {integers in A but not in B that are *not* whole numbers}

20. {even whole numbers in B but not in A}

1-4 Axioms for the Real Numbers

There are two basic **operations** used in working with real numbers, *addition* and *multiplication*. Each of these operations is called a **binary operation** because it pairs any two real numbers with a third real number. The addition operation, +, pairs any two real numbers a and b with another real number, $a + b$, called the **sum** of the two real numbers. Multiplication assigns to any two real numbers a and b their **product** denoted by $a \times b$, $a \cdot b$, $a(b)$, $(a)(b)$, or simply ab. In the sum $a + b$, a and b are called **addends;** in the product ab, a and b are called **factors.**

The properties of these operations in $\mathcal{R}$ all stem from a few basic statements, called **axioms** or **postulates,** that are assumed to be true. These familiar assumptions are listed on this page and the next.

Notice that parentheses, ( ), are used in some of the axioms to indicate an order of operations. For example, $(a + b) + c$ represents the result of first adding a and b, and then adding c to the sum. In some cases, such as in the distributive law where $(ab) + (ac)$ occurs, the parentheses usually are omitted. Thus, $(ab) + (ac) = ab + ac$.

Axioms of Addition and Multiplication in $\mathcal{R}$

Let a, b, and c denote real numbers (a, b, and $c \in \mathcal{R}$).

1. $a + b$ is a unique real number.	**Closure Axiom for Addition**
2. $(a + b) + c = a + (b + c)$	**Associative Axiom for Addition**
3. $a + b = b + a$	**Commutative Axiom for Addition**

4. There exists an element $0 \in \mathcal{R}$ such that for each $a \in \mathcal{R}$, $0 + a = a$ and $a + 0 = a$. — **Axiom of 0 (Identity Element for Addition)**
5. There exists an element $-a \in \mathcal{R}$ for each $a \in \mathcal{R}$, such that $a + (-a) = 0$ and $(-a) + a = 0$. — **Axiom of Additive Inverses**
6. ab is a unique real number. — **Closure Axiom for Multiplication**
7. $(ab)c = a(bc)$ — **Associative Axiom for Multiplication**
8. $ab = ba$ — **Commutative Axiom for Multiplication**
9. There exists an element $1 \in \mathcal{R}$, $1 \neq 0$, such that for each $a \in \mathcal{R}$, $a \cdot 1 = a$ and $1 \cdot a = a$. — **Axiom of 1 (Identity Element for Multiplication)**
10. There exists an element $\frac{1}{a} \in \mathcal{R}$ for each nonzero $a \in \mathcal{R}$ such that $$\frac{1}{a} \cdot a = 1 \text{ and } a \cdot \frac{1}{a} = 1.$$ — **Axiom of Multiplicative Inverses**
11. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ — **Distributive Axiom**

The word "unique" used in two of the axioms means "one and only one," and has this implication:

Substitution Principle

Since $a + b$ and ab are unique, changing the numeral by which a number is named in an expression involving sums or products does not change the value of the expression.

For example, since $8 + 2 = 10$ and $10 - 3 = 7$, you know that

$$(8 + 2) - 3 = 10 - 3 = 7.$$

In $\mathcal{R}$, 0 and 1 are called the **identity elements** for addition and multiplication, respectively. $-a$ and $\frac{1}{a}$ are called the **additive inverse** and the **multiplicative inverse** (or **reciprocal**) of a, respectively. Note also that you should always read $-a$ as "the additive inverse of a" or "the negative of a." The expression "negative a" should not be used since $-a$ may represent either a negative number, a positive number, or zero.

The way you use the symbol = in sentences is consistent with the following assumptions.

Axioms of Equality

Let a, b, and c be any elements of $\mathcal{R}$.

$a = a$	**Reflexive Property**
If $a = b$, then $b = a$.	**Symmetric Property**
If $a = b$ and $b = c$, then $a = c$.	**Transitive Property**

How do you add or multiply three or more real numbers? If a, b, c, d, . . . are real numbers, we define $a + b + c$ to be $(a + b) + c$, abc to be $(ab)c$, $a + b + c + d$ to be $(a + b + c) + d$, and so on. Of course, because addition and multiplication are associative and commutative, you may add the addends in a sum, or multiply the factors in a product, of three or more numbers in any convenient groups of two and in any order, and still obtain the same result. For example,

$$8 + 9 + 17 + 11 = (8 + 17) + (9 + 11) = 25 + 20 = 45.$$

Oral Exercises

Complete the given statement as an illustration of the given axiom for addition, multiplication, or equality.

1. $3 \times 5 =$ _?_ (Commutative Axiom for Multiplication)
2. $7 + (?) =$ _?_ (Axiom of Additive Inverses)
3. $(5)(-2)$ is _?_. (Closure Axiom for Multiplication)
4. If $a = 4$, then _?_. (Symmetric Property of Equality)
5. $(2 \times 3) \times 8 =$ _?_. (Associative Axiom for Multiplication)
6. $(a + 6) + 7 =$ _?_. (Commutative Axiom for Addition)
7. If $x = y$ and $y = 9$, then _?_. (Transitive Property of Equality)
8. $0.2(?) =$ _?_. (Axiom of Multiplicative Inverses)
9. $-3(?) =$ _?_. (Axiom of Identity Element for Multiplication)
10. $3(?) = 3a + 3b$ (Distributive Axiom)

Written Exercises

In Exercises 1–12 state the axiom or property that justifies the given statement. Assume that each variable denotes a real number.

A

1. $(3)(-1) = (-1)(3)$
2. $-5 + 0 = -5$
3. If $x \in \mathcal{R}$, then $-2x \in \mathcal{R}$.
4. $(6 + x) + 4 = 6 + (x + 4)$

5. $(z - 1) + 3 = 3 + (z - 1)$

6. If $r = s$, then $r - 2 = s - 2$.

7. $2(x + 5) = 2x + 10$

8. $-\frac{2}{3} + \frac{2}{3} = 0$

9. If $x + 3 = y$ and $y = 4$, then $x + 3 = 4$.

10. If $w - 2 = 7$, then $7 = w - 2$.

11. $\frac{1}{3} + (-4) \in \mathcal{R}$

12. $(-1)(-1) = 1$

In Exercises 13–16 determine the value of the expression.

13. $27 + 5 + (-7) + (-11) + 19$

14. $-12 + 52 + 17 + (-5)$

15. $(3)(5)(-2)(7)$

16. $(2)(-13)(-21)(5)$

In Exercises 17–25, give the solution set of each sentence over $\mathcal{R}$.

B 17. $-5 + y = 0$

18. $-3x = 5$

19. $-2t = -1$

20. $-(-z) = 17$

21. $-6(-v) = 12$

22. $u + 4 = 0$

23. $\frac{1}{5}w = 8$

24. $-7 + (-r) = 0$

25. $-\frac{1}{2}k = 16$

C 26. Give an axiom that justifies each step in the following proof.

1. $(a + b)(a + b) = a(a + b) + b(a + b)$
2. $= (a^2 + ab) + (ba + b^2)$
3. $= (a^2 + ab) + (ab + b^2)$
4. $= [(a^2 + ab) + ab] + b^2$
5. $= [a^2 + (ab + ab)] + b^2$
6. $= [a^2 + (1 \cdot ab + 1 \cdot ab)] + b^2$
7. $= [a^2 + (1 + 1)ab] + b^2$
8. $= (a^2 + 2ab) + b^2$
9. $(a + b)(a + b) = a^2 + 2ab + b^2$

Sophie Germain
1776–1831

Sophie Germain became interested in mathematics as a child in Paris. During the French Revolution and the Reign of Terror, she studied mathematics against her family's objections. She became a co-worker of the mathematicians Lagrange and Gauss. Later, she won the grand prize offered by the Academy of Science for her presentation of the mathematical theory of the vibration of elastic surfaces. She is also known for her contribution to the theory of numbers. Her work led to her recognition as one of the founders of the field of mathematical physics. Germain won the respect of other mathematicians of the time. She attended the meetings of the *Institute de France*, but was denied membership in the Academy of Science because only men were accepted.

Self-Test 1

VOCABULARY

set (p. 1)
member or element of a set (p. 1)
subset (p. 1)
empty set or null set (p. 2)
statement (p. 3)
variable (p. 3)
replacement set or domain of a variable (p. 3)
values of a variable (p. 3)
open sentence (p. 3)
solution set or truth set (p. 3)
solution or root of an open sentence (p. 3)
number line (p. 4)
graph of a number (p. 4)
coordinate of a point (p. 4)
real numbers (p. 4)
binary operation (p. 6)
sum (p. 6)
product (p. 6)
addends (p. 6)
factors (p. 6)
axiom or postulate (p. 6)
identity element for addition (p. 7)
identity element for multiplication (p. 7)
additive inverse (p. 7)
multiplicative inverse or reciprocal (p. 7)

1. Use set notation to represent the sentence: "The set consisting only of 0 is not contained in the set whose members are 3, 5, and 7." *Obj. 1, p. 1*

Specify the solution set over $\{1, 2, 3, 4, 5\}$.

2. $2x + 3 = 11$ *Obj. 2, p. 1*

3. $3x - 15 = 0$

4. Graph $\{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{3}{4}\}$ on a number line. *Obj. 3, p. 1*

5. Graph the subset of $A = \{-\frac{2}{3}, -5, 0, 1.2, 4, \sqrt{5}\}$ that contains all the whole numbers in A.

State the axiom that justifies each statement.

6. $-5 + 5 = 0$ *Obj. 4, p. 1*

7. $(2 \cdot 6)(7) = (2)(6 \cdot 7)$

8. If $x \in \mathcal{R}$, then $(x - 2) + 3 = 3 + (x - 2)$.

9. If $x, y \in \mathcal{R}$, then $x(y - 2) = xy - 2x$.

10. Solve over $\mathcal{R}$: $\frac{1}{3}(-x) = 6$.

Check your answers with those at the back of the book.

Properties of Operations in $\mathcal{R}$

***OBJECTIVES** for Sections 1-5 through 1-8:*
1. *Understand the meaning and methods of* direct proof.
2. *Understand important theorems giving properties of real numbers.*
3. *Apply these theorems in simplifying expressions for sums, products, differences, and quotients.*

1-5 Theorems and Proof: Addition

The basic properties of $\mathcal{R}$ stated in Section 1-4 imply other properties of $\mathcal{R}$. These implications are stated as **theorems**. A theorem consists of two parts, a **hypothesis** (or **premise**) and a **conclusion**. The hypothesis states what is assumed to be true, and the conclusion states something which logically follows from the assumptions. To give a **direct proof** of a theorem, you start with its hypothesis and by a logical chain of steps arrive at its conclusion. Here is an example of a direct proof.

Theorem. For all real numbers b and c,
$$(b + c) + (-c) = b.$$

PROOF

First note that the hypothesis is that b and c denote real numbers. Reasoning from this assumption, you have:

Statement	*Reason*
1. b and c are real numbers.	Hypothesis
2. $b + c$ is a real number.	Closure axiom for addition
3. $-c$ is a real number.	Axiom of additive inverses
4. $(b + c) + (-c)$ is a real number.	Closure axiom for addition
5. $(b + c) + (-c) = b + [c + (-c)]$	Associative axiom for addition
6. $c + (-c) = 0$	Axiom of additive inverses
7. $(b + c) + (-c) = b + 0$	Substitution principle
8. $b + 0 = b$	Axiom of 0
9. $(b + c) + (-c) = b$	Transitive property of equality (or substitution principle)

Observe that each step in the sequence in the foregoing proof is guaranteed either by hypothesis or by an axiom. Frequently, simple

steps involving closure, substitution, and other basic properties of equality are omitted. For example, the foregoing proof might be replaced by the following:

Statement	*Reason*
1. b and c are real numbers.	Hypothesis
2. $-c$ is a real number.	Axiom of additive inverses
3. $(b + c) + (-c) = b + [c + (-c)]$	Associative axiom of addition
4. $\qquad = b + 0$	Axiom of additive inverses
5. $\qquad = b$	Axiom of 0
6. $(b + c) + (-c) = b$	Transitive property of equality

Theorems that have been proved can then be used to help prove other theorems.

Theorem. For all real numbers a, b, and c,

$$\text{if } a + c = b + c, \text{ then } a = b.$$

PROOF

Statement	*Reason*
1. a, b, and c are real numbers and $a + c = b + c$.	Hypothesis
2. $(a + c) + (-c) = (b + c) + (-c)$	Substitution principle
3. $\qquad = b$	Theorem proved above
4. $(a + c) + (-c) = b$	Transitive property of equality
5. $(a + c) + (-c) = a$	Theorem proved above with a in place of b
6. $a = b$	Substitution principle

From one theorem you can sometimes quickly deduce a closely related theorem, called a **corollary.** Because addition in $\mathcal{R}$ is a commutative operation, you can easily prove the following corollary of the preceding theorem (see Exercise 27, page 16).

Corollary. For all real numbers a, b, and c,

$$\text{if } c + a = c + b, \text{ then } a = b.$$

The preceding theorem and its corollary can be restated in the following combined form.

Cancellation Property of Addition

For all real numbers a, b, and c, if

$$a + c = b + c \quad \text{or} \quad c + a = c + b,$$

then $a = b$.

The next theorem is useful in computing sums. Its proof uses the cancellation property of addition.

Property of the Negative of a Sum

For all real numbers a and b,

$$-(a + b) = (-a) + (-b).$$

That is, the negative of a sum of real numbers is the sum of the negatives of the numbers.

PROOF

Plan: Show that $(a + b) + [(-a) + (-b)] = 0$. Then use the axiom of additive inverses ($(a + b) + [-(a + b)] = 0$) and the cancellation law to obtain the desired result.

Statement	*Reason*
1. a and b are real numbers	Hypothesis
2. $(a + b) + [(-a) + (-b)]$ $= [a + (-a)] + [b + (-b)]$	Commutative and associative axioms of addition
3. $= 0 + 0$	Axiom of additive inverses
4. $= 0$	Axiom of 0
5. $(a + b) + [(-a) + (-b)] = 0$	Transitive property of equality
6. $(a + b) + [-(a + b)] = 0$	Axiom of additive inverses
7. $\therefore (a + b) + [-(a + b)]$ $= (a + b) + [(-a) + (-b)]$	Transitive property of equality
8. $-(a + b) = (-a) + (-b)$	Cancellation property of addition

The property of the negative of a sum can be used to simplify expressions for sums of real numbers, given that you can compute the sum of positive real numbers.

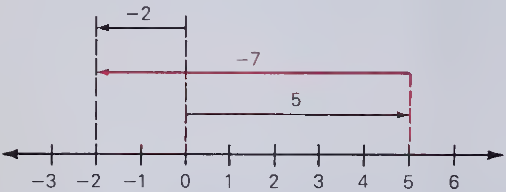
EXAMPLE 1 Simplify $(-12) + (-3)$.

SOLUTION $(-12) + (-3) = -(12 + 3) = -15.$

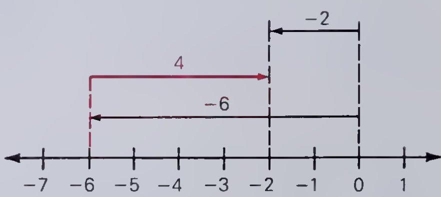
EXAMPLE 2 Simplify $12 + (-16)$.

SOLUTION
$$\begin{aligned} 12 + (-16) &= 12 + [-(12 + 4)] \\ &= 12 + [(-12) + (-4)] \\ &= [12 + (-12)] + (-4) \\ &= 0 + (-4) \\ &= -4 \end{aligned}$$

Sums of real numbers can be pictured by displacements along the number line. Notice that arrows **(vectors)** in the positive direction are used to represent positive numbers and arrows in the negative direction are used to represent negative numbers. The sum of two numbers can be pictured by attaching the initial end of the arrow representing the second addend to the terminal end of the arrow representing the first.



$5 + (-7) = -2$



$-6 + 4 = -2$

Oral Exercises

State as a reason for each of the given sentences being true either (a) one of the theorems proved in this section or (b) one of the axioms for addition.

1. $-(4 + 1) = (-4) + (-1)$
2. $(2 + 5) + (-5) = 2$
3. $(6 + (-3)) + 0 = 3$
4. If $x + 4 = 7 + 4$, then $x = 7$.
5. $((-8) + (-2)) + 2 = -8$
6. If $4 + (-x) = 4 + 2$, then $-x = 2$.
7. $-(3 + (-5)) = -3 + 5$
8. If a and b are real numbers, so is $a + b$.

Written Exercises

In Exercises 1–8 replace the ? with a variable or numeral so that a true statement results. Assume that each variable denotes a real number.

EXAMPLE $-(3 + t) = -3 + \underline{\ ?\ }$ **SOLUTION** $-(3 + t) = -3 + (-t)$

A 1. $11 + (-2) = (\underline{?} + 2) + (-2)$

2. If $y + 4 = -1 + 4$, then $y = \underline{?}$

3. $-(5 + (-t)) = -5 + \underline{?}$

4. $(t + 7) + (-7) = t + (7 + \underline{?})$

5. If $r + (-2) = -2$, then $r = \underline{?}$

6. If $x + 4 = 0$, then $x = \underline{?}$

7. $(z + 3) + (-6) = \underline{?} + (z + 3)$

8. $(-4) + (5 + 4) = \underline{?}$

Simplify each expression.

9. $42 + (-17)$

10. $-58 + 23$

11. $-\frac{1}{2} + \frac{5}{2}$

12. $261 + (-85)$

13. $-15.2 + 7.9$

14. $-28.35 + (-19.8)$

15. $5 + (-28) + (-9) + 33$

16. $-7 + 41 + (-26) + (-54)$

17. $36 + [-(14 + 58)] + 72$

18. $-19 + 35 + [-(-8 + 63)] + 2$

19. $-(29 + 15) + [-(4 + 18)]$

20. $12 + [-(-1 + 9) + (-10)]$

Find the sum of the numbers listed in each column.

21.	22.	23.	24.
-6	-263	-79.2	12.8
-39	82	63.7	-13.3
11	165	-6.4	-25.1
57	-71	-20.5	16

State the axiom or theorem which justifies each step in the following proofs.

B 25. Prove: For all real numbers b and c, $[b + (-c)] + c = b$.

1. b and c are real numbers.
2. $-c$ is a real number.
3. $[b + (-c)] + c = b + [(-c) + c]$
4. $\qquad = b + 0$
5. $\qquad = b$

26. Prove: For all real numbers a, $-(-a) = a$. (Cancellation property of inverses)

1. a is a real number.
2. $-a$ and $-(-a)$ are real numbers.
3. $-(-a) = -(-a) + 0$
4. $\qquad = -(-a) + [(-a) + a]$
5. $\qquad = [-(-a) + (-a)] + a$
6. $\qquad = 0 + a$
7. $\qquad = a$

Prove each theorem when each variable denotes a real number.

27. If $c + a = c + b$, then $a = b$.

28. If $a + (-c) = b + (-c)$, then $a = b$.

29. If $b + a = a$, then $b = 0$. (Uniqueness of additive identity)

30. If $a + b = 0$, then $b = -a$. (Uniqueness of additive inverse)

31. $-[(-a) + (-b)] = a + b$ (*Hint:* Use the result of Exercise 26.)

32. If $a = b$ and $c = d$, then $a + c = b + d$.

33. If $a + c = b + d$, and $c = d$, then $a = b$.

34. If $x + 2 = 9$, then $x = 7$.

35. If $x + (-5) = -2$, then $x = 3$.

36. If $x + 5 = -4$, then $x = -9$.

C

37. If $x + b = a$, then $x = a + (-b)$

38. If $x = a + (-b)$, then $x + b = a$.

39. If $2x + b = x$, then $x = -b$.

1-6 Properties of Products

Multiplication in $\mathcal{R}$ has properties similar to those of addition in $\mathcal{R}$. Compare the three theorems stated below with those for addition on pages 12 and 13. (See Exercises 25–28 on page 20 for proofs.)

Theorem. For all real numbers b and all nonzero real numbers c,

$$(bc)\frac{1}{c} = b.$$

Cancellation Property of Multiplication

For all real numbers a and b and all nonzero real numbers c, if

$$ac = bc \qquad \text{or} \qquad ca = cb,$$

then $a = b$.

Property of the Reciprocal of a Product

For all nonzero real numbers a and b,

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}.$$

That is, the reciprocal of a product of nonzero real numbers is the product of the reciprocals of the numbers.

EXAMPLE Simplify (a) $24 \cdot \frac{1}{3}$ and (b) $8 \cdot \frac{1}{24}$.

SOLUTION a. $24 \cdot \dfrac{1}{3} = (8 \cdot 3) \cdot \left(\dfrac{1}{3}\right) = 8 \cdot \left(3 \cdot \dfrac{1}{3}\right) = 8 \cdot 1 = 8$

b. $8 \cdot \dfrac{1}{24} = 8 \cdot \left(\dfrac{1}{8 \cdot 3}\right) = 8 \cdot \left(\dfrac{1}{8} \cdot \dfrac{1}{3}\right) = \left(8 \cdot \dfrac{1}{8}\right) \cdot \dfrac{1}{3} = 1 \cdot \dfrac{1}{3} = \dfrac{1}{3}$.

Some real numbers, such as the identity element 1, have special properties with respect to multiplication. Two other such numbers are 0 and -1.

Multiplicative Property of Zero

For all real numbers a,

$$a \cdot 0 = 0 \qquad \text{and} \qquad 0 \cdot a = 0.$$

PROOF

1. a is a real number.	Hypothesis
2. $0 + 0 = 0$	Axiom of 0
3. $a \cdot (0 + 0) = a \cdot 0$	Substitution property
4. $a \cdot 0 + a \cdot 0 = a \cdot 0$	Distributive axiom
5. $a \cdot 0 = a \cdot 0 + 0$	Additive identity axiom
6. $a \cdot 0 + a \cdot 0 = a \cdot 0 + 0$	Transitive property of equality
7. $a \cdot 0 = 0$	Cancellation property of addition
8. $0 \cdot a = 0$	Commutative axiom of multiplication

The next theorem leads to the familiar rules for multiplying additive inverses.

Multiplicative Property of -1

For all real numbers a,

$$a(-1) = -a \qquad \text{and} \qquad (-1)a = -a.$$

PROOF

Plan: Show that $(-1)a$ satisfies $x + a = 0$; because $-a$ is the unique solution of this equation (see Exercise 30 on page 16), the conclusion $(-1)a = -a$ will follow. The steps in the proof are given on the following page.

PROOF

1. a is a real number.	Hypothesis
2. $a = 1 \cdot a$	Axiom of 1
3. $(-1)a + a = (-1)a + 1 \cdot a$	Substitution
4. $\qquad\qquad = (-1 + 1)a$	Distributive axiom
5. $\qquad\qquad = 0 \cdot a$	Axiom of additive inverses
6. $\qquad\qquad = 0$	Multiplicative property of 0
7. $(-1)a + a = 0$	Transitive property of equality

Thus, $(-1)a$ satisfies $x + a = 0$, and must, therefore, equal $-a$. Because multiplication in $\mathcal{R}$ is commutative, $a(-1) = (-1)a$, so that $a(-1) = -a$.

The following examples show how the foregoing theorem and the fact that $(-1)(-1) = -(-1) = 1$ (Exercise 26 on page 15) enable you to simplify expressions for products involving negative numbers.

EXAMPLE 1 $15 \cdot (-3) = 15 \cdot [3 \cdot (-1)] = (15 \cdot 3)(-1) = (45)(-1) = -45$

EXAMPLE 2 $(-8)(-7) = (-1 \cdot 8)(-1 \cdot 7) = [(-1)(-1)](8 \cdot 7) = 1 \cdot 56 = 56$

EXAMPLE 3 $(-2)(5) + 3(-1) = -1(2)(5) + (-3) = -1(10) + (-3)$
$= -10 + (-3) = -13$

These examples suggest how to deduce the following corollary of the multiplicative property of -1. (See Exercises 32–34 on page 20.)

Properties of Negatives in Products

For all real numbers a and b,

$$(-a)b = -ab, \qquad a(-b) = -ab, \qquad (-a)(-b) = ab.$$

Can you explain why the following statements are true?

A product of several nonzero real numbers of which an even number are negative is a positive number.

A product of several nonzero real numbers of which an odd number are negative is a negative number.

The fact that $(-1)(-1) = 1$ means that the reciprocal of -1 is -1;

that is, $-1 = \frac{1}{-1}$. You can use this fact to show that for every nonzero real number a,

$$\frac{1}{-a} = -\frac{1}{a};$$

that is, *the reciprocal of the negative of a is the negative of the reciprocal of a.*

$$\frac{1}{-a} = \frac{1}{(-1)a} = \frac{1}{-1} \cdot \frac{1}{a} = -1 \cdot \frac{1}{a} = -\frac{1}{a}$$

Thus,

$$\frac{1}{-5} = -\frac{1}{5}, \quad \text{and} \quad \frac{1}{-7} = -\frac{1}{7}.$$

Oral Exercises

State as a reason for each of the following sentences either (a) one of the theorems or properties of multiplication given in this section or (b) one of the axioms for multiplication.

1. $(-1)(15) = -15$
2. $(-7)(2)(0) = 0$
3. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$
4. $(-8)(9) = -72$
5. $(4)(\frac{1}{4}) = 1$
6. If $3x = 12$, then $x = 4$.
7. $(-7)(-3) = 21$
8. $(1)(42) = 42$
9. $(8)(-5) = -40$

Written Exercises

A Simplify each expression.

1. $(4)(-6)(-7)$
2. $(-5)(6)(14)$
3. $(-21)(6)(-\frac{1}{14})$
4. $(-32)\left(\frac{-1}{2}\right)\left(\frac{-1}{4}\right)$
5. $(-7)(2) + (-5)(-8)$
6. $-13(-5 + 2)$
7. $45[\frac{1}{3} + (-\frac{1}{5})]$
8. $-60[(-\frac{1}{4}) + \frac{1}{5}]$
9. $(-42)(\frac{1}{12})(-\frac{1}{7})$
10. $(-\frac{1}{4})(-36)(-\frac{1}{12})$
11. $15(-12)(-\frac{1}{8})(-1)$
12. $28(-\frac{3}{4})(6)(0)$
13. $(1.4)(-5)(-6)(-2)$
14. $(-2.8)(-7)(5)(3)$
15. $\left(-\frac{1}{4}\right)\left(\frac{1}{-3}\right)\left(\frac{1}{-5}\right)(90)$

State whether the value of the given expression is positive, negative, or zero. Do not compute.

16. $(-8.3)(6)(5)(-4.9)$
17. $(-73)(-10)(-5)(-1)$
18. $(-8)(-23)(1)(-11)$
19. $(-16)(-199)(0)(25)$
20. $17(35)[7 + (-7)]$
21. $(-\frac{1}{4})(-\frac{1}{8})(67)(148)$
22. $(-72)\left(\frac{-1}{2} + \frac{1}{-3}\right)\left(\frac{1}{4}\right)$
23. $(-0.5)(4.2)(-8.7)$
24. $-0.41(-0.7 - 8.5)$

Prove each of the following theorems for $a, b, c \in \mathcal{R}$.

B 25. $bc\left(\frac{1}{c}\right) = b \quad (c \neq 0)$

26. If $ac = bc$, then $a = b$. $(c \neq 0)$

27. If $ca = cb$, then $a = b$. $(c \neq 0)$

28. $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b} \quad (a, b \neq 0)$

29. If $ab = a$, and $a \neq 0$, then $b = 1$.
(Uniqueness of multiplicative identity)

30. If $ab = 1$, then $b = \frac{1}{a}$.
(Uniqueness of multiplicative inverse)

31. $\frac{1}{\frac{1}{a}} = a \quad (a \neq 0)$

32. $(-a)b = -ab$

33. $a(-b) = -ab$

34. $(-a)(-b) = ab$

35. If $a = b$, then $\frac{1}{a} = \frac{1}{b}$. $(a, b \neq 0)$

36. $-a(b + c) = -(ab) + [-(ac)]$

C 37. If $xb = a$, then $x = a \cdot \frac{1}{b}$. $(b \neq 0)$

38. If $x = a \cdot \frac{1}{b}$, then $bx = a$. $(b \neq 0)$

39. If $ax + b = c$, then $x = \frac{1}{a}[c + (-b)]$. $(a \neq 0)$

40. If $\frac{1}{x} = a\left(\frac{1}{b}\right)$, then $x = b\left(\frac{1}{a}\right)$. $(a, b, x \neq 0)$

41. If $\frac{1}{x} = ab$, then $x = \frac{1}{ab}$. $(a, b, x \neq 0)$

42. If $a\left(\frac{1}{x}\right) = b$, then $x = a\left(\frac{1}{b}\right)$. $(a, b, x \neq 0$.

1-7 Properties of Differences

Two other operations are defined in terms of the basic operations of addition and multiplication in $\mathcal{R}$. In this section, we shall consider the first of these. The **difference** between a and b, $a - b$, is defined as follows:

> **Relationship between Addition and Subtraction**
>
> For all real numbers a and b,
>
> $$a - b = a + (-b).$$

You can interpret this to say "to **subtract** b from a, add the additive inverse of b to a." For example,

$$-3 - (-2) = -3 + 2 = -1,$$

and

$$8 - 11 = 8 + (-11) = -3.$$

Since

$$\begin{aligned} a - b + b &= [a + (-b)] + b \\ &= a + [(-b) + b] \\ &= a + 0, \\ &= a, \end{aligned}$$

you can see that $a - b$ is the number which when added to b produces a.

Since $a - b$ is the number that you add to b to obtain a, you can interpret $a - b$ on the number line as follows (see Figure 2): $a - b$ tells the **number of units and the direction of the displacement** *from* the graph of b *to* the graph of a.

Figure 2

Because $\mathcal{R}$ is closed under addition, it follows from the definition of a difference that $\mathcal{R}$ is also closed under subtraction. If you notice that $7 - 2 = 5$ whereas $2 - 7 = -5$, you can see that subtraction is not commutative in $\mathcal{R}$. The fact that $(3 - 2) - 1 = 1 - 1 = 0$ while $3 - (2 - 1) = 3 - (1) = 2$ demonstrates that subtraction is not associative in $\mathcal{R}$.

Knowing the relationship between addition and subtraction, you can prove several theorems about subtraction. For example, Exercise 19 on page 22 outlines a proof of the fact that *multiplication is distributive with respect to subtraction.* Thus, for each real number y,

$$7(y - 5) = 7 \cdot y - 7 \cdot 5 = 7y - 35.$$

Oral Exercises

State each expression in simplified form, using subtraction wherever possible.

1. $x + (-y) + (-z)$
2. $n + k(-t)$
3. $r(-p + (-q))$
4. $-v(r + s)$
5. $a + (-b) - (-c)$
6. $n(d + [-(e + f)])$
7. Is there an identity element with respect to subtraction? That is, is there a real number b such that $x - b = x$ and $b - x = x$ for all real numbers x?
8. Does each real number x have a *subtractive* inverse? That is, does there exist a real number y such that $x - y = 0$ and $y - x = 0$?

Written Exercises

Simplify each expression.

A

1. $-6-(-14)$
2. $11-17$
3. $-125-163$
4. $271-389$
5. $427-38$
6. $12-(-56)$
7. $-29+14-(-5)$
8. $37-(-12)-19$
9. $-3(6-13)$
10. $10(-26-15)$
11. $-2[(4-7)-5]+3$
12. $9[-6-(7-2)]-18$
13. $4[-2(4+3)-9(-1)]$
14. $-[6-2(-3+7)]-5(4-3)$
15. $-2[3(-5+7)-6][-4(8)]$

Show that the number represented by the given numeral satisfies the given equation.

16. $(-k+4)(k-9)=2k-8;\ 7$
17. $(5-x)(x-3)=(x-2)(6-x)-3;\ -2$
18. $(z+4)(7-z)=10z-(2z+3)(5-z)-5;\ -4$

Justify each step in the proofs of the following theorems. Assume that each variable denotes a real number.

B

19. $a(b-c)=ab-ac$

PROOF

1. a, b, and c are real numbers.
2. $b-c=b+(-c)$
3. $a(b-c)=a[b+(-c)]$
4. $=ab+a(-c)$
5. $=ab+(-ac)$
6. $=ab-ac$
7. $a(b-c)=ab-ac$

20. $a-(-b)=a+b$

PROOF

1. a and b are real numbers.
2. $-b$ is a real number.
3. $a-(-b)=a+[-(-b)]$
4. $-(-b)=b$
5. $a-(-b)=a+b$

Prove each theorem. Assume that each variable denotes a real number.

21. $(a-b)+b=a$
22. $-a(b+c)=-ab-ac$
23. $-a(b-c)=ac-ab$
24. If $a-c=b-c$, then $a=b$.
25. If $c-a=c-b$, then $a=b$.
26. If $a=b$, then $a-c=b-c$.
27. If $a=b$, then $c-a=c-b$.
28. If $a-b=0$, then $a=b$.
29. If $x-b=a$, then $x=a+b$.
30. If $x=a+b$, then $x-b=a$.

C

31. $(a-b)(c+d)=ac-bc+ad-bd$
32. $(a-b)(c-d)=ac-bc-ad+bd$
33. $a[b-(c+d)]=ab-ac-ad$
34. $a[b-(c-d)]=ab-ac+ad$

1-8 Properties of Quotients

Still another operation in $\mathcal{R}$ is defined in terms of multiplication. The **quotient** of a and b, $a \div b$ or $\frac{a}{b}$, $b \neq 0$, is defined as follows:

Relationship between Multiplication and Division

For all real numbers a and all nonzero real numbers b,

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

For example,

$$\frac{-24}{3} = (-24)\left(\frac{1}{3}\right) = (-8 \cdot 3)\frac{1}{3} = -8 \cdot \left(3 \cdot \frac{1}{3}\right) = -8 \cdot 1 = -8$$

and

$$20 \div (-\tfrac{1}{5}) = 20 \cdot (-5) = -100.$$

Since

$$\begin{aligned}(a \div b) \cdot b &= \left(a \cdot \frac{1}{b}\right) \cdot b \\ &= a \cdot \left(\frac{1}{b} \cdot b\right) \\ &= a \cdot 1 \\ &= a,\end{aligned}$$

you can see that $a \div b$ is the number which when multiplied by b produces a.

Notice that *division by 0* is *not defined*. This is because:

1. If $a \neq 0$, $\frac{a}{0} = c$ would imply that $0 \cdot c = a$, but $0 \cdot c = 0$ for every real number c (page 17).
2. If $a = 0$. $\frac{a}{0} = c$ would not be unique, since $0 \cdot c = 0$ for every real number c.

Because $\mathcal{R}$ is closed under multiplication and $a \div b = a \cdot \frac{1}{b}$, the set $\mathcal{R}$ is closed with respect to division, **excluding division by zero.** Exercises 19–33 on page 25 state several additional properties of division, among which is the fact that division is distributive over addition.

What is the value of an expression like $12 - 15 + 6 \cdot 8 \div 12$ that has no grouping symbols to show the order in which operations are to be performed? To assign a value to such an expression, you take the

following steps, performing the operations within each grouping symbol, if any, beginning with the innermost grouping symbol and working out to the entire expression.

1. Perform multiplications and divisions in order from left to right.
2. Then perform additions and subtractions in order from left to right.

EXAMPLE 1 Simplify $12 - 15 + 6 \cdot 8 \div 12$.

SOLUTION
$$\begin{aligned} 12 - 15 + 6 \cdot 8 \div 12 &= 12 - 15 + 48 \div 12 \\ &= 12 - 15 + 4 \\ &= -3 + 4 \\ &= 1 \end{aligned}$$

EXAMPLE 2 $\frac{1}{5}[-6 + 2(18 - 5)]$

SOLUTION
$$\begin{aligned} \tfrac{1}{5}[-6 + 2(18 - 5)] &= \tfrac{1}{5}[-6 + 2(13)] \\ &= \tfrac{1}{5}(-6 + 26) \\ &= \tfrac{1}{5}(20) \\ &= 4 \end{aligned}$$

Oral Exercises

Tell which operation you would perform first. Simplify each expression.

1. $5 + 3 \cdot 8$
2. $8 \div 4 + 3 \cdot 5$
3. $14 - 6 \div 3 + 13$
4. $36 \div (4 \cdot 3) + 6$
5. $15 \div (\frac{1}{2} + \frac{1}{2})$
6. $5 + [(1 - \frac{2}{5}) \div \frac{3}{5}]$

Written Exercises

Simplify each expression.

A

1. $51 \div (-17)$
2. $-144 \div 16$
3. $-225 \div (-15)$
4. $-306 \div (-18)$
5. $\left(35 + \dfrac{1}{2}\right) \div \dfrac{-1}{6}$
6. $35 + \dfrac{1}{2} \div \dfrac{-1}{6}$
7. $-12 \cdot \frac{1}{4} \div (-\frac{1}{5})$
8. $14 \cdot (-15) \div (-\frac{1}{2})$
9. $(17 + 8) \div (4 - 9)$
10. $17 + 8 \div 4 - 9$
11. $[-5 + 7(8 - 13)] \div [2(1 - 4)]$
12. $[4(5 - 3) - 2(4 - 7)] \div [2(-1 - 6)]$
13. $[12(2 - \frac{1}{3}) - 5] \div (\frac{1}{2} - 3)$
14. $(\frac{2}{3} - 2) \div [3(7 - 5) - \frac{2}{3}]$

Show that the number represented by the given numeral satisfies the given equation.

15. $5 - \dfrac{x + 3}{2} = \dfrac{x - 5}{-2} + 1;\ 7$
16. $\dfrac{z + 3}{8} - \dfrac{z - 5}{2} = 2z + 10;\ -3$

17. $y(y-5)+\dfrac{y-10}{y-1}=-9y;\ -2$

18. $\left(\dfrac{2}{w}-3\right)\left(\dfrac{w}{5}-2\right)=11-2w;\ 4$

Justify each step in the proofs of the following theorems. Assume that each variable denotes a real number.

B 19. $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}\quad(c\neq 0)$

PROOF

1. a, b, and c are real numbers; $c\neq 0$.
2. $\dfrac{a+b}{c}=(a+b)\cdot\dfrac{1}{c}$
3. $\qquad=a\cdot\dfrac{1}{c}+b\cdot\dfrac{1}{c}$
4. $\qquad=\dfrac{a}{c}+\dfrac{b}{c}$
5. $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$

20. $\dfrac{-a}{b}=-\dfrac{a}{b}$

PROOF

1. a and b are real numbers; $b\neq 0$.
2. $\dfrac{-a}{b}=-a\left(\dfrac{1}{b}\right)$
3. $\qquad=-\left(a\cdot\dfrac{1}{b}\right)$
4. $\qquad=-\dfrac{a}{b}$
5. $\dfrac{-a}{b}=-\dfrac{a}{b}$

Prove each of the following theorems. Assume that each variable denotes a real number. You may use the result of a previous exercise in any proof.

21. $\dfrac{a}{a}=1\quad(a\neq 0)$

22. $\dfrac{-a}{a}=-1\quad(a\neq 0)$

23. $(ab)\div b=a\quad(b\neq 0)$

24. If $ax=b$, then $x=\dfrac{b}{a}$. $(a\neq 0)$

25. If $x=\dfrac{b}{a}$, then $ax=b$. $(a\neq 0)$

26. $\dfrac{ac}{bc}=\dfrac{a}{b}\quad(b,c\neq 0)$

27. $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd}\quad(b,d\neq 0)$

28. $a+\dfrac{c}{b}=\dfrac{ab+c}{b}\quad(b\neq 0)$
(See Exercises 19 and 23.)

29. $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{b+a}{ab}\quad(a,b\neq 0)$
(See Exercise 26.)

30. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}\quad(b,d\neq 0)$
(See Exercise 29.)

C 31. $\dfrac{a}{b}\cdot\dfrac{b}{a}=1\quad(a,b\neq 0)$

32. $\dfrac{1}{\frac{a}{b}}=\dfrac{b}{a}\quad(a,b\neq 0)$
(See Exercise 31.)

33. $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ad}{bc}\quad(b,c,d\neq 0)$ (See Exercise 32.)

Self-Test 2

VOCABULARY

theorem (p. 11)
hypothesis or premise (p. 11)
conclusion (p. 11)
direct proof (p. 11)
corollary (p. 12)
vectors (p. 14)
difference (p. 20)
quotient (p. 23)

1. Prove: If $\frac{a}{c} = \frac{b}{c}$, then $a = b$. $(c \neq 0)$ *Obj. 1, p. 11*

2. Complete this statement of the Property of the Negative of a Sum: *Obj. 2, p. 11*
$-(a + b) = (?) + (?)$

Simplify each expression.

3. $-[2(-4 + 7) - (3 + 11)] - 8$ *Obj. 3, p. 11*
4. $-2[-3(2 - 9) - 6] \div (-5)$
5. $3(-12) + 24 \div 8 + 6 \cdot (-\frac{1}{2})$

Check your answers with those at the back of the book.

-30

Chapter Summary

1. Some of the familiar set symbols include { } (indicates a set), $\in$ (denotes a member of a set), $\subset$ (indicates a subset), $=$ (denotes the same set), and $\emptyset$ (the empty set).
2. A *variable* is a symbol which may represent any one of the members of a set called the *replacement set* of the variable. A sentence containing a variable is called an *open sentence.* You can convert an open sentence about numbers to a statement by replacing the variable with a numeral for one of the values of the variable. Values of a variable that convert a sentence to a true statement are *solutions* of the sentence.
3. *Axioms,* or *postulates,* are statements whose truth is assumed. These statements are used to prove *theorems.* The "if-clause" of a theorem is called the *hypothesis* of the theorem and the "then-clause" is called the *conclusion* of the theorem. By reasoning from the hypothesis to the conclusion, you can give a *direct proof* of a theorem.
4. The axioms for the set $\mathcal{R}$ of real numbers (pages 6–7) determine the properties of the system of real numbers.
5. Equality among real numbers is *reflexive, symmetric,* and *transitive.* (See page 8.)

Careers

in Data Processing

Computers do a variety of jobs, from keeping track of a business firm's finances to guiding rockets. However, before a computer can do any job, it must be *programmed*, or given instructions. This is the task of a computer programmer. The initial task in writing a program is to analyze the job so that it can be done in a sequence of logical steps. At this stage, the programmer must be sure that the computer being used is equipped to perform each step, and that the steps are arranged in order.

The programmer then translates the steps into a *program*, a list of instructions in a language which the computer is designed to "understand." There are many different programming languages, and the one used for a particular program depends upon the job and the computer to be used. Every programming language has a limited number of instructions, each telling the computer to perform a specific operation.

If you enjoy working puzzles and have the ability to work within formal logical systems, you might like computer programming. In this book you will learn about a programming language called BASIC and you will have the opportunity to write some programs.

Programming in design (above) and traffic control (below).

EXAMPLE Follow these directions.

1. Let $x = 0$.
2. Find the value of $3x + 2$.
3. Add 1 to the value of x.
4. Go to step 2.
5. Repeat steps 2 to 4, stopping after $x = 10$ in step 2.

SOLUTION

x	$3x + 2$
0	2
1	5
2	8
⋮	⋮
10	32

Exercises

Follow these directions.

1. Let $d = 5$.
2. Let $n = 1$.
3. Write $\frac{n}{d}$ as a decimal.
4. Add 1 to the value of n.
5. Go to step 3.
6. Repeat steps 3 to 5, stopping after $n = d - 1$ in step 3.

Change Step 1 above to "Let $d = 7$." Follow Steps 2–6.

Chapter Review

Indicate the correct answer by writing the appropriate letter.
In Exercises 1–4, indicate whether the statement is (a) true or (b) false.

1. $\emptyset \in \{1, 2\}$ **2.** $8 \in$ {the odd integers} *1-1*

3. $\{1, 2, 4\} \subset \{1, 2, 3, 4\}$ **4.** $\{1, 3, 5\} =$ {the odd integers}

Specify the solution set of the given open sentence over $\{1, 2, 3, 4, 5\}$. *1-2*

5. $4t = 12$

a. $\{1\}$ **b.** $\{4\}$ **c.** $\{4, 3\}$ **d.** $\{3\}$

6. $3z = 4z - z$

a. $\{1\}$ **d.** $\{1, 2\}$ **c.** $\{3, 4, 5\}$ **d.** $\{1, 2, 3, 4, 5\}$

7. Set $A = \{-3, -2, -1, 0, 1, 2, 3\}$. Which set is shown by the graph below? *1-3*

a. A **b.** {negative integers in A} **c.** {nonnegative integers in A}
d. {positive integers in A}

Choose the axiom that justifies the given statement.

8. $3 \times 5 = 5 \times 3$ *1-4*

a. Closure axiom for multiplication
b. Associative axiom for multiplication
c. Commutative axiom for multiplication
d. Axiom of 1

9. $3 + (8 + 2) = (3 + 8) + 2$

a. Closure axiom for addition
b. Associative axiom for addition
c. Commutative axiom for addition
d. Axiom of 0

Simplify each expression.

10. $-3 + 15 + (-17)$ *1-5*

a. -35 **b.** -5 **c.** 1 **d.** -1

11. $-(38 + 13) + 41$

a. 10 **b.** -10 **c.** 15 **d.** -5

12. $-3(4-8)$ 1-6

a. -12 b. -36 c. 12 d. 24

13. $28(-\frac{3}{4})(5)0$

a. -120 b. 35 c. 0 d. -35

14. $a+(-b)+(-c)$ 1-7

a. $a-b+c$ b. $a-b-c$ c. $a+b-c$ d. $a+b+c$

15. $m+n-(-p)$

a. $m+n+p$ b. $m-n+p$ c. $m-n-p$ d. $m+n-p$

16. $r(-s+(-t))$

a. $-rs+rt$ b. $rs-rt$ c. $-rs-rt$ d. $rs+rt$

17. $-324 \div 18$ 1-8

a. -3 b. -18 c. 18 d. 24

18. $-21 \div (-4-3)$

a. -3 b. 7 c. 3 d. -7

Chapter Test

Replace each $\underline{?}$ with one of the symbols $=$, $\in$, or $\subset$ to make a true statement.

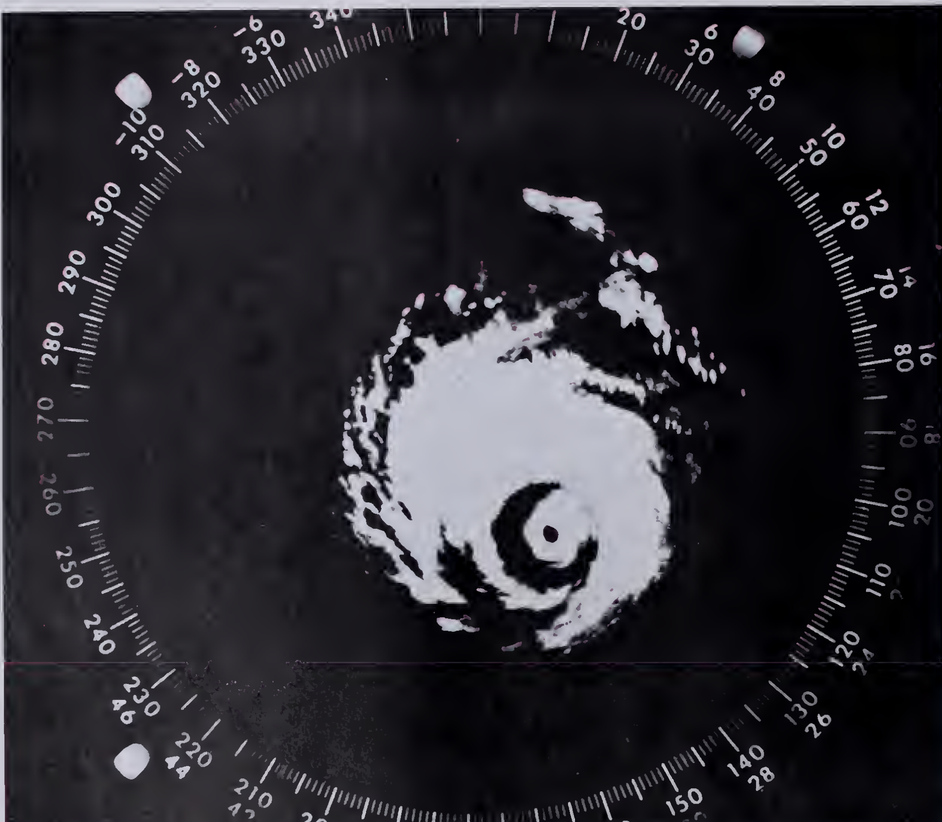
1. $\{2, 3, 5, 7\} \underline{?} \{\text{integers}\}$
2. $45 \underline{?} \{\text{multiples of } 5\}$ 1-1
3. Solve $3x+4=19$ if $x \in \{1, 2, 3, 4, 5\}$. 1-2
4. Solve $2x-7=1$ if $x \in \{1, 2, 3, 4, 5\}$.
5. Graph $\{-2, 0, 4\}$ on a number line. 1-3

State the axiom justifying each statement.

6. $8 \times (9 \times 7) = (8 \times 9) \times 7$
7. $4(3+5) = 4 \times 3 + 4 \times 5$ 1-4

Simplify each expression.

8. $31+(-62)+21$
9. $-15+(-24+30)$ 1-5
10. $(-40)(6)(-\frac{1}{8})$
11. $-15(41-21)$ 1-6
12. $7-(-15)$
13. $-2[4-(3+2)]$ 1-7
14. $(-3+7) \div (4 \cdot 5)$
15. $[2+2(-8)] \div (-14+7)$ 1-8



A hurricane viewed on a radarscope.

2

Review of Essentials

Solving and Applying Equations

***OBJECTIVES** for Sections 2-1 through 2-3:*
1. Simplify expressions for sums and differences of polynomials.
2. Solve first-degree equations in one variable.
3. Apply linear equations to solve word problems.

2-1 Sums and Differences of Polynomials

A **monomial** in the variable x is an expression of the form

$$ax^n,$$

where $a \in \mathcal{R}$, and n denotes a positive integer. The number denoted by a is called the **coefficient** (or **numerical coefficient**) of the monomial.

The symbol

$$x^n$$

represents a **power** of x, where x is called the **base** and n the **exponent**. In general, the ***n*th power of x** denotes the product of n factors, each equal to x. For example:

$x^1 = x$
$x^2 = x \cdot x$ (also read "x-squared" or "the square of x")
$x^3 = x \cdot x \cdot x$ (also read "x-cubed" or "the cube of x")
$x^4 = x \cdot x \cdot x \cdot x$ (also read "x-fourth" or "x to the fourth")

In the monomial ax^n, if $a \neq 0$, you call n the **degree** of the monomial. Thus, for the monomial $-x^4$, the coefficient is -1 and the degree is 4. Monomials such as -3, 5, and 0 are called **constant monomials,** and, with the exception of the zero monomial, 0, are assigned *degree zero.* The zero monomial has *no degree.*

A monomial such as

$$4x^2y^3,$$

which contains more than one variable, is assigned as degree the sum of the exponents of the variables. Thus, the degree of $4x^2y^3$ is $2 + 3$, or 5, while its coëfficient is 4.

An expression such as

$$5x^3 + 0x^2 + (-2x) + (-5),$$

which consists of a string of monomials connected by plus $(+)$ signs, is called a **polynomial.** A polynomial of two terms is a **binomial;** a polynomial of three terms is a **trinomial.** The monomials in the expression are called the **terms** of the polynomial and the coefficients of the terms are called the **coefficients** of the polynomial. Thus, the terms of the foregoing polynomial are $5x^3$, $0x^2$, $-2x$, and -5, while the coefficients are 5, 0, -2, and -5. Another way to write the polynomial is

$$5x^3 - 2x - 5,$$

where the term with 0 coefficient is omitted and the connecting + signs are taken as understood.

Two monomials are said to be **like** or **similar** if they are exactly the same or if they differ only in numerical coefficients. Thus,

$$5x^5, \qquad -3x^5, \qquad \text{and} \qquad x^5$$

are *like* monomials, while

$$5x^2, \qquad 5x^4, \qquad \text{and} \qquad 5x^5y$$

are *unlike.* A polynomial is said to be in **simple form** when no two of its terms are like terms. For example,

$$2x^3 - 5x + 7$$

is in simple form, whereas

$$2x^3 - 3x - 2x + 7$$

is not. The terms of a simplified polynomial are usually written in order of decreasing degree from left to right.

The **degree** of a polynomial in simple form is defined to be the degree of its nonzero term of highest degree. Thus, the polynomial $2x^3 - 5x + 7$ is of degree 3. A polynomial of degree 2 that contains a single variable is called a **quadratic polynomial.**

Given any two polynomials such as

$$4x^2 - 3x \qquad \text{and} \qquad x^2 + 2x - 1,$$

you call the expression

$$(4x^2 - 3x) + (x^2 + 2x - 1)$$

the **sum** of the polynomials, and the expression

$$(4x^2 - 3x) - (x^2 + 2x - 1)$$

their **difference.** To replace the sum or difference by polynomials in simple form, you use the following rules.

Rules for Adding and Subtracting Polynomials

1. To **add** polynomials, add the coefficients of similar terms in the polynomials.
2. To **subtract** one polynomial from another, subtract the coefficient of each term in the one polynomial from the coefficient of the similar term in the other polynomial.

Using these rules, you find

$$\begin{aligned}(4x^2 - 3x) + (x^2 + 2x - 1) &= (4 + 1)x^2 + (-3 + 2)x + (0 + (-1)) \\ &= 5x^2 + (-1)x + (-1) \\ &= 5x^2 - x - 1\end{aligned}$$

and

$$\begin{aligned}(4x^2 - 3x) - (x^2 + 2x - 1) &= (4 - 1)x^2 + (-3 - 2)x + (0 - (-1)) \\ &= 3x^2 + (-5)x + (1) \\ &= 3x^2 - 5x + 1.\end{aligned}$$

Because it can be proved, by using properties of the real numbers, that the equation

$$(4x^2 - 3x) + (x^2 + 2x - 1) = 5x^2 - x - 1$$

is a true statement for *every* numerical replacement of the variable, the two *members* of the equation (the expressions related by the = symbol) are called **equivalent expressions.** Whenever you replace a given polynomial by an equivalent polynomial in simple form, you say that you have **simplified** the given polynomial.

Oral Exercises

State the coefficients and the degree of each polynomial.

1. $4x^3 - 3x + 7$
2. $-9n^4 - n^3 + n^2 + 17$
3. $-xy^2 + 3xy - 5x$
4. $6p^2q^3 - 5p^2q^2 - 2p^2q + 8$

Each of the letters A, B, C, D, and E stands for the given algebraic expression. State the indicated sum or difference in simplified form.

A: $5x^2$ B: $-3x^2 + 1$ C: $x - 5$ D: $x^3 - 7x$ E: $4x^3 - x$

5. $A + B$ **6.** $B + C$ **7.** $A + D$ **8.** $C + E$ **9.** $D + E$

10. $B + D$ **11.** $A - B$ **12.** $E - C$ **13.** $B - C$ **14.** $D - B$

Written Exercises

In Exercises 1–6 add the given polynomials.

A

1. $$\begin{array}{r} 4y^3 - 5y^2 + y + 8 \\ \underline{y^3 - y^2 - y - 6} \end{array}$$

2. $$\begin{array}{r} -x^4 + 3x^3 + 7x^2 \\ \underline{x^4 - 3x^2 - x + 5} \end{array}$$

3. $$\begin{array}{r} t^4 - t^3 - 3t^2 \\ \underline{5t^3 - 4t^2 + t - 1} \end{array}$$

4. $$\begin{array}{r} -4z^4 + 6z^3 - z^2 - 7 \\ \underline{z^5 - 4z^4 + z^2 + 2} \end{array}$$

5. $$\begin{array}{r} w^5 - 4w^3 + w^2 - 5 \\ \underline{9w^5 + w^3 - w - 3} \end{array}$$

6. $$\begin{array}{r} 7m^2n^2 - 8mn^2 + 6mn - 1 \\ \underline{-5m^2n^2 + 3mn^2 + 9mn - 2} \end{array}$$

7–12. In Exercises 1–6 subtract the second polynomial from the first.

Simplify each expression.

13. $(-2x + 4) - (3 - 6x)$

14. $(5 - 3y) - (7 + 9y)$

15. $(3r + s) - (r - s) - (r + 3s)$

16. $(5p + q) - (p + 3q) - (p - 2q)$

17. $(z^2 - 6z - 10) - (2z^2 + 4z - 11)$

18. $(-3v^2 - 7v + 2) - (4v^2 + 6v - 2)$

19. $(r^3 - 4r^2 - 5) + (r^3 - r + 7) - (2r^3 - 3r^2 + 5r - 1)$

20. $(r^3 - 8) - (-r^3 + 4r^2 - 12r - 1) + (r^2 - 7r + 2)$

21. $(5t^4 - 8t^2 - 1) - (t^4 + t^3 - 6) + (-2t^4 + 2t^3 - 3t - 9)$

22. $(x^4 + 4x^3 - 8x) + (-x^3 - 5x^2 + 1) - (-2x^4 + x^2 - 7x - 10)$

23. $(3x^2 - xy + y^2) - (x^2 - xy + 2y^2) + (x^2 - xy + y^2)$

24. $(c^2 - 6cd - 2d^2) + (7c^2 - cd + 8d^2) - (-c^2 + 5cd - 6d^2)$

B

25. $[2q^2 - (3q + 2)] - [-q^2 - (5q + 3)]$

26. $[3x - (7x - y)] - [(4x + y) - (2x + 9y)]$

27. $-5[t - 8(3 - t)] + 4[7t + 9(t - 10)]$

28. $4[3t - 6(t + 1)] - 3[-7t - 2(3t + 4)]$

29. What polynomial must be added to $7x^4 - 5x^3 - 8$ to obtain the polynomial $x + 1$?

30. From what polynomial must $4a^3 - a^2 + 7a$ be subtracted in order to obtain $2a^3 - a + 5$?

C

31. $3[x^3 - 5(x^2 - 3x)] - [x + 4(x^2 - 2x)]$

32. $[2c^2 - 3(cd - d^2)] - 2[cd - (c^2 + d^2)]$

33. $[4r^3 - 5(r^2 + 3r - 2)] - [r^2 + 2(r^3 - 6r - 5)]$

34. $[3t^6 - 2(t^4 + 4t^2 - 3)] - [2t^4 - (8t^2 + t^6 - 5)]$

2-2 Transforming Equations

The equations below have the same solution set over $\mathfrak{R}$, namely, $\{2\}$.

$$3x - 2(x + 3) = 2x - 8 \qquad \text{and} \qquad x = 2$$

Equations that have the same solution set over a given set are called **equivalent equations** over that set.

To solve an equation, either you identify the root by inspection or else you perform a sequence of **transformations** on the equation until you arrive at an equivalent equation whose solution is evident by inspection. The properties of real numbers guarantee that the following transformations of a given equation always produce an equivalent equation.

Transformations Producing an Equivalent Equation

1. Substituting for either member of the given equation an expression equivalent to it.
2. Adding to or subtracting from each member of the given equation the same polynomial in any variable(s) appearing in the equation.
3. Multiplying or dividing each member by the same nonzero number.

EXAMPLE Solve $6z - 3(z + 1) = z + 1$ over $\mathfrak{R}$.

SOLUTION

1. Copy the equation. $\qquad 6z - 3(z + 1) = z + 1$
2. Use the distributive axiom to help simplify the left member. $\qquad 6z - 3z - 3 = z + 1$, $\quad 3z - 3 = z + 1$
3. **Subtract z** from each member. $\qquad 3z - 3 - z = z + 1 - z$, $\quad 2z - 3 = 1$
4. **Add 3** to each member. $\qquad 2z - 3 + 3 = 1 + 3$, $\quad 2z = 4$
5. **Divide** each member **by 2** (or multiply each member by $\frac{1}{2}$). $\qquad 2z \div 2 = 4 \div 2$, $\quad z = 2$

Because errors may occur in transforming equations, you should always check each solution in the original equation.

$$\begin{aligned} 6z - 3(z + 1) &= z + 1 \\ 6(2) - 3(2 + 1) &\stackrel{?}{=} 2 + 1 \\ 12 - 3(3) &\stackrel{?}{=} 3 \\ 12 - 9 &\stackrel{?}{=} 3 \\ 3 &= 3 \end{aligned}$$

$\therefore$ the solution set is $\{2\}$. Answer.

Note that, in the foregoing example, the replacement set of the variable was specified as $\mathcal{R}$. From here on, in this book, *unless otherwise stated, all open sentences are to be solved over* $\mathcal{R}$.

The "operational" properties of equality upon which the transformations used to solve equations are based are contained in the following theorem (see Exercises 40–43 on page 37).

For all real numbers a, b, and c, if $a = b$, then:

$a + c = b + c$, $c + a = c + b$	**Additive Property**
$ac = bc$, $ca = cb$	**Multiplicative Property**
$a - c = b - c$	**Subtraction Property**
$\frac{a}{c} = \frac{b}{c}$, provided $c \neq 0$	**Division Property**

Oral Exercises

State the transformation(s) you would use to solve each of the following equations.

EXAMPLE $2y + 3 = 7$

SOLUTION Subtract 3 from both members; divide both members by 2.

1. $x - 7 = 16$
2. $5 + x = -2$
3. $\frac{2y}{3} = 5$
4. $4c - 7 = 17$
5. $\frac{d}{4} + 2 = -1$
6. $9 - x = -8$
7. $\frac{1}{5}a - 6 = \frac{1}{2}$
8. $3 - 2b = 11$
9. $8 - \frac{1}{5}y = 2$

State the value of the variable printed in red in terms of the other variables.

10. $ax = b$
11. $\frac{p}{v} = q$
12. $r + w = -t$
13. $k - n = s$
14. $\frac{1}{cz} = d$

Written Exercises

State the transformation used to produce the second equation from the first and the third from the second.

A **1.** $5 - 8v = 21$; $-8v = 16$; $v = -2$
2. $4c - 3 = -c + 12$; $5c - 3 = 12$; $5c = 15$
3. $\frac{z}{3} + 11 = -7$; $\frac{z}{3} = -18$; $z = -54$
4. $9 - \frac{x}{4} = 1$; $-\frac{x}{4} = -8$; $x = 32$

5. $5 - 3(4 - t) = -7$; $5 - 12 + 3t = -7$; $3t - 7 = -7$
6. $5p - 8 = 6p + 5$; $-8 = p + 5$; $-13 = p$
7. $4m + 11 = 3(m - 2)$; $4m + 11 = 3m - 6$; $m + 11 = -6$
8. $-7k - 4 = -3k + 10$; $-4k - 4 = 10$; $-4k = 14$

Solve each of the following equations.

9. $5x - 6 = 34$
10. $8 - 3y = 50$
11. $2t - (3 + t) = 8$
12. $3a - (a - 7) = 15$
13. $5p - 3 = 27 - p$
14. $4w - 3(1 - w) = -17$
15. $z - 2(3 + z) = 19$
16. $\frac{2}{5}r = -6$
17. $\frac{3}{7}q = 12$
18. $\frac{1}{2}n + 4 = 10$
19. $5k - 3(4k + 1) = 5(3 - k)$
20. $2(c + 6) = -7c - 3(5 - 2c)$
21. $-5[2x - (3x + 1) - x] = 5$
22. $4[m - 3(2 - m) - 5] = -10(5 - m)$
23. $\frac{b}{7} - \frac{5b}{7} = 8$
24. $\frac{4u}{5} + \frac{6u}{5} = -12$
25. $\frac{3}{2}(7 + u) - \frac{5}{2}(2u + 1) = -6$
26. $\frac{2}{3}(2d - 5) - \frac{5}{3}(4 - d) = 2$

Solve each equation for the variable shown in red. That is, find an equivalent equation in which the symbol shown in red is alone in the left member.

B 27. $2ax - 3c = 5c$
28. $5ry + 4 = 3r$
29. $3bn - 2b = bn$
30. $6t - 4(b + t) = -10b$
31. $v = -gt + k$
32. $A = \frac{1}{2}(b + c)h$
33. $\frac{a}{b} = \frac{c}{d}$
34. $A = P(1 + RT)$
35. $A = \pi rl + \pi r^2$

Solve each equation for the variable shown in red. Use the resulting equation to find the value of that variable for the given values of the other variables.

36. $xy - 3z = 2yz$; $y{:}2$; $z{:}6$
37. $3t(m + n) + t(m - n) = 6m - n$; $m{:}3$; $n{:}2$
38. $na + 5ab = 4b$; $a{:}6$; $b{:}-9$
39. $m(c - d) - 3m(c + d) = 6d - 3c$; $c{:}2$; $d{:}-7$

Prove each of the following theorems.

C 40. For all real numbers a, b, and c, if $a = b$, then $a + c = b + c$.
41. For all real numbers a, b, and c, if $a = b$, then $a - c = b - c$.
42. For all real numbers a, b, and c, if $a = b$, then $ac = bc$.
43. For all real numbers a and b and nonzero real numbers c, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
44. For all real numbers b and c and nonzero real numbers a, if $ax + b = c$, then $x = \frac{c - b}{a}$.

2-3 Applying Equations

Algebraic expressions and sentences can be used as descriptions or **models** for many real-life situations. Such models can be used to help solve practical problems.

EXAMPLE 1 In the Central City Water Reclamation Center, one purifier can process twice as much sewage per minute as a second purifier, and a third purifier can process 50 more liters per minute than the first purifier. If the three units working together can purify 1100 L/min, how many liters per minute can each purifier process?

SOLUTION

1. **Read the problem carefully and decide what numbers are asked for.** The problem asks for the number of liters of sewage each of three machines can purify per minute.
2. **Choose a variable to represent one of the numbers asked for or described in the problem.** Let x represent the number of liters per minute the second machine can process. Then $2x$ represents the capacity per minute of the first machine, and $2x + 50$ represents the capacity per minute of the third machne.
3. **Write an open sentence showing the relationship(s) given in the problem.**

$$\underbrace{\text{Capacity of machine 1}}_{2x}\quad \underbrace{\text{added to}}_{+}\quad \underbrace{\text{capacity of machine 2}}_{x}\quad \underbrace{\text{added to}}_{+}\quad \underbrace{\text{capacity of machine 3}}_{2x + 50}\quad \underset{=}{\overset{\text{is}}{\downarrow}}\quad \underbrace{\text{total capacity of plant.}}_{1100}$$

4. **Solve the open sentence.** Knowing the solution of the open sentence, you can determine the numbers asked for in the problem.

$$\begin{aligned} 2x + x + 2x + 50 &= 1100 \\ 5x + 50 &= 1100 \\ 5x &= 1050 \\ x &= 210 \end{aligned}$$

$\therefore$ the only solution of the sentence is 210. This means that the second machine can process 210 L/min, the first machine can process 420 L/min, and the third machine can process 470 L/min.

5. **Check your results with the requirements stated in the problem.**

Capacity of first machine = twice second machine

$$\begin{aligned} 420 &\stackrel{?}{=} 2(210) \\ 420 &= 420 \end{aligned}$$

Capacity of third machine = 50 more than first machine

$$\begin{aligned} 470 &\stackrel{?}{=} 50 + 420 \\ 470 &= 470 \end{aligned}$$

Total capacity = sum of the capacities of all three machines

$420 + 210 + 470 \stackrel{?}{=} 1100$

$1100 = 1100$

$\therefore$ the three machines can process 420, 210, and 470 L/min, respectively. Answer.

The five steps taken to solve the preceding problem form a useful plan in solving any problem. Notice how they are used in the solutions of Examples 2 and 3 on pages 39 and 41.

Formulas from the social and physical sciences also are often useful in solving practical problems.

EXAMPLE 2 Six hours after the fishing boat Mackerel leaves harbor on a southerly course, a helicopter leaves the same harbor to pick up an injured crewmember on the boat. The helicopter flies at a rate of 100 km/h greater than the rate of the boat. If the boat reverses direction at the time the helicopter leaves the harbor and if the helicopter arrives at the ship in 1 h, find the rate of the boat and the rate of the helicopter.

SOLUTION

1. The problem asks for the rate of the boat and the rate of the helicopter.
2. Let r = the rate of the boat. Then $100 + r$ = the rate of the helicopter. Using the relationship distance = rate $\times$ time, or

$$d = rt,$$

you have the following facts about the situation.

	r	t	d
Boat	r	$6 - 1 = 5$	$5r$
Helicopter	$r + 100$	1	$r + 100$

3. $\underbrace{\text{Distance of boat from harbor}}_{5r}$ is $\underbrace{\text{distance of helicopter from harbor.}}_{r + 100}$

$$5r = r + 100$$

4–5. Completing the solution and checking the results is left for you.

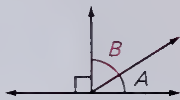
You will find that the rate of the boat is 25 km/h and that of the helicopter is 125 km/h. Answer.

In solving problems about geometric figures, sketches picturing the facts of the problem may help you to see relationships. You may wish to review some familiar facts about geometry shown in the table which follows.

Facts from Geometry–Plane Figures

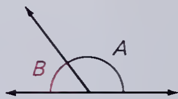
1. **Complementary angles** A and B

   $m\angle A + m\angle B = 90°$



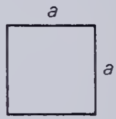
   Supplementary angles A and B

   $m\angle A + m\angle B = 180°$



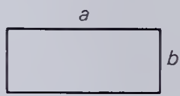
2. **Square** with **side** of length a

   Perimeter: $P = 4a$
   Area: $A = a^2$



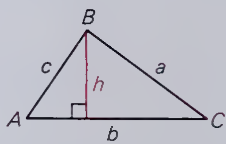
3. **Rectangle** with **length** a and **width** b

   Perimeter: $P = 2a + 2b$
   Area: $A = ab$



4. **Triangle** ABC with sides of **length** a, b, and c, with **base** b and **altitude** h

   Perimeter: $P = a + b + c$
   Area: $A = \frac{1}{2}bh$
   Sum of angles: $m\angle A + m\angle B + m\angle C = 180°$



   a. **Isosceles triangle:**

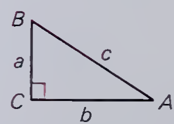
      two congruent sides
      two congruent (base) angles

   b. **Equilateral triangle:**

      three congruent sides
      three congruent angles (measuring 60° each)

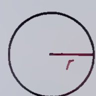
   c. **Right triangle** with **hypotenuse** c:

      $a^2 + b^2 = c^2$ (Pythagorean Theorem)



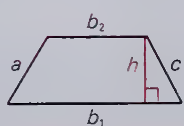
5. **Circle** with **radius** r or **diameter** $d = 2r$

   Circumference (perimeter): $C = 2\pi r$ or $C = \pi d$
   Area: $A = \pi r^2$ or $A = \frac{1}{4}\pi d^2$



6. **Trapezoid** with **legs** of length a and c, **bases** of length b_1 and b_2, and **altitude** h

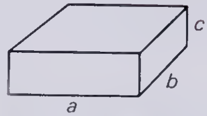
   Perimeter: $P = a + b_1 + b_2 + c$
   Area: $A = \frac{1}{2}h(b_1 + b_2)$



Facts from Geometry–Space Figures

1. **Rectangular prism** with length a, width b, and height c

   Surface area: $2(ab + ac + bc)$
   Volume: $V = abc$

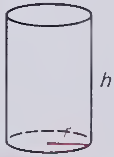


2. **Sphere** with radius r or diameter $d = 2r$

   Surface area: $A = 4\pi r^2$ or $A = \pi d^2$
   Volume: $V = \frac{4}{3}\pi r^3$ or $V = \frac{1}{6}\pi d^3$

3. **Right circular cylinder** with height h, radius of base r, and area of base B

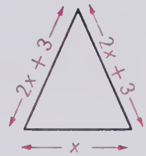
   Lateral area: $L = 2\pi rh$
   Total surface area: $S = 2\pi rh + 2\pi r^2$
   Volume: $V = \pi r^2 h$ or $V = Bh$



EXAMPLE 3 The length of each of the congruent sides of an isosceles triangle is 3 cm greater than twice the length of the base. If the perimeter of the triangle is 91 cm, find the length of each side of the triangle.

SOLUTION

1. The problem asks for the length of each side of the triangle.
2. Let x = the length of the base. (Note that the symbol = is used here to mean "represent.") Then you can express the length of each of the congruent sides by $2x + 3$.



3. $$\underbrace{\text{Perimeter of triangle}}_{x + 2(2x + 3)} \quad \underset{\downarrow}{\text{is}} \quad \underbrace{91 \text{ cm.}}_{91}$$
   $$x + 2(2x + 3) \stackrel{\downarrow}{=} 91$$
4. $$\begin{aligned} x + 2(2x + 3) &= 91 \\ x + 4x + 6 &= 91 \\ 5x + 6 &= 91 \\ 5x &= 85 \\ x &= 17 \end{aligned}$$

   The lengths of the other sides of the triangle then are $2(17) + 3 = 37$ cm.
5. Is the perimeter 91 cm? $$\begin{aligned} 17 + 2(37) &\stackrel{?}{=} 91 \\ 17 + 74 &\stackrel{?}{=} 91 \\ 91 &= 91 \end{aligned}$$

$\therefore$ the lengths of the sides of the triangle are 17 cm, 37 cm, and 37 cm. Answer.

Oral Exercises

For each of the following problems, use the given information to set up an equation that may be solved for the given variable. State the equation but do not solve it.

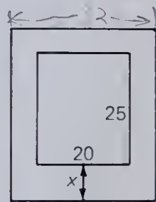
1. The sum of two consecutive integers is 157. Let n equal the smaller integer.
2. The length of a rectangle is 5 more than twice its width, and the perimeter of the rectangle is 58. Let w equal the width.
3. One of the two acute angles of a right triangle has degree measure 6 less than 3 times the other. Let x equal the degree measure of the first acute angle.
4. A motorist drives 165 km in 3 h. The first hour of travel was at a constant speed. The last two hours of travel were at a constant speed 15 km/h greater. Let r equal the speed during the first hour.

Problems

Solve each of the following problems.

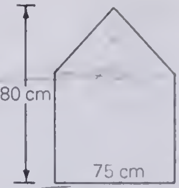
A

1. The sum of two integers is 85, and one integer is 27 greater than the other. Find the integers.
2. Henrietta can use 1120 cm^2 of mounting board to mount a photograph *vertically* that measures $20\text{ cm} \times 25\text{ cm}$. If she leaves a border of 4 cm on the top and sides, what border can she leave at the bottom of the photograph?



3. In Exercise 2, if Henrietta leaves a border of 3 cm at the top and 7 cm at the bottom, what will be the widths of the side borders if they are equal?
4. Find three consecutive integers whose sum is 162.
5. Find three consecutive integers such that the sum of the first and third is 134.
6. Find two consecutive *even* integers such that the sum of the first and three times the second is 190.
7. Find three consecutive *odd* integers such that the sum of the last two is 7 less than three times the first.
8. One base of a trapezoid is 3 cm longer than the other. If the height is 14 cm and the area is 189 cm^2, what are the lengths of the bases?
9. The vertex angle of an isosceles triangle has degree measure 12 less than either of the two (congruent) base angles. Find the degree measures of the angles of the triangle.

10. The length of a rectangle is 7 less than four times its width. If the perimeter is 116 cm, find the dimensions of the rectangle.



11. A window in the shape of a rectangle surmounted by an isosceles triangle is to have a width of 75 cm and a total height of 180 cm. If the total area of the window is to be 12,000 cm^2, what should be the altitude of the triangle?

12. Find the degree measures of two complementary angles, one of which has degree measure 2 less than three times the degree measure of the other.

13. A car gets 12 km/L of gasoline on the highway and 9 km/L in the city. During a recent week the car went 348 km and used 35 L of gas. How much of this gas was used for city driving?

14. From downtown a suburban phone call costs 15¢ more than a local call. One month Dr. Thorne's phone bill showed 30 local calls and 42 suburban calls, and the total bill was \$14.22. What is the cost of one local call?

B 15. Thirty minutes after Mary Jones leaves her house on her moped, her husband discovers that she has left her briefcase at home and sets out after her by car. If the moped travels at 24 km/h and the car at 40 km/h, how long will it take for Mary's husband to overtake her?

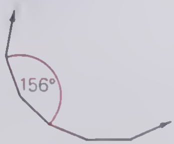
16. Two trains start simultaneously at opposite ends of a route 350 km long, each heading for the other's starting point. If one train travels at 110 km/h and the other at 90 km/h, how long will it be before they pass each other?

17. In Exercise 16, if the train traveling at 90 km/h leaves 20 min earlier, how long after the train traveling at 110 km/h leaves will they pass each other?

18. An empty chemical storage tank is to be filled by three pipes, the first two of which transmit 930 L/h of liquid and the third of which transmits 560 L/h. If the third pipe starts working 90 min after the first two begin and the tank has a capacity of 4000 L, how long will it take after the first two pipes begin working to fill the tank?

19. The formula for the degree measure M of each interior angle of a regular polygon with n sides is

$$M = \frac{(n - 2)180}{n}.$$



How many sides does a regular polygon have if each of its interior angles has degree measure 156?

20. Individual tickets to a raffle cost 35¢. A pair of consecutively numbered tickets may be bought for 50¢. If 100 tickets were sold and \$32.20 was collected, how many people purchased pairs of consecutively numbered tickets?

C **21.** Herb bought golf balls and paid a number of dollars for all of them that was 6 more than the number of balls he bought. Harry bought 2 more of the same kind of golf balls than Herb and paid a total in dollars that was 7 more than the number of balls he bought. How much does one golf ball cost?

22. A hardware dealer paid \$54 for a shipment of shovels and \$135 for a larger shipment of another kind of shovel, each of which cost twice as much as the first kind. Altogether there were 27 shovels in both shipments. How many shovels were there in each shipment?

23. What speed must a motorist average on a return trip from a city 100 km from home in order to have an average speed of 32 km/h for the round trip, if the average speed going out was 20 km/h? (Hint: Average speed = total distance ÷ total time.)

Self-Test 1

VOCABULARY

monomial (p. 31)
numerical coefficient of a monomial (p. 31)
power (p. 31)
base (p. 31)
exponent (p. 31)
constant monomial (p. 32)
binomial (p. 32)
trinomial (p. 32)
degree of a monomial (p. 32)
polynomial (p. 32)
coefficients of a polynomial (p. 32)
like monomials (p. 32)
similar monomials (p. 32)
simple form of a polynomial (p. 32)
degree of a polynomial (p. 32)
quadratic polynomial (p. 32)
equivalent expressions (p. 33)
equivalent equations (p. 35)
transformations (p. 35)
model (p. 38)
complementary angles (p. 40)
supplementary angles (p. 40)
isosceles triangle (p. 40)
equilateral triangle (p. 40)
right triangle (p. 40)
circle (p. 40)
trapezoid (p. 40)
rectangular prism (p. 41)
sphere (p. 41)
cylinder (p. 41)

Simplify each expression. Assume that each variable denotes a real number.

1. $(-3x^2 - 2x + 5) + (7x^2 - x - 1)$ *Obj. 1, p. 31*
2. $4(y^2 - 3) - 3(2y^2 - 6)$
3. $(n^2 + 4) - (n^2 - 3n + 8) - (n - 3n^2)$
4. $(xy - 2y^2 + 3x^2) - (-xy - 2y^2 + x^2)$

Solve over $\mathcal{R}$.

5. $3(4 - t) - 6 = 5t - 7(t - 1)$ *Obj. 2, p. 31*
6. $6(3 - 2r) + 22 = -4(r - 2)$
7. Solve for x: $a(b - 3x) = c - 5ax$
8. Each of the two congruent sides of an isosceles triangle is 5 less than twice the length of the base. If the perimeter of the triangle is 25, find its dimensions. *Obj. 3, p. 31*
9. If the members of a marching band are arranged in rows of 9 players, there would be 2 less rows than if each row contained 8 players. How many players are there in the band?
10. During a trip of 400 km, Fred averaged 25 km/h in cities and 70 km/h on highways. If the trip took a total of 7 hours, how much time did Fred spend driving in cities?

Check your answers with those at the back of the book.

Order in the Set of Real Numbers

***OBJECTIVES** for Sections 2-4 through 2-7:*

1. ***Solve linear inequalities.***
2. ***Apply linear inequalities to practical problems.***
3. ***Give simple indirect proofs.***
4. ***Solve inequalities involving absolute value.***

2-4 Properties of Order

The symbol $<$ is read **"is less than."** It is used to show the relative **order** of two real numbers. You say that

$$3 < 5 \text{ (read "3 is less than 5")},$$

because there is a positive number, 2, such that $3 + 2 = 5$. The statement $3 < 5$ can be written equivalently as $5 > 3$ (read "5 **is greater than** 3"). In general, we have the following definition:

> If a and b are real numbers, then
>
> $$a < b \qquad (\text{or } b > a)$$
>
> if and only if there is a positive real number c such that
>
> $$a + c = b.$$

Note the phrase "if and only if," which condenses two statements into one. In this case it means:

If there is a positive real number c such that $a + c = b$, then $a < b$;

and

If $a < b$, then there is a positive real number c such that $a + c = b$.

These statements are called **converses** of each other, as are any two "If . . . , then . . ." statements each of which can be obtained from the other by interchanging hypothesis and conclusion.

We make the following assumption about order in the set of real numbers.

Comparison Axiom

If a and b are real numbers, then one and only one of the following statements is true:

$$a > b, \qquad a = b, \qquad a < b.$$

The set of *positive real numbers* is denoted by the symbol $\mathcal{R}_+$. One further assumption we make is that the sum of two positive real numbers is a positive real number and the product of two positive real numbers is a positive real number.

Closure Axiom for $\mathcal{R}_+$

If a and $b \in \mathcal{R}_+$, then

$$a + b \in \mathcal{R}_+ \qquad \text{and} \qquad ab \in \mathcal{R}_+;$$

that is, $\mathcal{R}_+$ is closed under addition and multiplication.

Using the definition of "less than," and the closure axiom for $\mathcal{R}_+$, we can prove the following three theorems about order in $\mathcal{R}$.

Transitive Property of Order

If a, b, and c are real numbers, and if $a < b$ and $b < c$, then $a < c$.

PROOF

1. a, b, and c are real numbers, $a < b$, and $b < c$. Hypothesis

2. There are positive real numbers e and f such that $a + e = b$ and $b + f = c$.	Definition of $<$
3. $a + e + f = c$	Substitution principle
4. $e + f$ is positive.	Closure axiom for $\mathcal{R}_+$
5. $a < c$	Definition of $<$

Additive Property of Order

If a, b, and c are real numbers, and if $a < b$, then $a + c < b + c$.

PROOF

1. a, b, and c are real numbers, and $a < b$.	Hypothesis
2. There is a positive real number d such that $a + d = b$.	Definition of $<$
3. $(a + d) + c = b + c$	Additive property of equality
4. $a + (d + c) = b + c$	Associative axiom of addition
5. $a + (c + d) = b + c$	Commutative axiom of addition
6. $(a + c) + d = b + c$	Associative axiom of addition
7. $a + c < b + c$	Definition of $<$

Multiplicative Property of Order

Let a, b, and $c \in \mathcal{R}$.

I. If $a < b$ and c is positive, then $ac < bc$.

II. If $a < b$ and c is negative, then $ac > bc$.

The proofs of the two parts of this theorem are left for you. (Exercises 29 and 30 on page 49.) Of course, the foregoing theorems are also true with $<$ replaced by $>$, and $>$ by $<$, throughout.

Graphically, given two different real numbers, the graph of the lesser lies to the *left* of the graph of the greater on a number line with positive direction to the right. Thus, if $x \in \mathcal{R}$, the graph of the solution set of $x < 2$ appears as in Figure 1.

Figure 1

Note that the endpoint is depicted by an open circle which indicates that the graph of that point is not in the set.

The additive and multiplicative properties of order imply that the following transformations on inequalities will produce **equivalent inequalities** over $\mathcal{R}$, that is, inequalities having the same solution set over $\mathcal{R}$.

Transformations Producing an Equivalent Inequality

1. Substituting for either member of the inequality an expression equivalent to that member.
2. Adding to or subtracting from each member of the inequality the same polynomial in any variable(s) appearing in the inequality.
3. Multiplying or dividing each member by the same *positive* number.
4. Multiplying or dividing each member by the same *negative* number and reversing the direction of the inequality symbol.

EXAMPLE Solve $3x + 2 < 5x - 6$ over $\mathcal{R}$ and graph the solution set.

SOLUTION

$3x + 2 < 5x - 6$	
$3x + 2 - 2 < 5x - 6 - 2$	Transformation 2
$3x < 5x - 8$	Transformation 1
$3x - 5x < 5x - 8 - 5x$	Transformation 2
$-2x < -8$	Transformation 1
$\dfrac{-2x}{-2} > \dfrac{-8}{-2}$	Transformation 4
$x > 4$	Transformation 1

$\therefore$ the solution set is $\{x\colon x > 4\}$. Answer.

Oral Exercises

State the solution of each inequality for the variable and state the transformation that justifies the step needed to arrive at the solution.

1. $x - 8 < 2$
2. $y + 4 > 1$
3. $-2 < a - 7$
4. $3d < 21$
5. $-5 > 4t$
6. $-2r < -18$
7. $-\frac{1}{2}p < 6$
8. $\frac{2}{3}q > 12$
9. $\frac{1}{8}x < 0$
10. Use the multiplicative property of order to explain why the product of two negative numbers is positive.
11. Use the multiplicative property of order to explain why the product of a negative number and a positive number is negative.

Written Exercises

Give the transformation or property of order that justifies each statement, *or* indicate that the statement is not true for all real numbers.

A

1. If $a > b$, then $a - 3 > b - 3$.

2. If $r < s$, then $5r < 5s$.

3. If $x + 3 < 1$, then $x < -2$.

4. If $c = d + 1$ and $c + 7 < 0$, then $d + 8 < 0$.

5. If $a < b$ and $c < d$, then $ac < bd$.

6. If $-3y > 21$, then $y < -7$.

7. If $\frac{n}{-4} > 2$, then $n > -8$.

8. If $z < y + 4$ and $y + 4 < -3$, then $z < -3$.

9. If $-x - 5 > 4$, then $-x < 9$.

10. If $\frac{-t}{5} < -3$, then $t > 15$.

Solve each inequality over $\mathcal{R}$ and graph its solution set.

11. $x - 3 < 5$

12. $-x + 2 < -4$

13. $3x + 2 > -10$

14. $5 - 6x < -7$

15. $\frac{y}{3} - 1 < 2$

16. $\frac{3v}{2} + 5 > 1 + \frac{v}{2}$

17. $5(2z + 3) - 6z > 7z$

18. $-\frac{5}{2}(n + 6) < 1 - \frac{n}{2}$

19. $4(2 - 3t) < -7t - 2$

20. $8\left(\frac{r}{4} - 3\right) > r - 17$

B

21. $5 - 3\left(\frac{x}{2} - 4\right) < \frac{1}{2}(x - 8)$

22. $\frac{1}{3}(5 - 2x) > -\frac{1}{2}(x - 5) - 1$

23. $\frac{2(5 - x)}{3} - 3 < \frac{-3(x - 2)}{2} - 6$

24. $\frac{3(2 - 5x)}{4} - 1 < 5 - \frac{3(3 - 2x)}{2}$

25. $t[(t - 4) - 3] > t^2 - 5t - 8$

26. $-z(z + 3) + z(z - 2) > -4z - \frac{1}{2}(z + 5)$

C

27. Prove: For all real numbers a, if $a > 0$, then $-a < 0$.

28. Prove: For all real numbers a, if $a < 0$, then $-a > 0$.

29. Prove: For all real numbers a, b, and c, if $a > b$ and $c > 0$, then $ac > bc$.

30. Prove: For all real numbers a, b, and c, if $a > b$ and $c < 0$, then $ac < bc$.

31. Prove: For all real numbers a, b, and c, if $a + c > b + c$, then $a > b$.

32. Is it true that for all real numbers a and b, if $a > b$, then $a^2 > b^2$? If so, prove it. If not, give a counterexample.

33. Prove: For all real numbers a, b, c, and d, if $a > b$, and $c > d$, then $a + c > b + d$. (Hint: Show that $a + c > b + c$ and $b + c > b + d$.)

34. Prove: For all *positive* real numbers, a, b, c, and d, if $a > b$ and $c > d$, then $ac > bd$. (Hint: Show that $ac > bc$ and $bc > bd$.)

2-5 Compound Sentences

Is it true that "$3 < 5$ or $3 = 5$"? The answer is "yes" because $3 < 5$. Of course, $3 = 5$ is *not* true, but the **compound sentence** "$3 < 5$ or $3 = 5$" is true because one part of it is true. A sentence such as

$$3 < 5 \qquad \textbf{or} \qquad 3 = 5$$

which is formed by joining two sentences with the word **or** is called a **disjunction** of sentences. For a disjunction to be true, *at least one* of the joined sentences must be true. Disjunctions such as "$3 < 5$ or $3 = 5$" are ordinarily written

$$3 \leq 5 \qquad (\text{alternatively, } 5 \geq 3)$$

(read "3 is less than or equal to 5" and "5 is greater than or equal to 3," respectively).

The graph of the solution set of the open sentence $x \leq 2$ over $\mathcal{R}$ is shown in Figure 2, where the heavy dot at the right endpoint of the graph indicates that the point *is* in the set.

Figure 2

A compound sentence such as

$$2 < 3 \qquad \textbf{and} \qquad 3 < 5$$

which is formed by joining two sentences by the word **"and"** is called a **conjunction** of sentences, and is true if and only if *both* sentences are true. For example, the foregoing conjunction is true, while the conjunction

$$2 < 3 \qquad \text{and} \qquad 5 < 3$$

is false, because $5 < 3$ is false.

Conjunctions of the form "$a < b$ and $b < c$" are ordinarily written

$$a < b < c$$

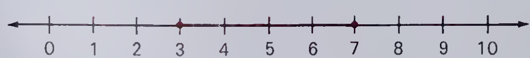
(read "a is less than b and b is less than c").

Conjunctions and disjunctions frequently are combined in compound sentences. For example,

$$3 \leq x \leq 7$$

represents "3 is less than **or** equal to x, **and** x is less than **or** equal to 7."

Figure 3



This sentence is true provided *both* disjunctions are true. Its graph over $\mathcal{R}$ appears in Figure 3, where both endpoints are shown as solid dots to denote that the endpoints are in the set.

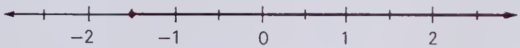
The transitive, additive, and multiplicative properties of order given on pages 46 and 47 also hold with $<$ replaced by $\le$ and $>$ by $\ge$. Similarly, the transformations shown on page 48 will produce equivalent sentences when used in sentences containing $\le$ or $\ge$.

EXAMPLE Solve $3 - (x + 2) \le 4 + x$ over $\mathcal{R}$ and graph its solution set.

SOLUTION

$$\begin{aligned} 3 - (x + 2) &\le 4 + x \\ 3 - x - 2 &\le 4 + x \\ -x + 1 &\le 4 + x \\ -2x + 1 &\le 4 \\ -2x &\le 3 \\ x &\ge -\tfrac{3}{2} \end{aligned}$$

$\therefore$ the solution set is $\{x\colon x \ge -\frac{3}{2}\}$ whose graph is shown. Answer.



As illustrated by Figure 3 and the graph in the foregoing example, the solution set of a *conjunction* is the *intersection* of the solution sets of the simple sentences in the conjunction, while the solution set of a *disjunction* is the *union* of the solution sets of the simple sentences involved. Thus,

$$\{x\colon 3 \le x \le 7\} = \{x\colon 3 \le x\} \cap \{x\colon x \le 7\},$$

and

$$\{x\colon x \ge -\tfrac{3}{2}\} = \{x\colon x > -\tfrac{3}{2}\} \cup \{x\colon x = -\tfrac{3}{2}\}.$$

When negation symbols are used with the symbols $<$ and $>$, the results are equivalent to compound sentences:

$a \not< b$ is equivalent to $a \ge b$,
$a \not> b$ is equivalent to $a \le b$.

Similarly, some negated compound sentences are equivalent to simple sentences. Thus,

$a \not\le b$ is equivalent to $a > b$,
$a \not\ge b$ is equivalent to $a < b$.

Oral Exercises

For each statement state an equivalent conjunction or disjunction.

1. $x \le y$ **2.** $3 \ge 5x + 1$ **3.** 4 [illegible] 5

4. $8 \ge 2z > 2$ **5.** $4 < 8z^2 \le 15$ **6.** $3x + 5 \le x^2$

Written Exercises

Solve each sentence over $\mathcal{R}$ and graph its solution set.

A

1. $2t - 5 \leq 3$
2. $4 - 3z < 10$
3. $4(y + 5) \geq y - 1$
4. $3(2 - k) < 5 - 2k$
5. $\frac{1}{3}x + 2 \geq \frac{5}{6}$
6. $4 - \frac{1}{2}n < n - 2$
7. $0 \leq a + 2 \leq 5$
8. $5 \leq 2b - 1 \leq 9$
9. $-6 < 3(c - 1) < 9$
10. $6 < 2(3 - x) < 18$
11. $2 \leq \frac{1}{4}s + 3 \leq 5$
12. $-3 < 1 - \frac{1}{2}r \leq 4$
13. $4n - 9 \leq 6(n - 2) + 7$
14. $3(v - 2) \geq \frac{1}{2}v + 9$

B

15. $4[2 - 3(m - 1)] \geq 11(2 - m)$
16. $\frac{2}{3}[2 - (3 - p)] \leq \frac{1}{2}(p - 3)$
17. $2x + 1 \geq -5$ and $7x - 1 < 20$
18. $8x \geq 0$ and $5(x - 4) \leq 15$
19. $\frac{y}{3} - 7 < -9$ or $\frac{y + 6}{2} > 5$
20. $9 + 4z \geq 3$ or $-(9 + 4z) \geq 3$
21. $2(4 - t) < 5$ and $-2(4 - t) < 5$
22. $3(2 - r) \leq 1$ and $-3(2 - r) \leq 1$

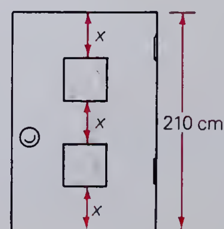
C

23. $-5 \leq 3 + 2x \leq 1$ or $0 \leq 3(x - 1) \leq 9$
24. $-6 < 2(y - 3) < 2$ or $4 < 3 - y < 9$
25. $-7 < 2r - 3 < -1$ or $-1 < 3r + 2 < 14$
26. $-7 < 2u + 1 < 9$ and $[u - 1 \geq 2$ or $2 - u \geq 5]$
27. $[3p - 1 < 8$ or $2(p - 1) < -6]$ and $-1 \leq 2p \leq 3$
28. $[2c - 1 > 11$ or $5c + 1 < 6]$ and $-3 < 3(c + 5) < 42$

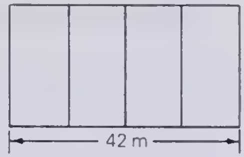
Problems

A

1. Applied to a strip of ground 20 m wide, a large package of lawn seed will cover 30 m more of the length of the strip than a small package and will cover at least 2000 m^2 of ground. What is the shortest length the small package will cover?
2. The two congruent sides of an isosceles triangle are each 5 cm less than twice the base. If the perimeter of the triangle is to be at most 60 cm, what is the maximum length of the base?
3. Two identical windows are to be cut into a door 210 cm tall so that equal widths are left above and below the windows as well as between them. If the windows must be 60 cm wide and must have a combined area of at least 8100 cm^2, what is the maximum width of the border above, below, and between the windows?



4. A farmer has 139 m of fencing materials to make a rectangular pen subdivided by partitions into 4 smaller pens (as shown). If the total length of the pens is to be 42 m, at most, how wide can the rectangle be?

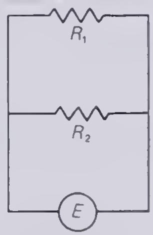


B 5. Lucy Morris jogs the first half of a course 880 m long in 2.5 min. How long can she take to run the second half of the course if she wants to achieve a minimum speed of 160 m/min over the entire course?

6. A subway train stops for equal amounts of time at 4 stations between points A and B, which are 40 km apart. The traveling time between these two points is 20 min. At most how long can the train spend at each station if it is to average at least 60 km/h for the entire trip (including stops)?

7. Frank and Edward start at opposite ends of a track 560 m long and run toward each other, Frank at 400 m/min and Edward at 300 m/min. Frank always arrives at point A of the track before Edward. What is the farthest distance point A could be from Frank's end of the track?

8. If two resistances R_1 and R_2 are connected in parallel to one voltage source, the currents I_1 and I_2, through the resistances, add up to the current at the voltage source, and the voltage E is such that: $E = I_1R_1 = I_2R_2$. If $R_1 = 20\ \Omega$ (ohms), $R_2 = 30\ \Omega$, and the current at the voltage source is at least 5 A (amperes), what is the minimum current through R_2?



9. One of two oil tanks, each originally containing 3000 L of oil, is being emptied at the rate of 15 L/min, while the other is being filled at the rate of 20 L/min. How many minutes will it take for the contents of the second tank to become double that of the first?

C 10. Henry can run the first 12 km of a 33 km course at an average speed of 16 km/h. What speed must he average over the last 21 km in order to run the entire course in less than 2.5 h?

11. The degree measure M of each interior angle of a regular polygon with n sides is given by the formula $M = \dfrac{(n-2)180}{n}$. How many sides must a polygon have in order for the degree measure of each interior angle to be at least 162?

12. A piece of wire 20 cm long is cut into 2 pieces, one of which is bent to form a square, the other of which is bent to form a circle. If the square must have an area of at least 9 cm², what is the largest area that the circle can have?

2-6 Additional Properties of Order

The results of Exercises 27 and 28 on page 49 can be stated in the following theorem about the order of opposites and 0.

Theorem. For all real numbers a,

$$\text{if } a > 0, \text{ then } -a < 0;$$
$$\text{if } a < 0, \text{ then } -a > 0.$$

Another useful inequality can be stated thus:

Theorem.

If a is a nonzero real number, then $a^2 > 0$.

The proof is simple. If $a > 0$, then $a \cdot a > a \cdot 0$, from which $a^2 > 0$. On the other hand, if $a < 0$, then $-a > 0$, and $(-a)(-a) > (-a)(0)$, or $a^2 > 0$.

A corollary of the foregoing theorem is that $1 > 0$, because for $a = 1$, $1^2 > 0$ and $1^2 = 1$. Accordingly, by the first theorem in this section, $-1 < 0$.

We can use the fact that $1 > 0$ to prove the following property of reciprocals.

Theorem. For all nonzero real numbers a,

$$\text{if } a > 0, \text{ then } \frac{1}{a} > 0;$$

$$\text{if } a < 0, \text{ then } \frac{1}{a} < 0.$$

To prove this theorem, we shall use a method of reasoning called an **indirect proof.** In an indirect proof, you begin by assuming that the conclusion of a theorem is false, even though the hypothesis is accepted as true. You then show that a sequence of logically correct steps leads you to contradict an accepted fact, such as the hypothesis, an axiom, or a previously proved theorem. Because the assumption that the conclusion of the theorem is false leads to a contradiction, you know that the conclusion cannot be false, and thus that the theorem must be true.

As an example of an indirect proof, let us prove the first part of the last theorem stated on page 54.

PROOF

Suppose that a is a real number such that $a > 0$. To show that $\frac{1}{a} > 0$, we shall show that assuming $\frac{1}{a}$ is **not** greater than 0 (in symbols, $\frac{1}{a} \not> 0$) leads to a contradiction.

If $\frac{1}{a} \not> 0$, then by the comparison axiom of inequality there are two cases to consider: **(1)** $\frac{1}{a} = 0$, and **(2)** $\frac{1}{a} < 0$.

Case 1: Assume that $\frac{1}{a} = 0$.

1. $\frac{1}{a} \cdot a = 0 \cdot a$	Multiplicative property of equality
2. $\frac{1}{a} \cdot a = 0$	Multiplicative property of 0
3. $1 = 0$	Axiom of multiplicative inverses

Case 2: Assume that $\frac{1}{a} < 0$.

1. $\frac{1}{a} < 0$ and $a > 0$	Hypothesis
2. $\frac{1}{a} \cdot a < 0 \cdot a$	Multiplicative property of order
3. $\frac{1}{a} \cdot a < 0$	Multiplicative property of 0
4. $1 < 0$	Axiom of multiplicative inverses

In each case, the last step contains a statement that contradicts the fact that $1 > 0$, which was deduced on page 54. Therefore, the assumption that $\frac{1}{a} \not> 0$ leads to contradictions and must be incorrect. Hence, $\frac{1}{a} > 0$.

The proof of the second part of the theorem is left as an exercise for you (Exercises 1–2 on page 56).

To Write an Indirect Proof of a Theorem

1. Assume that the conclusion of the theorem is false.
2. Reason from this assumption until you obtain a statement contradicting the hypothesis, an axiom, or a previously proved theorem.
3. Point out that the assumption must be incorrect, so that the conclusion of the theorem must be true.

Oral Exercises

For each of the following, state the assumption with which you would begin an indirect proof. Assume that each variable denotes a real number, and no denominator is 0.

1. If $a^2 > 0$, then $a \neq 0$.
2. If $a + c \neq b + c$, then $a \neq b$.
3. If $a \neq b$, then $\frac{1}{a} \neq \frac{1}{b}$.
4. If $\frac{1}{a} \neq \frac{1}{b}$, then $a \neq b$.
5. If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$.
6. If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.
7. If $a > b$ and $c < 0$, then $ac < bc$.
8. If $a > b$, then $\frac{1}{a-b} \geq 0$.

Written Exercises

Justify each step in the indirect proofs of the following theorems. For the step designated "contradiction" tell what fact is contradicted. Assume that each variable denotes a real number.

Theorem I. If $a < 0$, then $\frac{1}{a} < 0$.

PROOF

A

1. Case 1: Assume that $\frac{1}{a} = 0$.
   1. $\frac{1}{a} \cdot a = 0 \cdot a$
   2. $\frac{1}{a} \cdot a = 0$
   3. $1 = 0$
   4. Contradiction

2. Case 2: Assume that $\frac{1}{a} > 0$.
   1. $\frac{1}{a} > 0$ and $a < 0$
   2. $\frac{1}{a} \cdot a < 0 \cdot a$
   3. $\frac{1}{a} \cdot a < 0$
   4. $1 < 0$
   5. Contradiction

Theorem II. If $a > b > 0$, then $\frac{a}{b} > 1$.

PROOF

3. Case 1: Assume that $\frac{a}{b} = 1$.

1. $\frac{a}{b} \cdot b = 1 \cdot b$
2. $\frac{a}{b} \cdot b = b$
3. $\left(a \cdot \frac{1}{b}\right)b = b$
4. $a\left(\frac{1}{b} \cdot b\right) = b$
5. $a \cdot 1 = b$
6. $a = b$
7. Contradiction

4. Case 2: Assume that $\frac{a}{b} < 1$.

1. $\frac{a}{b} < 1$ and $a > b > 0$
2. $\frac{a}{b} \cdot b < 1 \cdot b$
3. $\frac{a}{b} \cdot b < b$
4. $\left(a \cdot \frac{1}{b}\right)b < b$
5. $a\left(\frac{1}{b} \cdot b\right) < b$
6. $a \cdot 1 < b$
7. $a < b$
8. Contradiction

In Exercises 5–16 prove the given theorem using either a direct or an indirect proof. Assume that each variable denotes a real number.

B 5. If $a > 1$, then $\frac{1}{a} < 1$.

6. If $a > 0$ and $\frac{1}{a} < 1$, then $a > 1$.

7. If $ac > bc$ and $c > 0$, then $a > b$.

8. If $ac > bc$ and $c < 0$, then $a < b$.

C 9. If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.

10. If $a > b > 0$, then $a^2 > b^2$.

11. If $a > 0$, $b > 0$, and $a^2 > b^2$, then $a > b$.

12. If $a < b < 0$, then $\frac{1}{a} > \frac{1}{b}$.

13. If $a < 0$, $b < 0$, and $a^2 > b^2$, then $a < b$.

14. If $a < 0$, $b < 0$, and $a > b$, then $a^2 < b^2$.

15. If a, b, c, and d are all *positive* with $ab < cd$ and $b > d$, then $a < c$. (Hint: First show that $ab > ad$.)

16. If a, b, c, and d are all *negative* with $ab < cd$ and $b < d$, then $a > c$. (Hint: First show that $ab > ad$.)

2-7 Absolute Value and Order

It is sometimes convenient to work with the nonnegative (0 or positive) one of the pair a and $-a$. We refer to this number as the **absolute value** of a and denote it by $|a|$. For example, $|-3|$ is the positive one of the pair 3 and -3, so $|-3| = 3$. Similarly, $|7| = 7$, $|-8| = 8$, and $|0| = 0$. Formally, we make the definition:

> If a is a real number, then
> $$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

EXAMPLE 1 Solve $|x - 3| = 4$ over $\mathcal{R}$.

SOLUTION By definition, you have the disjunction

$$\begin{aligned} (x - 3) &= 4 \quad &\textbf{or}\quad -(x - 3) &= 4, \\ x - 3 &= 4 \quad &\textbf{or}\quad x - 3 &= -4, \\ x &= 7 \quad &\textbf{or}\quad x &= -1. \end{aligned}$$

$\therefore$ the solution set is $\{-1, 7\}$. Answer.

Graphically, an expression of the form $|a - b|$ or $|b - a|$ represents the distance (nondirected) between the graph of a and the graph of b, as shown in Figure 4. In particular, $|a|$ or $|-a|$ represents the distance between the origin and the graph of a or $-a$ (Figure 5).

Figure 4

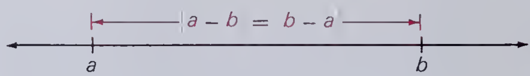
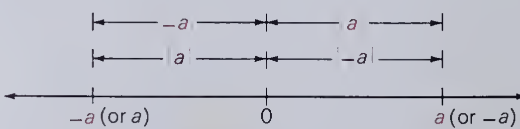


Figure 5



EXAMPLE 2 Solve $|x - 5| \leq 3$ over $\mathcal{R}$ and graph its solution set.

SOLUTION By inspection, $|x - 5|$ represents the distance between the graph of 5 and the graph of x. The sentence asserts that this distance is 3 or less. Since 2 and 8 are 3 units from 5, you see at once that $|x - 5| \leq 3$ is equivalent to $2 \leq x \leq 8$. Thus, by inspection, the solution set is $\{x\colon 2 \leq x \leq 8\}$, whose graph is as shown.

More formally,

$$|x - 5| \leq 3$$

is equivalent to the compound sentence

$$\begin{array}{rcr} (x - 5) \leq 3 & \text{and} & -(x - 5) \leq 3, \\ x - 5 \leq 3 & \text{and} & x - 5 \geq -3, \\ x \leq 8 & \text{and} & x \geq 2. \end{array}$$

Thus the solution set can be represented as

$$\{x\colon x \leq 8\} \cap \{x\colon x \geq 2\}$$

or

$$\{x\colon 2 \leq x \leq 8\},$$

whose graph is shown on page 58. Answer.

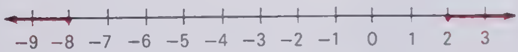
Note from the foregoing example that sentences of the form $|x - a| \leq b$ are equivalent to conjunctions. On the other hand, sentences of the form $|x - a| \geq b$ are equivalent to disjunctions.

EXAMPLE 3 Solve $|z + 3| \geq 5$ over $\mathcal{R}$ and graph its solution set.

SOLUTION Write $|z + 3| \geq 5$ as $|z - (-3)| \geq 5$. Then, by inspection, the distance between the graph of z and the graph of -3 must be 5 or greater. Since -8 and 2 are each 5 units from -3, the solution set must be

$$\{z\colon z \leq -8\} \cup \{z\colon z \geq 2\},$$

whose graph is shown below.



More formally,

$$|z + 3| \geq 5$$

is equivalent to

$$\begin{array}{rcr} (z + 3) \geq 5 & \text{or} & -(z + 3) \geq 5, \\ z + 3 \geq 5 & \text{or} & z + 3 \leq -5, \\ z \geq 2 & \text{or} & z \leq -8. \end{array}$$

Thus, again, the solution set is

$$\{z\colon z \leq -8\} \cup \{z\colon z \geq 2\},$$

whose graph is shown above. Answer.

Oral Exercises

State the value of each expression.

1. $|-8|$
2. $|3| + |-3|$
3. $|-7 + 7|$
4. $|5 - 2|$
5. $|4| - |9|$
6. $|6| - |-6|$
7. $4 - |4|$
8. $|-2| - |-2|$
9. $|-2||5| + 10$
10. $|-8| + |8|$
11. $|0|$
12. $3 - |-3|$
13. What is the solution set of the sentence $|x| = -2$?
14. Express the condition "$x > 5$ or $x < -5$" as a single inequality using absolute value.

Written Exercises

Simplify each expression.

A

1. $-|3 - 5|$
2. $|-8| - |2|$
3. $3|-4 - 7|$
4. $|-9| + |9|$
5. $|-6| - |6|$
6. $|4 - 1| - 3|2 - 7|$

Solve each open sentence over $\mathcal{R}$ and graph its solution set.

7. $|t| = 3$
8. $|x| < 6$
9. $|b - 2| \geq 1$
10. $|r + 4| = 5$
11. $|y - 5| < 2$
12. $|z + 7| \leq 3$
13. $|2n - 3| > 7$
14. $|2m + 1| \leq 5$
15. $2|x - 3| \geq 4$
16. $\frac{1}{2}|k + 9| < 3$
17. $|3x + 2| > 1$
18. $\dfrac{|z - 4|}{3} = 1$

B

19. $-3|a + 1| \leq -15$
20. $|5 - 2c| > 7$
21. $\frac{2}{5}|y + 3| < 9$
22. $|\frac{3}{2}z + 6| \geq 3$
23. $\left|\dfrac{2p - 7}{3}\right| > 1$
24. $\left|\dfrac{5 - 3v}{2}\right| < 4$

Tell whether or not each of the following statements is true for *all* real numbers *a*, *b*, and *c*. If it is not true, give a counterexample.

25. $|a + b| = |a| + |b|$
26. $|ab| = |a||b|$
27. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
28. $|a - b| = |a| - |b|$
29. If $|a| = |b|$, then $a = b$.
30. $|a - b|^2 = (a - b)^2$
31. $a|b + c| = |ab + ac|$
32. $\dfrac{|a + b|}{c} = \left|\dfrac{a + b}{c}\right|$
33. If $|a - b| = 0$, then $a = b$.

Solve each sentence over $\mathcal{R}$ and graph its solution set.

C

34. $2 \leq |x| < 5$
35. $3 < |y| \leq 4$
36. $2 < |x - 1| < 3$
37. $1 \leq |x - 3| \leq 4$
38. $3 < |2y - 4| < 5$
39. $4 \leq |8 - 2y| \leq 10$

Self-Test 2

VOCABULARY

is less than (p. 45)
order (p. 45)
is greater than (p. 45)
converse (p. 46)
equivalent inequalities (p. 48)
compound sentence (p. 50)
disjunction (or) (p. 50)
conjunction (and) (p. 50)
intersection of sets (p. 51)
union of sets (p. 51)
indirect proof (p. 54)
absolute value (p. 58)

Solve each open sentence over $\mathcal{R}$ and graph its solution set.

1. $x + 8 > 5$ 2. $5 - 4y > -3$ *Obj. 1, p. 45*

3. $3(n + 2) \leq 2(8 - n)$ 4. $-2 < 3k - 5 \leq 7$

5. An equilateral triangle is made from a 50 cm length of wire. A square, each of whose sides is at least 8 cm long, is made from the remaining wire. What is the maximum length of a side of the equilateral triangle? *Obj. 2, p. 45*

6. Use an indirect proof to show that if $ac > c$ and $c > 0$, then $a > 1$. *Obj. 3, p. 45*

Solve each open sentence over $\mathcal{R}$ and graph its solution set.

7. $|z - 4| \leq 5$ 8. $|2x + 9| > 5$ *Obj. 4, p. 45*

Check your answers with those at the back of the book.

The Bernoulli Family
17th–18th centuries

This Swiss family produced more mathematicians than any other family in the history of mathematics. Within four generations at least eight members achieved distinction in mathematics and physics. Of these, brothers Jacques I (1654–1705) and Jean I (1667–1748) were the most famous.

Jacques's main interest was in calculus, probability, and graphs; one curve even bears his name—the "Lemniscate of Bernoulli," described by the polar equation $r^2 = a^2 \cos 2\theta$.

Jean helped a great deal to spread the calculus throughout Europe. He also published works in chemistry, physics, and astronomy. His son Daniel (1700–1782) has been called "the founder of mathematical physics."

Careers

in Electronics

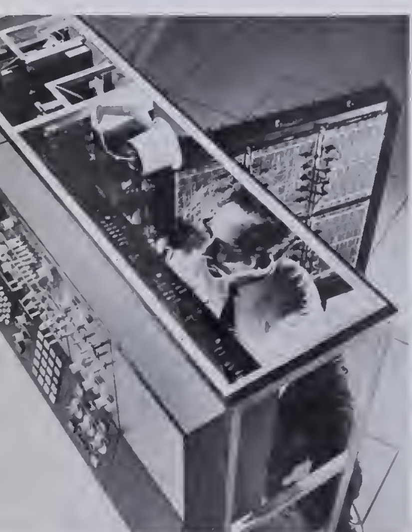
Electronic components are a vital part of many of the machines used today, including computers, radio and television equipment, quality control equipment for industry, satellites, and telephone equipment. Electronics engineers must often solve equations about circuits. One important equation in electronics is **Ohm's Law,**

$$V = IR,$$

where V, measured in volts, is the difference in potential energy between two points when a current I, measured in amperes (A), flows through a resistor, or wire, of resistance R. The unit used to measure resistance is the ohm (Ω).

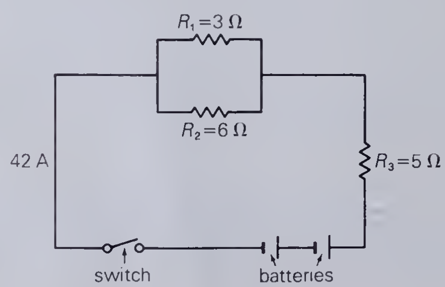
If two or more resistors are connected one after the other, , they are said to be *in series*. If two or more resistors are connected like this , they are said to be *in parallel*. In a series circuit, the total resistance is simply the sum of the individual resistances. However, in a parallel circuit the sum of the reciprocals of the individual resistances is the reciprocal of the total resistance.

Circuit repair (above) and customer engineer at work (below).



EXAMPLE In the circuit diagram below, resistors R_1 and R_2 are connected in parallel, and resistor R_3 is connected to the circuit in series. A current of 42 A is flowing through the circuit. To find the combined resistance R of R_1 and R_2, we know that $\frac{1}{R} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Therefore $R = 2\ \Omega$.

We can now determine the potential difference across each resistor by applying Ohm's Law. Since the potential difference across each resistor connected in parallel is the same, we have $V = 42 \cdot 2 = 84$ V for each of the resistors R_1 and R_2. For resistor R_3, $V = 42 \cdot 5 = 210$ V.



Chapter Summary

1. One polynomial is added to or subtracted from another by adding or subtracting coefficients of like terms. The result of this procedure is a simpler polynomial *equivalent* to the original sum or difference.

2. Transformations that produce an equivalent equation are:
   1. Substituting for either member of the given equation an expression equivalent to it.
   2. Adding to or subtracting from each member of the given equation the same polynomial in any variable(s) appearing in the equation.
   3. Multiplying or dividing each member by the same *nonzero* number.

3. To solve an applied problem stated in words:
   1. Read it carefully and decide what numbers are asked for.
   2. Choose a variable to represent each number asked for or described in the problem.
   3. Write an open sentence showing the relationship given in the problem.
   4. Solve the open sentence.
   5. Check the results with the requirements stated in the problem.

4. The real numbers are ordered. (See the *comparison axiom* and the *transitive, additive,* and *multiplicative properties* on pages 46–47.)

5. Transformations that produce an equivalent inequality are:
   1. Substituting for either member of the inequality an expression equivalent to that member.
   2. Adding to or subtracting from each member of the inequality the same polynomial in any variable(s) appearing in the inequality.
   3. Multiplying or dividing each member by the same *positive* number.
   4. Multiplying or dividing each member by the same *negative* number and reversing the direction of the inequality symbol.

6. Compound sentences involving the word "or" are *disjunctions,* and are true whenever at least one of the simple sentences involved is true.
7. Compound sentences involving the word "and" are *conjunctions,* and are true whenever both of the simple sentences involved are true.
8. To write an *indirect proof,* you start with the assumption that the conclusion of a theorem is false and reason from this to a contradiction of the hypothesis, an axiom, or a previously proved theorem.
9. The *absolute value* of a real number a is defined by

$$|a| = \begin{cases} a, \text{ if } a \geq 0, \\ -a, \text{ if } a < 0. \end{cases}$$

Chapter Review

Subtract the second polynomial from the first. Simplify the resulting expression.

1. $(-3x + 2) - (4 - 5x)$ 2-1

   a. $-8x - 2$ b. $2x - 2$ c. $-2x + 2$ d. $8x - 2$

2. $(4a + b) + (a - b) - (a + 3b)$

   a. $5a - b$ b. $4a - 3b$ c. $2a - b$ d. $2a + 3b$

Solve the equations over $\mathcal{R}$.

3. $7y + 2 = 51$ 2-2

   a. 3 b. 5 c. 7 d. 9

4. $4x - (4 + x) = 11$

   a. 3 b. 5 c. 7 d. 9

A rectangle is 4 cm longer than it is wide and has a perimeter of 32 cm. In Test Items 5–6, let x represent the width of the rectangle.

5. Which equation expresses the relation of x to the perimeter correctly? 2-3

   a. $x(x + 4) = 32$ b. $2x + 2(x + 4) = 32$ c. $2(x - 4) + 2x = 32$

6. What is the length of the rectangle?

   a. 6 cm b. 8 cm c. 5 cm d. 10 cm

Choose the symbol that replaces _?_ to make a true statement.

7. If $a < b$ and $c > 0$, then ac _?_ bc. 2-4

   a. $>$ b. $<$ c. $=$ d. $\leq$

8. If $a < b$ and $c < 0$, then ac _?_ bc.

   a. $>$ b. $<$ c. $=$ d. $\geq$

9. Which one of the following sentences is equivalent to $x - 3 \leq 7$? 2-5

   a. $x - 3 = 7$ or $x - 3 < 7$ b. $x - 3 = 7$ and $x - 3 < 7$

   c. $x = 7$ or $x - 3 < 7$ d. $x = 7$ and $x - 3 < 7$

10. Which one of the following sentences is equivalent to $-4 < x < 8$?

    a. $-4 < x$ or $x < 8$ b. $-4 < x$ and $x < 8$ c. $-4 < x$ or $8 < x$

In Test Items 11–12 state the assumption with which you would begin an indirect proof of the given statement.

11. If $a \leq 0$ and $b \leq 0$, then $a + b \leq 0$. 2-6

    a. $a + b = 0$ b. $a + b > 0$ c. $a + b < 0$

12. If $a \geq b$ and $c \leq 0$, then $ac \leq bc$.

a. $ac = bc$ b. $ac \geq bc$ c. $ac > bc$

13. Evaluate $|-8| + |8|$. 2-7

a. 0 b. -64 c. 64 d. 16

14. Which one of the following sentences is equivalent to $|x - 2| > 4$?

a. $x - 2 > 4$ or $-x + 2 > 4$ b. $x - 2 > 4$ or $-x - 2 > 4$

c. $x - 2 = 4$ or $-x + 2 > 4$ d. $x - 2 > 4$ or $x + 2 < -4$

Chapter Test

Simplify each expression.

1. $(3x^2y^2 - 2xy + 7) + (x^2y^2 + 2xy) - (3x^2y^2 + 4)$ 2-1
2. $(8z^3 + 5z^2 - 2z + 7) - (3z^3 - 12z - 2)$

Solve over $\mathcal{R}$.

3. $4k - (3 + k) = 9$ 4. $2(x + 7) = 6$ 2-2

5. Find two consecutive even integers such that the sum of twice the first and the second equals 116. 2-3

6. One base of a trapezoid is 5 cm longer than the other. If the height of the trapezoid is 16 cm and its area is 200 cm², what are the lengths of the bases?

Solve the inequality over $\mathcal{R}$ and graph its solution set.

7. $3 - 4x < -5$ 8. $\frac{1}{3}(y - 1) < 2 + \frac{y}{3}$ 2-4

9. $-4 < 2(c - 1) < 8$ 10. $-2 < 1 - \frac{y}{2} \leq 3$ 2-5

11. Give an indirect proof that if $x \geq 4$, then $2x + 3 \geq 11$. 2-6

12. Evaluate $\frac{|3 - 4| - |2|}{|8 - 17|}$. 2-7

13. Solve the inequality $5 \leq |2x - 3|$ over $\mathcal{R}$ and graph its solution set.

Apartment buildings in Creteil, a newly built suburb of Paris.

3 Linear Functions and Relations

Specifying Functions and Relations

***OBJECTIVES** for Sections 3-1 and 3-2:*
1. ***Determine the range corresponding to the domain of a function whose rule is given.***
2. ***Picture a function or relation by means of a mapping diagram and also by means of a graph.***

3-1 Functions

When you count to ten by 2's, you are really pairing the integers 1, 2, 3, 4, 5 with their doubles: 2, 4, 6, 8, 10. Such a set of *ordered pairs*,

$$\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\},$$

in which each *first component* is paired with exactly one *second component* according to a given rule, is called a **function**. The set of first components is called the **domain**, and the set of second components the **range**, of the function. Here the domain is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$. If we let x represent an element of the domain and y of the range, the rule for pairing in this case is $y = 2x$.

Some letters commonly used to name a function are f, g, h, F, G, and H. Thus, you might designate the "counting-by-2's" function as follows:

$$f = \{(x, y): y = 2x\},$$

read "f is the set of ordered pairs, x, y, such that y is two times x."

You might also use arrow notation:

$$f: x \to 2x,$$

read "the function f that assigns to x the value $2x$." Of course, in specifying a function you must also identify its domain.

The symbol $f(x)$, read "f of x," is often used instead of y to indicate the second component of the ordered pair in f whose first component is x. Thus you can write for f, above:

$$f = \{(x, f(x)): f(x) = 2x\}.$$

Then to denote "the value of f when $x = 3$," for example, you write $f(3)$. Here, $f(3) = 2 \times 3$, or 6.

Often a function is described simply by giving the rule, or formula, for obtaining the second component, $f(x)$. *The domain, unless otherwise specified, will be assumed to be the set of all real numbers x for which the rule produces one, and only one, real value for $f(x)$.*

EXAMPLE 1 If $f(x) = 2x + 1$, find (a) $f(3)$ and (b) $f(-7)$.

SOLUTION a. $f(3) = 2 \cdot 3 + 1 = 7$.
b. $f(-7) = 2 \cdot (-7) + 1 = -13$.

EXAMPLE 2 State the domain and range of the function $f: x \to \frac{1}{x^2}$.

SOLUTION The domain is the set of all real numbers except 0 because the rule $\frac{1}{x^2}$ produces a real value for each real value of x except 0. The range is the set of positive real numbers, since $\frac{1}{x^2} > 0$ for all real values of x (except 0). Answer.

EXAMPLE 3 If $f: x \to 5x + 3$ and $g: x \to x^2$, find $g(f(2))$ and $f(g(2))$.

SOLUTION First, to find $f(x)$ when $x = 2$, we have $f(2) = 5 \cdot 2 + 3 = 13$. Then, $g(f(2)) = g(13)$. Since $g(x) = x^2$, $g(13) = 13^2 = 169$.
Similarly, $g(2) = 2^2 = 4$, and $f(g(2)) = f(4) = 5 \cdot 4 + 3 = 23$.
$\therefore g(f(2)) = 169$ and $f(g(2)) = 23$. Answer.

Oral Exercises

1. Are (3, 6) and (6, 3) the same ordered pair of numbers? Why or why not?
2. Are (2, 3.5) and $(2, \frac{7}{2})$ the same ordered pair of numbers? Why or why not?

State the domain and the range of each function and give a rule for the pairing, such as $y = 2x$.

3. $\{(1, -1), (2, -2), (3, -3), (4, -4)\}$
4. $\{(-3, -4), (0, -1), (1, 0), (3, 2)\}$
5. $\{(-3, -1), (0, 2), (3, 5), (4, 6)\}$
6. $\{(2, \frac{1}{2}), (3, \frac{1}{3}), (-2, -\frac{1}{2}), (4, \frac{1}{4})\}$
7. $\{(2, 5), (7, 20), (50, 149), (1, 2)\}$
8. $\{(-2, -8), (1, 1), (\frac{1}{2}, \frac{1}{8}), (3, 27)\}$

State $f(-3)$ for each of the following functions.

9. $f\colon x \to -x + 1$
10. $f\colon x \to -(x + 1)$
11. $f\colon x \to -2x^2$
12. $f\colon x \to (-2x)^2$
13. $f\colon x \to \dfrac{2x}{3} - 1$
14. $f\colon x \to \dfrac{2x - 1}{3}$

State the domain and range of each function f whose rule is given.

15. $f(x) = |x + 2|$
16. $f(x) = \dfrac{1}{x - 2}$
17. $f(x) = -x^2$

Written Exercises

If $g\colon x \to -3x^2 - 4x + 7$, find each of the following.

A

1. $g(0)$
2. $g(1)$
3. $g(-2)$
4. $g(\frac{1}{2})$
5. $g(-4)$
6. $-g(4)$
7. $g(c)$
8. $g(-c)$

For each of the following functions, give a rule for the pairing.

9. $\{(2, 6), (5, 15), (-1, -3)\}$
10. $\{(1, -1), (5, 3), (8, 6)\}$
11. $\{(6, 2), (12, 4), (2, \frac{2}{3})\}$
12. $\{(4, -8), (3, -6), (0, 0), (-5, 10)\}$
13. $\{(2, 9), (4, 19), (7, 34), (-3, -16)\}$
14. $\{(2, 0), (40, 19), (14, 6), (-8, -5)\}$

B

15. $\{(-2, 4), (\frac{1}{2}, \frac{1}{4}), (3, 9), (0, 0)\}$
16. $\{(4, -\frac{1}{4}), (3, -\frac{1}{3}), (1, -1), (\frac{1}{2}, -2)\}$
17. $\{(1, 2), (4, 17), (10, 101), (-5, 26)\}$
18. $\{(3, \frac{1}{2}), (5, \frac{1}{4}), (-1, -\frac{1}{2}), (\frac{1}{3}, -\frac{3}{2})\}$

If $f(x) = 3x$ and $g(x) = x^2 - 1$, find each of the following.

19. $f(g(2))$
20. $g(f(2))$
21. $g(f(-1))$
22. $f(g(\frac{1}{2}))$
23. $g(f(-\frac{1}{3}))$
24. $f\left(g\left(\dfrac{1}{a}\right)\right)$

C

25. If f is a function such that $f(x + 2) = f(x) + f(2)$ for all $x \in \mathcal{R}$, show that (a) $f(0) = 0$, and hence (b) $f(-2) = -f(2)$.
26. Give an example of a function (a) that satisfies the hypothesis of Exercise 25; (b) that does not satisfy the hypothesis of Exercise 25.
27. If f is a function such that $f(-x) = -f(x)$, for all $x \in \mathcal{R}$, show that $f(0) = 0$.
28. If f is a function such that $f(x) = x^2 + 1$ for all $x \in \mathcal{R}$, find $f(2 + h) - f(h)$.

3-2 Picturing Functions and Relations

In mathematics a **relation** is defined as *any* set of ordered pairs. The set of all the first coordinates is called the domain and the set of all the second coordinates the range.

Every function is a relation, but not every relation is a function. For example, compare these relations:

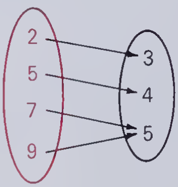
$$R = \{(x, y)\colon y^2 = x\}$$
$$F = \{(x, y)\colon y = x^2\}$$

In the relation R, when $x = 4$ you obtain two different pairs with the same first component, (4, 2) and (4, −2). In fact, each positive value, a, of x is paired with two values for y, namely, $\sqrt{a}$ and $-\sqrt{a}$. Hence, the relation R is *not* a function, since it does not pass the test, "with each x, only one y." In F the situation is reversed; each positive value of y is paired with two different values of x, for example: (4, 16) and (−4, 16). Nevertheless, F is a function because each x-value is associated with exactly one y-value.

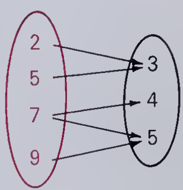
The rule for the pairing in a relation can be thought of as "mapping" the domain onto the range.

EXAMPLE 1 Tell whether or not each mapping diagram pictures a function, and give your reasons.

a.



b.



SOLUTION

a. The mapping represents a function because each element in the domain is mapped onto just one element in the range. Answer.

b. The mapping does not represent a function because the element 7 in the domain is mapped onto two different elements, 4 and 5, in the range. Answer.

Another way to picture a relation depends on the fact that a **plane rectangular** (or **Cartesian**) **coordinate system** establishes a one-to-one correspondence between the set of points in the plane and the set of ordered pairs of real numbers. The horizontal x-axis serves to locate values in the domain, while the vertical y-axis locates the corresponding values in the range.

The four red dots in Figure 1 represent the graph of the four ordered pairs in the relation $\{(-2, 1), (1, -2), (1, 3), (2, 4)\}$. This relation is not a function, because there are

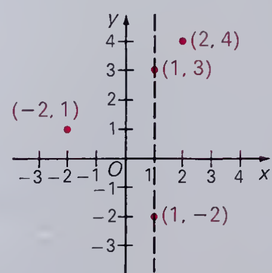


Figure 1

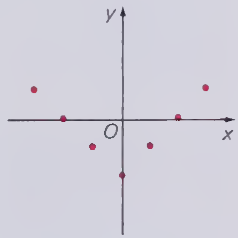
two points in the graph that have the same **abscissa,** or x-coordinate, paired with different **ordinates,** or y-coordinates: $(1, -2)$ and $(1, 3)$. This fact suggests a simple test to determine whether or not a graph represents a function:

> If you can draw a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

EXAMPLE 2 Draw the graph of the function $f: x \rightarrow |x| - 2$, with domain $\{-3, -2, -1, 0, 1, 2, 3\}$.

SOLUTION First make a table of values for x and y. Then graph these ordered pairs.

x	$y = \|x\| - 2$
-3	1
-2	0
-1	-1
0	-2
1	-1
2	0
3	1

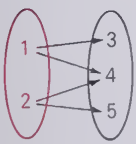


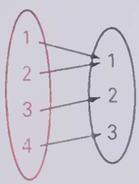
Oral Exercises

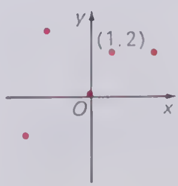
In Exercises 1–4 tell whether or not the given relation is a function.

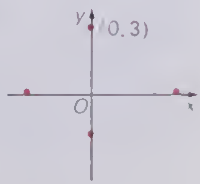
1. $\{(-3, 1), (0, 0), (2, 2)\}$
2. $\{(1, 5), (-2, 0), (1, -1), (3, 6)\}$
3. $\{(-1, 3), (-3, 2), (3, 5), (-3, 0)\}$
4. $\{(2, 7), (-1, 7), (0, 7), (4, 7)\}$
5. Can a mapping diagram in which the range has more values than the domain picture a function?
6. *Must* a mapping diagram in which the domain has more values than the range picture a function?

In Exercises 7–10 state the ordered pairs in the relation pictured by the mapping diagram or graph.

7. 

8. 

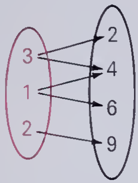
9. 

10. 

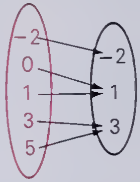
In Exercises 11–20 tell whether or not the relation pictured by the mapping diagram or graph is a function.

11–14. Use the relations pictured in Exercises 7–10.

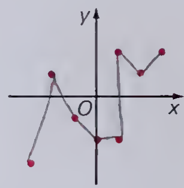
15.



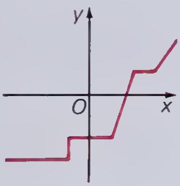
16.



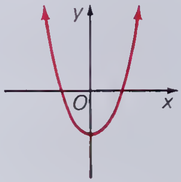
17.



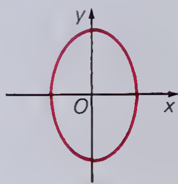
18.



19.



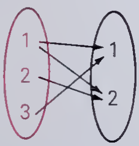
20.



Written Exercises

In Exercises 1–4 write the set of ordered pairs in the relation pictured by the mapping diagram and tell whether or not the relation is a function.

A **1.**



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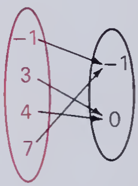

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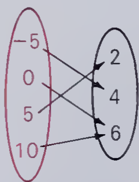

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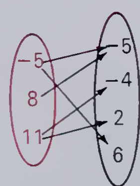


In Exercises 5–10 draw a mapping diagram for the given relation and determine whether or not the relation is a function.

5. $\{(3, 1), (4, 2), (5, 3)\}$

6. $\{(-2, 1), (1, 1), (-2, 2)\}$

7. $\{(3, 2), (2, -1), (1, 2)\}$

8. $\{(-3, 5), (4, 5), (1, 4), (7, 4)\}$

9. $\{(1, -1), (2, -1), (3, -1), (4, -1)\}$

10. $\{(2, 3), (2, -1), (2, 4), (2, 5)\}$

11–16. Graph the relations in Exercises 5–10. If the relation is not a function, draw a vertical line that intersects the graph in more than one point.

Graph each relation over the domain $\{-2, -1, 0, 1, 2, 3\}$ and tell whether or not the relation is a function.

17. $\{(x, y)\colon y = 2x\}$

18. $\{(x, y)\colon y = -x + 1\}$

19. $\{(x, y): |y| = x + 2\}$

20. $\{(x, y): |y| = 2|x| - 1\}$

B 21. $\{(x, y): y = x^2 - 4\}$

22. $\{(x, y): y = x^2 - x + 1\}$

23. $\{(x, y): |y| = -|x| + 3\}$

24. $\left\{(x, y): |y| = \dfrac{6}{|x| + 1}\right\}$

Graph each relation when the domain is the subset of the integers $\{-3, -2, -1, 0, 1, 2, 3\}$ for which there are real values for y in the set $\{-3, -2, -1, 0, 1, 2, 3\}$.

C 25. $\{(x, y): |y| < 2|x|\}$

26. $\{(x, y): |x - y| < 3\}$

27. $\left\{(x, y): |y| < \dfrac{|x|}{|x| - 1}\right\}$

28. $\{(x, y): |x| + |y| \leq 3\}$

29. $\{(x, y): x^2 + y^2 \leq 9\}$

30. $\{(x, y): x^2 - y^2 > 1\}$

Self-Test 1

VOCABULARY function (p. 67), domain (p. 67), range (p. 67), relation (p. 70), coordinate system (p. 70), abscissa (p. 71), ordinate (p. 71)

For $f: x \rightarrow -2x$ and $g: x \rightarrow x^2 + 1$, find:

1. $f(-3)$
2. $g(-2)$
3. $g(f(3))$
4. $f(g(3))$

Obj. 1, p. 67

Let $r = \{(3, 0), (1, 2), (1, -2), (-1, 0)\}$.

5. Picture r by means of a mapping diagram.
6. Picture r by means of a graph.

Obj. 2, p. 67

Check your answers with those at the back of the book.

Hilda Geiringer
1894–1973

Hilda Geiringer was born and educated in Vienna. She taught at Berlin University, and after moving to the United States in 1939 she taught at Bryn Mawr College and at Wheaton College in Massachusetts. She published many articles dealing with probability and statistics, and also studied fluids and the plasticity of solids. She later did research at Harvard University, where she compiled and edited the works of her husband, Richard von Mises, after his death.

Graphs of Linear Equations and Inequalities

OBJECTIVES for Sections 3-3 and 3-4:
1. Graph a linear equation.
2. Use the graph of the associated linear equation to graph a linear inequality.

3-3 The Graph of a Linear Equation

A *solution* of the open sentence

$$3x + y = 6$$

is any ordered pair (x, y) of real numbers that satisfies the equation. The *solution set* of the equation consists of all its solutions, that is, all ordered pairs in the relation

$$L = \{(x, y)\colon 3x + y = 6, x, y \in \mathbb{R}\}.$$

By transforming $3x + y = 6$ into the equivalent equation

$$y = 6 - 3x$$

and assigning values to x, you can then easily obtain solutions (members of L), such as the following, which are graphed in red in Figure 2:

$$(-1, 9), (0, 6), (2, 0), (3, -3)$$

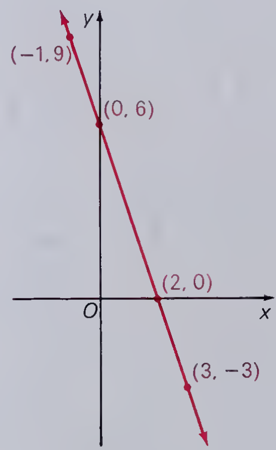


Figure 2

While it is not possible to graph the infinite set of pairs in L, the straight line on which all four of these points appear to lie suggests a pattern for the graph of the solution set of the equation $3x + y = 6$. The graph is a line that consists of *all* the points, and *only* those points, that represent the members of L. We call this line the *graph* of L. In general:

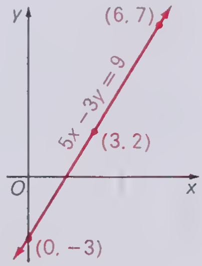
> The graph of an equation of the form $Ax + By = C$, where A, B, $C \in \mathbb{R}$ and A and B are not both zero, is a straight line. Conversely, every straight line in the plane is the graph of a *linear equation in two variables*, that is, an equation of the form $Ax + By = C$, where A and B are not both zero.

When graphing a linear equation, you need graph only two solutions in order to determine its unique line. It is safer, however, to plot a third point just as a check.

EXAMPLE 1 Graph $5x - 3y = 9$.

SOLUTION First transform the equation into an equivalent one that expresses y in terms of x.

$$5x - 3y = 9$$
$$-3y = 9 - 5x$$
$$y = -3 + \tfrac{5}{3}x$$



It is usually simpler to graph the equation if you assign values to x that will make the solution (x, y) a pair of integers. Thus the values 0, 3, and 6 yield the solutions $(0, -3)$, $(3, 2)$, and $(6, 7)$. Graph the points and the line through them. Answer.

In the relation

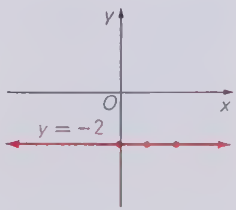
$$\{(x, y)\colon Ax + By = C\},$$

if neither A nor B is zero, then each value of x is paired with exactly one value of y and hence the relation is a function. If $A = 0$, it is still a function, as you can see from the following example.

EXAMPLE 2 Graph $Ax + By = C$ when $A = 0$, $B = 2$, $C = -4$. Explain why the graph represents a function.

SOLUTION The equation to be graphed is $0 \cdot x + 2y = -4$, or

$$y = -2.$$

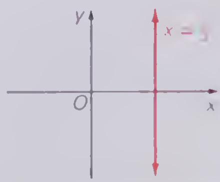


The solution set consists of all ordered pairs (x, y) where x is any real number and y is always -2; for example, $(0, -2)$, $(1, -2)$, $(2, -2)$. Graph the line through these points. Since no vertical line intersects the horizontal-line graph representing $y = -2$ in more than one point, this graph represents a function. Answer.

If $B = 0$ and $A \neq 0$, then the graph of the solution set will be a *vertical* line. Such a relation, of the form $\{(x, y)\colon Ax + 0 \cdot y = C\}$, is not a function.

EXAMPLE 3 Graph $x = \frac{3}{2}$.

SOLUTION The solution set consists of all ordered pairs (x, y) such that $x = \frac{3}{2}$ and y is any real number; for example, $(\frac{3}{2}, -1)$, $(\frac{3}{2}, 0)$ and $(\frac{3}{2}, 1)$. Graph the points and the line through them. Answer.



Oral Exercises

Tell whether or not the given equation is that of a straight line.

1. $y = \frac{1}{2}x$
2. $y = -3$
3. $y = |x - 2|$
4. $-x + 15y = -3$
5. $y = \frac{2}{x} + 4$
6. $x = \frac{5}{2}$
7. $x^2 - y^2 = 1$
8. $xy = 6$

State which point(s) with the given coordinates lie on the graph of the given equation.

9. $y + 3x = 2$; $(-2, 8)$, $(2, 0)$, $(-1, 7)$, $(0, 1.5)$
10. $x - 2y = -3$; $(1, -2)$, $(-1, 1)$, $(1, 2)$, $(5, 4)$
11. $2x - y + 3 = 7$; $(-1, 2)$, $(0, 4)$, $(2, 0)$, $(2, -10)$
12. $2y + \frac{3}{2}x = 5$; $(6, -2)$, $(10, -5)$, $(4, -\frac{1}{2})$, $(5, -2)$

For each equation give the coordinates $(x, 0)$ and $(0, y)$ of the points, if any, where the graph intersects each axis.

13. $x - y = 5$
14. $4x + 3y = 12$
15. $2x - 5y = 15$
16. $y = -2$

State a value for _?_ that makes the ordered pair a solution of the given equation.

17. (_?_, 6); $5x - 2y = 8$
18. $(-5,$ _?_); $6x + 7y = -2$
19. (0, _?_); $-3x + 4y = 18$
20. (9, _?_); $y = -6$
21. (4, _?_); $x = 4$
22. (_?_, a); $2x + 11y = 3$
23. Give a necessary and sufficient condition on the coefficients A and/or B in the general linear equation $Ax + By = C$ that will ensure that the equation defines a function.

Written Exercises

In Exercises 1–16 graph the equation.

A 1–8. Use the equations in Oral Exercises 9–16 above.

9. $\frac{1}{3}x + \frac{1}{2}y = 4$
10. $0.2x - 0.5y = -1$
11. $4x + \frac{y}{3} + 6 = 0$
12. $3(x - 6) - 9(y + 2) = 0$

B

13. $y = |x|$
14. $y = |x| - 2$
15. $y = |x - 2|$
16. $y = |x| + x$

In Exercises 17–20 determine k in each equation so that the given ordered pair will be a solution.

17. $3x - ky = 17$; $(3, 2)$
18. $7y = k(x + 2)$; $(-4, 4)$
19. $x - \frac{1}{4}y + 3k = 0$; $(5, 2)$
20. $kx + (k - 2)y = 13$; $(-1, -2)$

In Exercises 21–24 graph the pair of equations in the same coordinate plane and give the coordinates of the point where the graphs seem to intersect. Then check whether or not these coordinates satisfy both equations.

C **21.** $x + y = 1, -x + y = 1$ **22.** $x - 3y = 6, 2x + y = 5$

23. $4x - y = 6, \frac{1}{2}x - y = -1$ **24.** $2x - y = -5; x + 3y = 1$

3-4 The Graph of a Linear Inequality

The *half-line* $\overset{\circ\!\!\longrightarrow}{PT}$ shown in red in Figure 3 is the graph of the relation

$$R = \{(x, y)\colon y > x\}$$

when the value of x is 2. You can see that every point on $\overset{\circ\!\!\longrightarrow}{PT}$ has 2 as its abscissa, and a real number greater than 2 for its ordinate. Note that the point P lies on the dashed line l, the graph of $y = x$, but not on $\overset{\circ\!\!\longrightarrow}{PT}$.

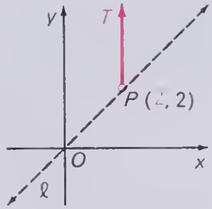


Figure 3

The pink-shaded region in Figure 4 consists of the totality of half-lines in the plane, such as $\overset{\circ\!\!\longrightarrow}{PT}$ of Figure 3, with "initial" points on the graph of $y = x$ and extending vertically upward from it. This region, called an **open half-plane** ("open" means the boundary is not included) is the graph of the given *linear relation* $\{(x, y)\colon y > x\}$. It is also the graph of the *linear inequality*

$$y > x.$$

Figure 4

Thus every point located *above* $y = x$, and no other point in the plane, has coordinates that satisfy the inequality $y > x$.

By similar reasoning you can see that the gray-shaded, open half-plane below l in Figure 4 is the graph of $y < x$.

In Figure 5 the shaded, **closed half-plane** ("closed" means the boundary line is included) is the graph of the inequality

$$y \geq x.$$

The graph consists of all those points, but no others, with coordinates that satisfy either $y > x$ or $y = x$. The fact that the boundary line, $y = x$, is shown solid rather than dashed indicates that the half-plane is closed rather than open.

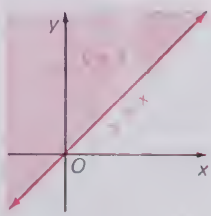
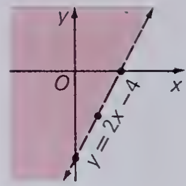


Figure 5

EXAMPLE 1 Graph the relation $\{(x, y)\colon 2x - y < 4\}$.

SOLUTION 1. Transforming to express y in terms of x, we have:

$$\begin{aligned} 2x - y &< 4 \\ -y &< 4 - 2x \\ y &> 2x - 4 \end{aligned}$$



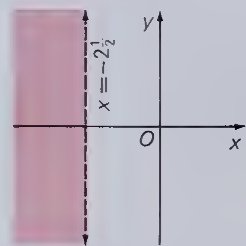
2. Draw the graph of $y = 2x - 4$ as a dashed line. Three solutions are $(0, -4)$, $(2, 0)$, and $(1, -2)$.

3. The graph is shown by the shaded open half-plane above the dashed line. Answer.

In general, the graph of a *linear inequality in two variables,* such as $Ax + By < C$, is either an open or closed half-plane (open for the signs $<$ and $>$, and closed for the signs $\leq$ and $\geq$). Its boundary is the graph of the *associated linear equation,* $Ax + By = C$.

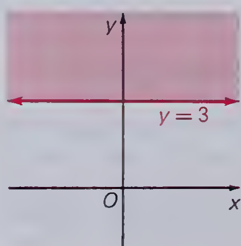
EXAMPLE 2 Graph the relation $\{(x, y)\colon 2x < -5\}$.

SOLUTION The boundary line is the graph of $2x = -5$, or $x = -2\frac{1}{2}$. The shaded, open half-plane is the graph of $\{(x, y)\colon 2x < -5\}$. Answer.



EXAMPLE 3 Graph the relation $\{(x, y)\colon y \geq 3\}$.

SOLUTION The boundary line is the graph of $y = 3$. The shaded, closed half-plane is the graph of $\{(x, y)\colon y \geq 3\}$. Answer.



Oral Exercises

In Exercises 1–9 tell whether the graph of the given inequality is an open or closed half-plane, and state the equation of the boundary line in the form $y = ax + b$ or $x = c$.

EXAMPLE $-x + 3y \leq 15$

SOLUTION The graph is a closed half-plane; $y = \frac{1}{3}x + 5$.

1. $y \leq 3$
2. $y > -x$
3. $x - y < 5$
4. $-2x + 6y \leq 4$
5. $y - 2 > 7$
6. $2x \geq -9$
7. $2x - 3y < b$
8. $2y - x \leq 6$
9. $2y + 10x - 3 \geq 0$

In Exercises 11–15 tell whether or not the point with the given coordinates is in the graph of the given inequality.

10. $y - x \geq 0$; $(2, 3)$
11. $3x + 7 \leq 5$; $(2, -1)$
12. $2y > 4x - 1$; $(1, -3)$
13. $-2x - 3y < 5$; $(1, 1)$
14. $x \geq -4$; $(2, -10)$
15. $8 < -y$; $(-3, 9)$

16. If the inequalities $y \geq 2x + 1$ and $y < 2x + 1$ are graphed on the same set of axes, describe the set of points that are common to the graphs.

17. Describe the set of points common to the graphs of $x \geq 0$ and $y \geq 0$.

Written Exercises

Graph each inequality as a shaded region on a coordinate plane.

A

1. $y > x$
2. $-x \leq 3y$
3. $x + y < 6$
4. $x - 2y > 4$
5. $2y + 4x \geq 3$
6. $-3y \leq 6$
7. $y + 2 < 3x$
8. $5 - 2y \geq 13 - 2x$
9. $-5x + 10y > 15$
10. $-14x < 7y$
11. $x + 6 \leq 3y$
12. $12x - 4y - 3 \geq 5$

In Exercises 13–18 graph both inequalities on the same plane, and shade the region containing all the points with coordinates that satisfy both inequalities.

B

13. $x \leq 2$ and $x \geq -3$
14. $x > -1$ and $x \geq y$
15. $y < 3$ and $y > x + 1$
16. $y \geq x - 2$ and $y \geq -2x - 2$
17. $2x < y + 3$ and $y < -x$
18. $y \leq -x + 4$ and $2y \leq x - 4$

C

19. Graph these inequalities on the same plane: $x > 2$; $y < 2x + 2$; $y > -x + 5$. Shade the region containing all the points with coordinates that satisfy all three inequalities.

Self-Test 2

VOCABULARY
solution of an open sentence (p. 74)
solution set of an open sentence (p. 74)
linear equation in two variables (p. 74)
linear inequality in two variables (p. 77)
open half-plane (p. 77)
closed half-plane (p. 77)

Graph each linear equation in the coordinate plane.

1. $\frac{2}{3}x + y = 3$ **2.** $y = 3$ **3.** $3x - 2y = 4$ *Obj. 1, p. 74*

Graph each linear inequality in the coordinate plane.

4. $-3x + y < -6$ **5.** $x + 26 \geq 8$ **6.** $8y < -24$ *Obj. 2, p. 74*

Check your answers with those at the back of the book.

Lines and Their Equations

OBJECTIVES for Sections 3-5 and 3-6:

1. ***Find the slope of the line through two given points.***
2. ***Find an equation of the line through two given points.***
3. ***Find an equation of a line when its slope and the coordinates of one of its points are given.***
4. ***Find an equation of a line when its slope and y-intercept are given.***

3-5 The Slope of a Line

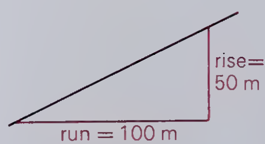


Figure 6

Figure 6 shows a hill that is rising steadily at a rate of 50 m of vertical "rise" for each 100 m of horizontal "run." The steepness, or *grade,* of such an incline is defined to be the ratio of rise to run, in this case, $\frac{50}{100}$ or 50%.

This hill can be represented mathematically by the line $y = \frac{1}{2}x$, shown in Figure 7. By observing the coordinates of P and Q, you can readily confirm that this line rises at a rate of 1 vertical unit for each 2 horizontal units. Hence its steepness, or slope, is

$$\frac{\text{rise}}{\text{run}} = \frac{1}{2}.$$

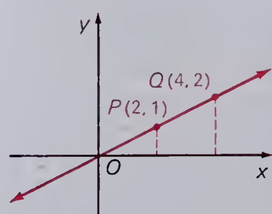


Figure 7

In general, as in Figure 8, you can use *subscript notation* to name any two points $P(x_1, y_1)$ (read "x sub one, y sub

one") and $Q(x_2, y_2)$ on a nonvertical line and form this ratio:

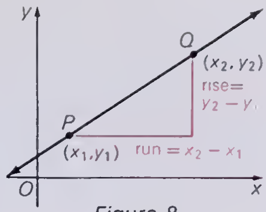


Figure 8

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{ordinate of } Q - \text{ordinate of } P}{\text{abscissa of } Q - \text{abscissa of } P} = \frac{y_2 - y_1}{x_2 - x_1}$$

The proof of the theorem stated below shows that this ratio does not depend upon the particular points chosen to be P and Q; therefore, it may be taken as the definition of slope. Thus in Figure 7, using the points P and Q, or O and P, you have

$$m = \frac{2-1}{4-2} = \frac{1}{2}, \qquad \text{or} \qquad m = \frac{1-0}{2-0} = \frac{1}{2}.$$

EXAMPLE 1 Find the slope of each line and graph both lines on the same plane.

a. $y = \frac{1}{2}x - 3$ b. $-3x + 6y = 2$

SOLUTION Find two pairs of coordinates for each line, and then use the slope formula.

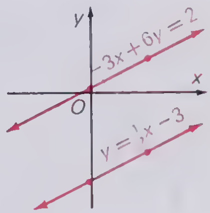
a. $(0, -3)$ and $(2, -2)$ satisfy $y = \frac{1}{2}x - 3$.

$$\therefore m = \frac{-2-(-3)}{2-0} = \frac{1}{2}.$$

b. Using $(0, \frac{1}{3})$ and $(2, \frac{4}{3})$ you get

$$m = \frac{\frac{4}{3} - \frac{1}{3}}{2-0} = \frac{1}{2}.$$

The graph is shown at the right.



Notice that if you transform Equation b in Example 1 to the form $y = \frac{1}{2}x + \frac{1}{3}$ and then compare the equations of the lines in Figure 7 and Example 1,

$$y = \tfrac{1}{2}x,$$
$$y = \tfrac{1}{2}x - 3,$$
$$y = \tfrac{1}{2}x + \tfrac{1}{3},$$

you see that *the coefficient of x is equal to the slope* in each case. This illustrates the following theorem, proved on page 82.

Theorem. For all A, B, and $C \in \Re$ and $B \neq 0$, the slope m of the line $Ax + By = C$ is $-\frac{A}{B}$.

PROOF

Since $B \neq 0$, the equation $Ax + By = C$ is equivalent to $y = -\frac{A}{B}x + \frac{C}{B}$. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the graph of the equation. Since the coordinates of both P and Q must satisfy this equation, we can write the following:

$$y_2 = -\frac{A}{B}x_2 + \frac{C}{B} \qquad (1)$$

$$y_1 = -\frac{A}{B}x_1 + \frac{C}{B} \qquad (2)$$

Subtracting (2) from (1), we obtain

$$y_2 - y_1 = -\frac{A}{B}(x_2 - x_1) \qquad (3)$$

Notice from (3) that $x_2 - x_1 \neq 0$, since otherwise we would have $x_2 - x_1 = 0$, $y_2 - y_1 = 0$, and P and Q would be the same point. Therefore, dividing (3) by $x_2 - x_1$, we get

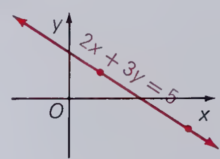
$$\frac{y_2 - y_1}{x_2 - x_1} = m = -\frac{A}{B}.$$

EXAMPLE 2 Find the slope m of $2x + 3y = 5$ and graph the line.

SOLUTION From the theorem above,

$$m = -\frac{A}{B} = -\frac{2}{3}.$$

Two pairs of coordinates that satisfy the given equation are (1, 1) and (4, −1). The graph is shown at the right.



The line graphed in Example 2 has a *negative slope.* In general, a line with **negative** slope **falls,** while one with **positive** slope **rises,** from left to right on the coordinate plane.

The slope of a horizontal line, such as $y = 1$ in Figure 9, is 0, because the numerator in the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, is 0 for any two points on the line.

On a vertical line, such as $x = 2$ in Figure 9, every point has the same x-coordinate. Hence the denominator $x_2 - x_1$ in the formula for m would always be 0 and, accordingly, **slope is not defined for a vertical line.**

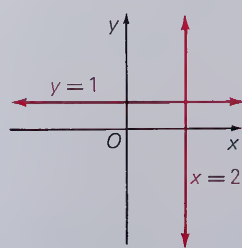


Figure 9

Oral Exercises

Find the slope of the line determined by the two given points.

1. (0, 0) and (2, 4)
2. (2, 4) and (5, 10)
3. (1, 1) and (−3, 3)
4. (−1, −3) and (1, 3)
5. $(4, \frac{3}{2})$ and (3, −2)
6. (5, 3) and (−49, 3)
7. From the answers to Exercises 1 and 2, what can you conclude about the points (0, 0), (2, 4), and (5, 10)?
8. What can you say about the line in Exercise 6?

In Exercises 9–17 give the slope, if any, for the line and tell whether the line rises or falls from left to right. If the line is vertical or horizontal, so state.

9. $y = 3x - 4$
10. $2y = 18$
11. $-3y = 5x$
12. $-x + 3 = 7$
13. $2y + 9x = 4$
14. $x = 4y - 6$
15. $\frac{2}{3}x - 5 - y = 0$
16. $x - \frac{1}{3}y = 2$
17. $3y + 2x = 5$

In Exercises 18–20 state values for A and/or B that will determine a line with the given slope.

18. $-2y = Ax - 1$; $m = \frac{1}{2}$
19. $4x + By = 5$; $m = \frac{2}{3}$
20. $Ax + By = 6$; $m = 0$

In Exercises 21–24 let ℓ_1 and ℓ_2 have slopes m_1 and m_2, respectively. Tell whether $m_1 > m_2$, $m_1 = m_2$, or $m_1 < m_2$ in each case.

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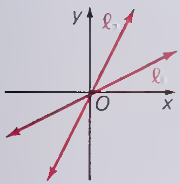

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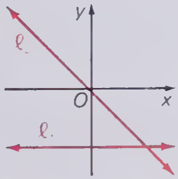

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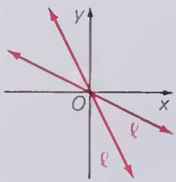

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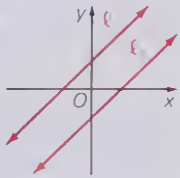


Written Exercises

Graph the line through the given points and determine its approximate slope by inspecting the graph. Check your results using the slope formula.

A

1. (2, 5) and (3, 7)
2. (−2, 1) and (2, −1)
3. (−3, 3) and (2, 3)
4. (−1, 5) and (3, −1)
5. (−2, −2) and (2, −3)
6. $(\frac{1}{5}, 1)$ and $(\frac{3}{5}, 3)$
7. (3, 5) and (3, −2)
8. $(1, -\frac{1}{2})$ and $(\frac{5}{3}, 4)$
9. (−3, −3) and (4, 4)

Find the slope $-\frac{A}{B}$, and graph the line by using the two points where it intersects the axes. Check that these points satisfy the slope formula.

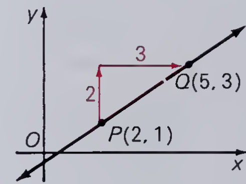
10. $2x - 3y = 6$ **11.** $-4x - 2y = 8$ **12.** $2y = 5x - 10$

13. $6x = 2y + 6$ **14.** $y + 2x + 3 = 0$ **15.** $\frac{x}{2} + 3y = 3$

For the line through the given point P and with given slope m, use m and the coordinates of P to determine the coordinates of a second point Q on the line. Then draw the line through P and Q.

EXAMPLE $P(2, 1)$; $m = \frac{2}{3}$

SOLUTION Since the slope is $\frac{2}{3}$, you can let the ordinate of Q be 2 more than the ordinate of P, and the abscissa be 3 more than the abscissa of P as shown in the diagram at the right, giving $Q(2 + 3, 1 + 2)$ or $Q(5, 3)$. Answer



16. $P(1, 3)$; $m = \frac{1}{2}$ **17.** $P(-1, 4)$; $m = -\frac{2}{3}$ **18.** $P(\frac{1}{2}, -3)$; $m = 2$

Determine the value of b that makes the given pairs of coordinates satisfy the given value for the slope m.

B **19.** $(2, -1)$ and $(3, b)$; $m = \frac{5}{2}$ **20.** $(b, 1)$ and $(-b, 4)$; $m = -4$

21. $(b, -2)$ and $(4b, -3)$; $m = \frac{1}{2}$ **22.** $\left(2, \frac{1}{b}\right)$ and $(\frac{4}{3}, -1)$; $m = 2$

23. $(b, -b)$ and $(1, b)$; $m = -3$ **24.** $\left(3, \frac{b}{2}\right)$ and $(b, -1)$; $m = \frac{1}{3}$

Use the following theorem to answer Exercises 25–26. Theorem: When neither of two lines is vertical, they are perpendicular if and only if the product of their slopes is -1.

C **25.** What is the slope of a line perpendicular to the line with equation $x + 2y = 8$?

26. Determine the value of A so that the graph of $Ax + 3y = 8$ is perpendicular to the line with equation $12x + 2y = 10$.

3-6 Finding an Equation of a Line

Since two points determine a unique line, you can use the slope formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

to determine an equation of a line through two given points, P and Q, with coordinates (x_1, y_1) and (x_2, y_2) if $x_2 \neq x_1$.

EXAMPLE 1 Find an equation of the line containing $P(3, -2)$ and $Q(4, 1)$.

SOLUTION The general equation of a nonvertical line, $Ax + By = C$, can be written equivalently in the form

$$y = -\frac{A}{B}x + \frac{C}{B}.$$

Since $-\frac{A}{B} = m$, we can simplify this to

$$y = mx + b,$$

where b is the constant term, $\frac{C}{B}$. First, we can calculate m from the slope formula,

$$m = \frac{1 - (-2)}{4 - 3} = 3.$$

Then we can substitute the coordinates of *either* P or Q in the equation $y = mx + b$ to find b. In this case, substituting $(3, -2)$ in $y = 3x + b$ yields

$$-2 = 3 \cdot 3 + b,$$
$$-11 = b.$$

Now we can use our known values for m and b to write an equation of the desired line: $y = 3x - 11$.

Check: Do $(3, -2)$ and $(4, 1)$ both satisfy $y = 3x - 11$?

$$-2 = 9 - 11, \qquad \text{and} \qquad 1 = 12 - 11.$$

$\therefore$ an equation of the line is $y = 3x - 11$. Answer.

From Example 1 you can see that a line ℓ is uniquely determined by its slope m and the coordinates of just one point $P(x_1, y_1)$ on ℓ. That is, any point Q other than P lies on ℓ if and only if the coordinates of Q together with those of P satisfy the slope formula for the given value of m.

Theorem. Given a line ℓ with slope m and a point $P(x_1, y_1)$ on ℓ. Then for P and any other point $Q(x, y)$:

1. If Q is on ℓ, then $\dfrac{y - y_1}{x - x_1} = m$.

2. If $\dfrac{y - y_1}{x - x_1} = m$, then Q is on ℓ.

If the equation for slope m in the theorem is rewritten as

$$y - y_1 = m(x - x_1),$$

we have the **point-slope form** of an equation of a line.

If the given point $P(x_1, y_1)$ of the line ℓ lies on the y-axis as in Figure 10, so that P is the point where ℓ intersects the y-axis, then $x_1 = 0$ and the ordinate y_1 of P is called the **y-intercept** of ℓ. (In Figure 10 the y-intercept of ℓ is 2.) Using the point-slope form of ℓ, we have

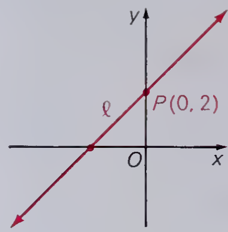


Figure 10

$$y - y_1 = m(x - 0)$$
$$y = mx + y_1.$$

Thus an equation of a line with slope m and y-intercept b is

$$y = mx + b.$$

This equation is called the **slope-intercept form** of an equation of a line.

The abscissa of the point in which a line ℓ intersects the x-axis is called the **x-intercept** of the line. The x-intercept of the line in Figure 10 is -2.

EXAMPLE 2 Find an equation of the line with y-intercept 3 and x-intercept 1.

SOLUTION To find m we use the axis-intersection points (0, 3) and (1, 0):

$$m = \frac{0 - 3}{1 - 0} = -3$$

Then, using the slope-intercept form, we have

$$y = -3x + 3.$$ Answer.

Oral Exercises

1. What is the relationship between the lines defined by the equations $y = 3x + 4$ and $y - 1 = 3(x + 1)$?
2. Can every line be defined by an equation of the form $Ax + By = C$? By an equation of the form $y = mx + b$?

State an equation of the line with the given slope m and y-intercept b.

3. $m = 3;\ b = -5$
4. $m = -1;\ b = 4$
5. $m = 0;\ b = -6$
6. $m = 8;\ b = 0$
7. $m = -\frac{1}{2};\ b = \frac{3}{2}$
8. $m = 0;\ b = 0$

State an equation of the line passing through the given point and having the given slope.

9. $(2, 5);\ m = 3$
10. $(-1, 7);\ m = -2$
11. $(\frac{1}{2}, 3);\ m = 6$
12. $(0, -\frac{2}{3});\ m = \frac{1}{3}$
13. $(3, -1);\ m = -\frac{3}{2}$
14. $(-2, -\frac{1}{4});\ m = \frac{1}{2}$

State the slope and y-intercept of the given line.

15. $-2x + y = -3$ **16.** $2y = 10x + 1$ **17.** $5y - x = 4$

18. $-3y = \frac{1}{2}x$ **19.** $4x + 3y = 7$ **20.** $x - 4y = 18$

Written Exercises

Write an equation of the line having slope m and y-intercept b.

A **1.** $m = 5;\ b = -7$ **2.** $m = 3;\ b = 0$ **3.** $m = -\frac{1}{2};\ b = 10$

4. $m = 0;\ b = 6$ **5.** $m = 0;\ b = -\frac{3}{2}$ **6.** $m = 2.1;\ b = 0$

Write an equation in the form $y = mx + b$ of the line passing through the point P with the given slope m.

7. $P(2, 11);\ m = 3$ **8.** $P(-3, 4);\ m = -5$ **9.** $P(3, -2);\ m = 0$

10. $P(-6, 7);\ m = \frac{4}{3}$ **11.** $P(3, 0);\ m = -\frac{5}{2}$ **12.** $P(-5, -1);\ m = -\frac{1}{2}$

13. $P(8, 9);\ m = 0$ **14.** $P(0, 0);\ m = -4$ **15.** $P(-1, 4);\ m = 3$

Write an equation in the form $y = mx + b$ of the line containing the two given points.

16. $(2, 3);\ (0, 5)$ **17.** $(5, -2);\ (0, 0)$ **18.** $(4, 1);\ (8, -1)$

19. $(6, -1);\ (-2, -1)$ **20.** $(-1, \frac{1}{3});\ (-\frac{5}{3}, -1)$ **21.** $(5, 1);\ (1, -2)$

22. $(4, 3);\ (1, 2)$ **23.** $(\frac{3}{4}, 2);\ (\frac{1}{8}, -\frac{1}{2})$ **24.** $(-2, 7);\ (1, 0)$

Determine an equation in the form $y = mx + b$ of the line satisfying the given conditions.

B **25.** Through the point $(1, 2)$ and having the same slope as the line through $(4, -1)$ and $(3, 6)$.

26. Through the point $(-2, 3)$ and having the same slope as the line through $(-1, -4)$ and $(1, 2)$.

27. Through the point $(\frac{1}{2}, 3)$ and having the same slope as the graph of $y = -4x - 3$.

28. Through the point $(15, -2)$ and having the same slope as the graph of $-x + 3y = 2$.

29. Through the point $(3, 0)$ and having the same slope as the graph of $3x + 2y = 5$.

30. Through the point $(5, 1)$ and having the same slope as the graph of $8y = x$.

31. With y-intercept -4 and having the same slope as the line through $(1, -3)$ and $(6, -2)$.

32. With y-intercept 0 and having the same slope as the line through $(-2, -5)$ and $(0, 2)$.

33. With y-intercept 3 and having the same slope as the graph of $2x + 3y = -8$.

34. With y-intercept -1 and having the same slope as the graph of $10x - 10y = 1$.

35. With slope $\frac{1}{2}$ and x-intercept 3.

36. With slope -5 and x-intercept 2.

37. With x-intercept 4 and y-intercept -3.

38. With x-intercept -7 and y-intercept -2.

C 39. Determine a so that the graph of $3x + ay = 9$ has the same slope as the line through $(7, -2)$ and $(5, -1)$.

40. Determine k so that the graph of $-2x + ky = 1$ has the same slope as the graph of $3x - 8y = 7$.

41. Find k so that the line through the points $(4, k)$ and $(-1, 3)$ has the same y-intercept as the graph of $x + 3y = 6$.

42. Show that the line having x-intercept a $(a \neq 0)$ and y-intercept b $(b \neq 0)$ has an equation of the form

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(called the intercept form).}$$

43. Show that the line passing through the points (x_1, y_1) and (x_2, y_2) has an equation of the form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{(called the two-point form).}$$

Self-Test 3

VOCABULARY

rise (p. 80)
run (p. 80)
slope (p. 81)
y-intercept (p. 86)
point-slope form (p. 86)
slope-intercept form (p. 86)
x-intercept (p. 86)

1. Find the slope of the line through the points $P(-2, 7)$ and $Q(6, 9)$. *Obj. 1, p. 80*
2. Find an equation of the line through the points $P(3, 0)$ and $Q(0, -5)$. *Obj. 2, p. 80*
3. Find an equation of the line through the points $(1, 4)$ and $(3, -6)$.
4. Find an equation of the line having slope -3 and passing through the point $(-5, 2)$. *Obj. 3, p. 80*
5. Find an equation of the line having slope 2 and y-intercept $-\frac{3}{2}$. *Obj. 4, p. 80*

Check your answers with those at the back of the book.

Applications of Linear Relations

OBJECTIVE for Section 3-7:
1. Solve problems involving direct variation.

3-7 Direct Variation

A function f in which the rule for pairing is given by a linear equation of the form

$$y = mx + b \qquad (m, b \in \mathcal{R})$$

is called a **linear function:**

$$f\colon x \to mx + b$$

Generally, the domain of a linear function is taken to be $\mathcal{R}$, and its graph is a nonvertical line.

EXAMPLE 1 a. Find a rule for specifying a linear function G if $G(5) = 1$ and $m = \frac{1}{10}$.
b. Does $(2, \frac{7}{10})$ belong to G?

SOLUTION a. Since $m = \frac{1}{10}$ and $(5, 1)$ is a pair in G, we can substitute these in $G(x) = mx + b$:

$$1 = \tfrac{1}{10} \cdot 5 + b, \qquad \text{or} \qquad b = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Hence the rule is $G\colon x \to \frac{1}{10}x + \frac{1}{2}$. Answer.

b. The pair $(2, \frac{7}{10})$ belongs to G because $\frac{7}{10} = \frac{1}{10} \cdot 2 + \frac{5}{10}$. Answer.

If $m = 0$, then $y = b$ for all $x \in \mathcal{R}$. In that case f is called a **constant function**, and its graph is the horizontal line through $(0, b)$.

If $b = 0$ and $m \neq 0$, we have $y = mx$. Then the function f is called a **direct variation**, and we say that **y varies directly as x**, or **y is directly proportional to x**, and that m is the **constant of variation** or the **constant of proportionality**. When the domain of a direct variation f is $\mathcal{R}$, the ordered pair $(0, 0)$ is always in f, and hence the graph of f is a line passing through the origin and having slope m $(m \neq 0)$.

Figure 11 shows the graph of the direct variation specified by

$$y = 2x \qquad (x \in \mathcal{R}).$$

For any two points (x_1, y_1) and (x_2, y_2) on the graph of this function, we have

$$y_1 = 2x_1 \qquad \text{and} \qquad y_2 = 2x_2.$$

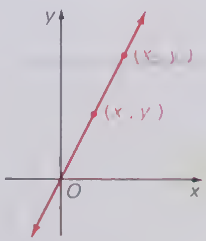


Figure 11

If $x_1, x_2 \neq 0$, you can transform these equations to

$$\frac{y_1}{x_1} = 2 \quad \text{and} \quad \frac{y_2}{x_2} = 2.$$

Therefore $$\frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

In general:

> If a linear function f is a direct variation, then for any two ordered pairs (x_1, y_1) and (x_2, y_2) in f, with $x_1, x_2 \neq 0$,
>
> $$\frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Such an equality of ratios, sometimes written as

$$y_1 : x_1 = y_2 : x_2,$$

is called a **proportion.** The terms y_1 and x_2 are called the **extremes,** and x_1 and y_2 the **means,** of the proportion. Since

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

can be transformed to the equivalent equation

$$y_1 x_2 = x_1 y_2,$$

you can see that:

> In any proportion the product of the means equals the product of the extremes.

EXAMPLE 2 If a 100 g package of herbal tea costs 50¢, is a 250 g package costing \$1.29 a better value?

SOLUTION Find what 250 g would cost at the rate given for 100 g, and then compare that figure with \$1.29.

$$\frac{100}{0.50} = \frac{250}{x}$$

$$100x = 125$$

$$x = 1.25 \text{ or } \$1.25 \text{ for } 250\text{ g}$$

Since \$1.29 > \$1.25, the 250 g package is not a better value! Answer

Oral Exercises

State whether or not the given function is linear, and if so, whether or not it is a direct variation.

1. $f\colon x \to -3x$
2. $f\colon x \to \frac{1}{2}x - 5$
3. $f\colon x \to \dfrac{1}{4x}$
4. $f\colon x \to \frac{1}{3}x^2$
5. $\left\{(x, g(x))\colon g(x) = \dfrac{5x}{9}\right\}$
6. $\{(x, y)\colon y = -2\}$
7. $\left\{(x, f(x))\colon f(x) = \dfrac{1 + 2x}{7}\right\}$
8. $\left\{(x, h(x))\colon h(x) = \dfrac{x}{14}\right\}$

State an equation of the form $y = kx$ and then state a proportion for the direct variation described in words.

EXAMPLE The circumference of a circle C varies directly as its diameter d.

SOLUTION $C = kd;\ \dfrac{C_1}{d_1} = \dfrac{C_2}{d_2}$

y = mx + b

9. The diagonal d of a square is directly proportional to the length s of a side.
10. The volume V of a pyramid with a given base varies directly as its height h.
11. The current I in a given electric circuit varies directly with the applied voltage E.
12. The rate of radioactive decay r of the isotope carbon-14 varies directly with the amount m.
13. The cost C of a tiled linoleum floor is directly proportional to the number of tiles n, each of which costs 40¢.

For each set of ordered pairs examine the ratio $\dfrac{y}{x}$ and tell whether or not the pairs are in direct variation.

14. $\{(1, -1), (-2, 2), (0, 0)\}$
15. $\{(1, 3), (2, 6), (-3, -1)\}$
16. $\{(5, 11), (3, 7), (10, 21)$
17. $\{(4, -6), (2, -3), (-8, 12)\}$

Written Exercises

Determine a rule for pairing in a linear function f if:

A

1. $f(0) = -1$ and $f(2) = 5$
2. $f(0) = 0$ and $f(-3) = -8$
3. $f(3) = 4$ and $f(7) = 0$
4. $f(-2) = 1$ and $f(-6) = 1$

Determine a rule for pairing in a linear function f if:

5. $f(6) = -9$ and $f(-10) = 15$
6. $f(\frac{1}{2}) = -1$ and $f(\frac{5}{2}) = 7$
7. $f(3) = 3$ and $f(-6) = -3$
8. $f(\frac{1}{3}) = 0$ and $f(\frac{2}{3}) = -1$

Determine whether or not the given pairs are in a direct variation. If so, give the slope of the graph of the variation.

9. $\{(2, -4), (-1, 2), (3, -6)\}$
10. $\{(1, 4), (2, \frac{1}{2}), (-1, -\frac{1}{2})\}$
11. $\{(0, 0), (3, 0), (-2, 0)\}$
12. $\{(1, 5), (2, 5), (3, 5)\}$
13. $\{(2, 5), (6, 15), (1, \frac{5}{2})\}$
14. $\{(4, 6), (5, 8), (6, 10)\}$
15. $\{(1, 3), (5, 7), (11, 13)\}$
16. $\left\{(a, 3a), \left(\frac{a}{3}, a\right), (3a, 9a)\right\}$

Solve each proportion for a.

17. $a:3 = 5:6$
18. $7:4 = a:8$
19. $a:2 = 7:3$
20. $9:a = 15:10$
21. $6:5 = 1:a$
22. $7:4 = 5:a$
23. $4:a = 3:5$
24. $a:6 = 7:15$

25. If y varies directly as x, and y is 8 when x is 5, find x when y is 20.
26. If y varies directly as x, and y is 27 when x is 24, find y when x is 72.
27. If the cost of installing a window varies directly with the area of the window, and it costs \$3.00 to install a window of area $4\ m^2$, how much would it cost to install a window of area $3\ m^2$?
28. If $C = \dfrac{3kx}{2}$, and C is 15 when x is 8, find C when x is $-\frac{1}{2}$.

B 29. If $y = \dfrac{kx}{3} + 2$, and y is -4 when x is 9, find x when y is 5.
30. If $y = k(3x - 1)$, and y is 10 when x is 3, what is y when x is 7?

For the proportion $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$ $(x_1, y_1, x_2, y_2 \neq 0)$, prove each of the following properties.

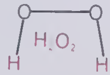
31. $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$
32. $\dfrac{y_1}{y_2} = \dfrac{x_1}{x_2}$
33. If $x_1 \neq x_2$, then $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{y_2}{x_2}$. (*Hint:* Let $y_1 = mx_1$.)

C 34. If the function g is a direct variation, prove that $g(r + s) = g(r) + g(s)$ for all $r, s \in \mathcal{R}$.
35. If g is a linear function defined by $g(x) = mx + b$, and $g(r + s) = g(r) + g(s)$ for some $r, s \in \mathcal{R}$, prove that b must be 0.

Problems

A

1. If a baseball player has 30 hits in 150 times at bat, how many hits should the player get in 1000 at-bats?
2. If 4 m of fabric costs \$11, how much would 5 m cost?
3. If hydrogen and oxygen combine at a rate of 1 part of hydrogen to 16 parts of oxygen by mass to form hydrogen peroxide (H_2O_2), how many grams of each element are there in 306 g of H_2O_2?



4. A municipal ordinance in a certain town decrees that there must be 1 teacher for every 18 students in the town's public schools. This year the student population was 2880, but a 12.5% drop is expected next year. How many teachers will the town need next year?
5. In a lab test, torsion pendulum A rotates back and forth 12 times in 52 s; pendulum B rotates 12 times in 60 s. How many times will pendulum B rotate in the time it takes pendulum A to rotate 15 times?
6. An object of mass 25 kg stretches a spring 9.5 cm. If the distance the spring is stretched is directly proportional to the mass of an attached object, what is the mass of an object that will stretch the spring 5.7 cm?

7. Emily Wharton receives \$1340 in dividends from 250 shares in a mutual fund. How many shares of the same mutual fund does Arthur Fletcher own if he receives \$2412 in dividends?

B

8. The average number of red cells in a cubic millimeter (mm^3) of a woman's blood is 4,500,000. In order to perform a test on a sample of Mary Brown's blood, a laboratory technician dilutes 1 part of blood with 299 parts of salt solution. How many red blood cells would the technician expect to find in 2.5 mm^3 of the diluted soluton?

In Problems 9 and 10, assume that if light and sound are sent at the same instant from the same source, then the distance of an observer from the source is directly proportional to the time elapsing from the moment the observer sees the light to the moment he or she hears the sound.

9. During a thunderstorm, Alice heard the sound of thunder associated with a bolt of lightning 3.5 s after she saw the lightning flash. Henry heard the thunder 4.2 s after he saw the lightning flash. If Alice was 1190 m from the lightning bolt, how far from it was Henry?
10. Tracking station A is 850 m from a rocket launching pad. It detects the first sound of the rocket engines 3 s before tracking station B, which is 1870 m from the pad. How long did it take the sound to reach station B?

Self-Test 4

VOCABULARY

linear function (p. 89)
direct variation (p. 89)
constant of proportionality (p. 89)
constant function (p. 89)
proportion (p. 90)
extremes (p. 90)
means (p. 90)

1. If y varies directly as x, and $y = 7$ when $x = 4$, find y when $x = 18$. *Obj. 1, p. 89*
2. If the gas consumption of a car varies directly as the distance traveled, and the car uses 5 L of gas to go 225 km, how much gas would it use to go 315 km?

Check your answers with those at the back of the book.

Chapter Summary

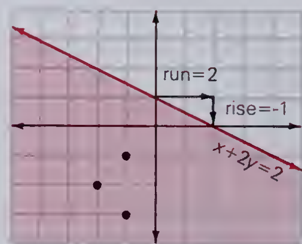


Figure 12

1. A *function* is a set of *ordered pairs* in which each *first component* is paired with exactly one *second component.* A *relation,* however, is any set of ordered pairs.
2. A function or relation may be graphed on a *coordinate plane.* For example, the black points in Figure 12 are the graph of the relation $r = \{(-1, -1), (-2, -2), (-1, -3)\}$.
3. A *solution* of an *open sentence in two variables* is any ordered pair for which the sentence is true. The *solution set* of the sentence is the set of all its solutions.
4. A *linear equation in two variables* is an equation of the form $Ax + By = C$, where $A, B, C \in \mathcal{R}$ and A and B are not both zero. The graph of a linear equation is a straight line. For example, the red line in Figure 12 is the graph of the equation $x + 2y = 2$.
5. The *graph of a linear inequality* is an *open* or *closed half-plane.* The boundary of the half-plane is the graph of the associated linear equation. For example, the half-plane shaded red in Figure 12 is the graph of $x + 2y \le 2$.
6. The *slope* of a nonvertical line is the ratio of *rise* to *run.* For example, the line in Figure 12 has slope $-\frac{1}{2}$. A horizontal line has zero slope, while a vertical line has *no* slope.
7. A *linear function* is a function in which the rule for pairing is given by a linear equation of the form $y = mx + b$, with $m, b \in \mathcal{R}$. If $b = 0$, then the linear function is a *direct variation.*
8. An equality of *ratios* is called a *proportion;* the product of the *means* equals the product of the *extremes.*

Chapter Review

1. What is the rule for the pairing in the function 3-1
$\{(\frac{1}{2}, 2), (1, 1), (2, \frac{1}{2}), (3, \frac{1}{3})\}$?

a. $y = 2x$ b. $y = x^2$ c. $y = \frac{1}{x}$ d. $y = \frac{1}{x + 1}$

2. If $f(x) = 2x$ and $g(x) = x^2 + 1$, find $f(g(2))$.

a. 17 b. 8 c. 10 d. 20

Is the statement (a) true or (b) false?

3. The relation $\{(-2, 1), (-1, 3), (0, 5), (1, 7)\}$ is a function. 3-2
4. The relation $\{(-2, 2), (-1, 3), (0, 8), (-1, 4)\}$ is not a function.
5. The relation $z = |x| + 1$, $x \in \mathcal{R}$, is a function.

6. Which point does not lie on the line with equation $2x - 3y = 1$? 3-3

a. (5, 3) b. (2, 1) c. (6, 4) d. $(-1, -1)$

7. For which value of k is (1, 3) a solution to $2x - ky = -4$?

a. 1 b. -1 c. 2 d. -2

8. Which point is not in the graph of the inequality $y + x > 2$? 3-4

a. (2, 1) b. $(-1, 5)$ c. (0, 1) d. (4, 0)

9. Which point is not in the graph of the inequality $3x - y \geq 1$?

a. (2, 1) b. (1, 2) c. $(-1, -1)$ d. $(0, -2)$

10. What is the slope of a line through (1, 2) and (3, 3)? 3-5

a. 2 b. $\frac{1}{2}$ c. 1 d. 3

11. For what value of z does the line with slope $-\frac{2}{3}$ pass through the points $(6, -3)$ and $(3, z)$?

a. 1 b. -1 c. 2 d. 4

12. Choose the equation for the line that passes through (4, 8) and has slope $-\frac{1}{2}$. 3-6

a. $y = -\frac{1}{2}x + 10$ b. $y = -\frac{1}{2}x + 8$ c. $y = -\frac{1}{2}x + 4$

13. Give an equation in slope-intercept form for the line passing through (0, 1) and $(3, -1)$.

a. $y = 3x - 10$ b. $y = -\frac{2}{3}x + 1$ c. $y = 2x + 1$

14. Solve $a:8 = 9:4$ for a. 3-7

a. 3 b. 6 c. 18 d. 12

15. If y varies directly as x, and y is 9 when x is 5, find x when y is 45.

a. 45 b. 81 c. 25 d. 54

Chapter Test

If $f(x) = \dfrac{x}{3x - 1}$, find each of the following.

1. $f(1)$ **2.** $f(-2)$ **3.** $f(0)$ **4.** $f(a)$ 3-1

Draw a mapping diagram for each of the relations and determine whether or not it is a function.

5. $\{(1, 3), (2, 4), (3, 3), (4, 2)\}$ 3-2

6. $\{(1, 2), (1, 3), (2, 4), (3, 5)\}$

7. Draw the graph of the relation in Test Item 5.

Graph the equation.

8. $2x + 2y = 6$ **9.** $2x - y = 4$ 3-3

10. $y = \frac{1}{2}$ **11.** $-x + 2y = -2$

Graph each inequality as a shaded region on a coordinate plane.

12. $x > 2y$ **13.** $x + 3y \leq 8$ 3-4

Find the slope of the line through the given points.

14. $(1, 3)$ and $(2, 5)$ **15.** $(2, 6)$ and $(4, 5)$ 3-5

Find the slope of the line with the given equation.

16. $3x + 2y = 12$ **17.** $2x - 4y = 9$ 3-6

18. Write an equation of the line with slope -3 and y-intercept -1.

Write an equation of the line passing through the point P and having slope m.

19. $P(3, -2)$; $m = \frac{2}{3}$ **20.** $P(-1, 3)$; $m = -2$

21. Write an equation for the line passing through $(2, 5)$ and $(4, -1)$.

22. Solve the proportion $a:7 = 5:30$ for a. 3-7

23. Solve the proportion $5:b = 17:51$.

24. The volume of water in a cylindrical tank is directly proportional to the height of the water measured on the tank wall. If 10,000 m^3 of water fills the tank to the 25 m mark, how much water is in the tank when it is filled to the 35 m mark?

25. If hydrogen and oxygen combine at a rate of 1 part of hydrogen to 8 parts of oxygen by mass to form water (H_2O), how many grams of each element are there in 432 g of water?

programming in BASIC

These optional sections are for use by students who have access to a computer that will accept the language BASIC. Pages 97–105 contain a four-part review of the fundamentals of BASIC using material from Chapter 3. Additional work with BASIC will appear at intervals throughout the text.

Part I

You can use a computer to find values of functions and relations.

First, you must translate the algebraic expressions into the BASIC language by using:

+ for addition		* for multiplication
− for subtraction		/ for division
	↑ for exponentiation	

The usual order of operations is:

> first, ↑
> then *, / in order from left to right
> then +, − in order from left to right

You may use parentheses whenever necessary to indicate a different order of operations. Thus:

$2x^2 - 3x - 5$ would be written as 2*X↑2−3*X−5

but

$\frac{a+b}{a-b}$ would be written as (A+B)/(A−B)

The expression A+B/A−B would mean: $a + \frac{b}{a} - b$

To find the value of $2x^2 - 3x - 5$ for a single value of the *variable* x, you may use this *program:*

```
10  LET X=−2
20  PRINT 2*X↑2−3*X−5
30  END
```

Notice that every line in a program must be numbered, and every program must end with an END statement. The LET statement in line 10 assigns the value on the right-hand side of the equals sign to the variable named on the left.

If you communicate with your computer by means of a terminal, type in the program on page 97, pressing the RETURN key after each line, and then type the *command* RUN (a command has no line number) followed by RETURN.

The PRINT statement in line 20 causes the computer to compute the value and then print it:

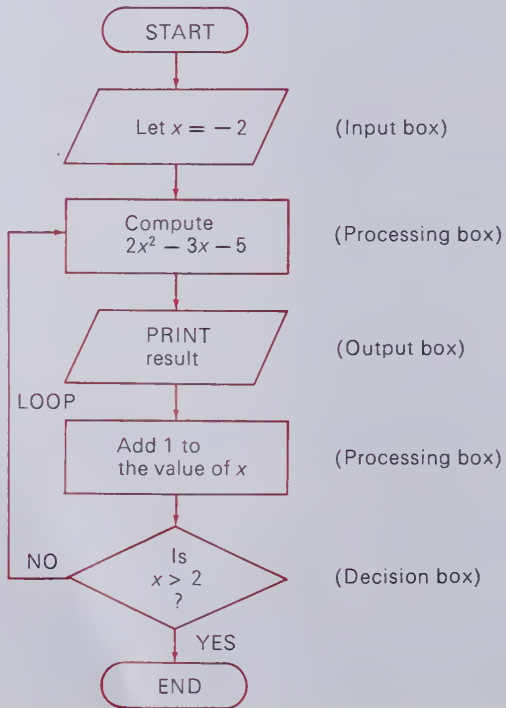
9

But to use the computer effectively, you want it to find several values in succession. To do this, you use a loop. Here is an example where the domain is $\{-2, -1, 0, 1, 2\}$:

loop

```
10  LET X=-2
20  PRINT 2*X↑2-3*X-5
30  LET X=X+1
40  IF X>2 THEN 60
50  GOTO 20
60  END
```

This program can be represented by the following flow chart (notice the shapes of the boxes):



The statement LET X = X + 1 in line 30 means that the computer takes the current value of X, adds 1 to it, and assigns this new value to X.

This program uses the following kinds of statement in addition to the LET, PRINT, and END statements:

IF...THEN... statement: This is a "conditional transfer" statement. If the condition is true, then the execution is transferred to the line specified. Otherwise, the computer goes on to the next line in the program.

GOTO statement: This is an "unconditional transfer." The execution is always transferred to the line specified.

If you run this program, you will see that only the values of the function are printed. You can make the computer print out the ordered pairs of the function

$$x \rightarrow 2x^2 - 3x - 5$$

in neatly labeled columns. The headings are made by using quotation marks and commas in a PRINT statement. Thus, add:

```
5   PRINT "X         ---->","2X↑2-3X-5"
```

and change:

```
20   PRINT X,2*X↑2-3*X-5
```

To get a listing of the program as it now stands, type the *command* LIST. You will see that the computer has inserted line 5 and changed line 20. In BASIC every line is given a number so that such changes can be made easily. Now run this revised program. In general:

Quotation marks in PRINT statements: The computer will copy and print out all the symbols and spaces that are typed within quotation marks. (If you need quotation marks within these, you must use the single ones, '.)

Commas in PRINT statements: The commas in lines 5 and 20 cause the computer to move the second item in the line over 15 spaces from the left margin. (In general, commas may be used to arrange print-outs in 5 columns across the page.)

Notice that whenever a numerical value is printed, space is left for a sign, but only the negative sign is printed.

BASIC has several built-in functions available, one of which is the absolute value function. $|x|$ is written as ABS(X) in BASIC. To print out some ordered pairs of the function

$$x \rightarrow |x|,$$

change:

```
 5   PRINT "X         ---->","ABS(X)"
20   PRINT X, ABS(X)
```

Then run the program and observe the results

Exercises

In these exercises use the domain $\{-3, -2, -1, 0, 1, 2, 3\}$. Write programs to find the values of these functions for the given domain.

1. $x \to x^2 + x$ 2. $x \to x^2 - 1$ 3. $x \to |x| - 1$ 4. $x \to 3 + 2|x|$

Part II

When you want to erase the entire program that you have been using, type the *command* SCR (for scratch) and press RETURN. This clears your working space for a new program.

To find ordered pairs to use in plotting a graph of an equation, you can transform the equation into an equivalent one that expresses y in terms of x and then use a computer program similar to that given on page 98. There is, however, a special shortcut for writing the kind of loop used in that program. Compare the following program with the earlier one:

```
 5  PRINT "X          ---->","2X↑2-3X-5"
10  FOR X=-2 TO 2
20  PRINT X, 2*X↑2-3*X-5
30  NEXT X
40  END
```

The FOR statement (line 10) and the NEXT statement (line 30) begin and end the loop, and the same variable must be used in both, in this case, the variable X.

When one loop is used inside another loop, they are called *nested loops*. We shall use nested loops in the next program. We shall give an interesting method of testing selected pairs of values of x and y within a given region to determine which pairs, if any, are solutions of a given open sentence. Run the following program, which finds some solutions of

$$x + y = 4.$$

```
 5  PRINT "SOME SOLUTIONS OF X+Y=4"
10  FOR Y=4 TO -4 STEP -1
20  FOR X=-4 TO 4
30  IF X+Y=4 THEN 60
40  PRINT "        ";  (Leave 8 spaces.)
50  GOTO 70
60  PRINT "(";X;",";Y;") ";  (Leave a single space.)
70  NEXT X
80  NEXT Y
90  END
```

This program uses several more features of the BASIC language:

STEP −1: This produces a countdown, in this program from 4 to −4. In a FOR statement, if no STEP value is given, the STEP value is assumed to be 1. The STEP value may be any BASIC expression.

Semicolons in PRINT statements: These cause the items to be printed close together, although space is always allowed for the sign of a numerical value.

Placing a comma or a semicolon at the end of a PRINT statement holds the terminal carriage on the same line.

This program tests all pairs of integral values of x and y where

$$x \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

and

$$y \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

The loops work like this:

$$\begin{aligned} Y &= 4, \text{ while } X = -4, -3, -2, -1, 0, 1, 2, 3, 4 \\ Y &= 3, \text{ while } X = -4, -3, -2, -1, 0, 1, 2, 3, 4 \end{aligned}$$

and so on. The number of values of x has been chosen so that the ordered pairs can be printed in 9 columns across the page. The number of values of y may be increased or decreased as desired, provided

$$-9 \leq y \leq 9.$$

(The spacing of this particular program allows for only single-digit integers.) Each column in the print-out occupies 8 spaces, and 9 columns of 8 spaces fill the 72 spaces that are available in the terminal print-out. Notice that if (X,Y) is *not* a solution, the computer "prints" 8 spaces in the column.

In order to make it easy to transfer these values to a graph, we have started Y at the top with a positive value and X at the left with a negative value.

Make the following changes and run the program again:

```
 5   PRINT "SOME SOLUTIONS OF X+Y<4"
30   IF X+Y<4 THEN 60
```

Change the inequality to X+Y<=4 and run the program again. The inequality symbols available in BASIC are:

< is less than	<= is less than or equal to
> is greater than	>= is greater than or equal to
<> is not equal to	

Exercises

1. Marion wanted a program that would give three integral ordered pairs that are solutions of

$$Ax + By = C,$$

A, B, C intergers, with x having the value 0 and two positive values. Work through the program shown below by hand and see if it does what is wanted. What value must B not have?

```
1  LET A=5
2  LET B=-3
3  LET C=9
5  PRINT "X","Y"
10 FOR X=0 TO 2*ABS(B) STEP ABS(B)
20 PRINT X, C/B-(A/B)*X
30 NEXT X
40 END
```

2. Write a program that will give the squares of the integers 0, 1, . . . , 10.
3. Write a program that will give the products of a given number and the integers 0, 1, . . . , 10.

Part III

If you want to evaluate a formula while working at a computer terminal, it is useful to INPUT the given values. Try this program:

```
10 PRINT "WHAT IS THE RADIUS OF THE CIRCLE";
20 INPUT R
30 PRINT "THE AREA IS";3.14159*R*R;"."
40 END
```

The INPUT statement (line 20) causes the computer to print a question mark and then wait for you to type in a number (an integer or a decimal). (The semicolon at the end of line 10 makes the computer print the question mark on the same line.) You type in the number and press RETURN. The computer then continues the program.

You can keep running the program, inputting a different value each time. If you know ahead of time what values you want to use, you can put in a READ statement and a DATA statement as in the following program:

```
10 READ R
20 PRINT "THE AREA OF A CIRCLE WITH RADIUS";R;
30 PRINT "IS";3.14159*R*R;"."
40 PRINT            (This "prints" a blank line here.)
50 GOTO 10
60 DATA 2,4,6,8,10,6.75
70 END
```

Run this program. The computer will "read" the first value in the DATA statement (line 60), do lines 20 and 30, skip a line (line 40 produces a line feed), READ the second value, and so on, until it has used all the values in line 60. It will then print that it is OUT OF DATA and stop.

We shall now write a program to find the slope of a line by using the formula on page 81. So far we have used only single letters as variables. The BASIC language, however, also allows us to use a single letter followed by a single digit:

A0, A1, . . . A9, . . . , Z1, . . . Z9

Therefore we may translate the expression for the slope of a line

$$\frac{y_2 - y_1}{x_2 - x_1}$$

into BASIC as:

(Y2−Y1)/(X2−X1)

BASIC allows you to INPUT several values at a time. Since the computer will print only one question mark, it is useful to precede the INPUT statement with a PRINT statement reminding you of what values to type in. You must type them in exactly the order you have indicated in your program, separated by commas. Study and run this program:

```
10  PRINT "INPUT X1,Y1,X2,Y2"
20  INPUT X1,Y1,X2,Y2
30  IF X1=X2 THEN 60
40  PRINT "SLOPE =";(Y2-Y1)/(X2-X1)
50  STOP
60  PRINT "NO SLOPE"
70  END
```

There can be only one END statement in a program, and so a **STOP statement** is used in line 50.

Exercises

1. Draw a flow chart for the program for finding the slope of a line when two points on it are given.
2. Write a program that will find the x-intercept and the y-intercept of a line with equation of the form $Ax + By = C$ when you input values of A, B, and C.
3. Transform the two-point form given in Exercise 43, page 88, to the form:

$$Ax + By = C$$

Then write a program that will print an equation of a line through two points when you input their coordinates.

Part IV

Another built-in function that BASIC has is the greatest integer function, which is written INT(X). This gives the greatest integer less than or equal to X. Try this program to see how it works.

```
10  PRINT "X","INT(X)"
20  FOR X=1 TO 3 STEP .25
30  PRINT X,INT(X)
40  NEXT X
50  END
```

We have seen how commas and semicolons work in PRINT statements. By using TAB(X) in a print statement, you can print any symbol in a specified space. Try this program:

```
10  FOR X=0 TO 5
20  PRINT TAB(X);"*"
30  NEXT X
40  END
```

For example, TAB(5);"*" prints * in the 6th space, counting from the left margin. Change line 20 to

```
20  PRINT TAB(X); X
```

and run the program. Since these are numerical values, space is allowed for a sign.

If the values of X are not integral, TAB works like INT. Try this program:

```
10  FOR X=1 TO 3 STEP .25
20  PRINT TAB(X);"*"
30  NEXT X
40  END
```

TAB can be used in plotting graphs. On the facing page is a program that will print out rough sketches of portions of graphs of some equations of the form:

$$Ax + By = C,\ A \neq 0,\ B \neq 0.$$

In order to make a nearly square grid, it is necessary to use two horizontal spaces to represent the unit that one line feed represents.

If the absolute value of the x-intercept is greater than 8, the program prints the value of the x-intercept and TOO WIDE and stops.

There is no built-in stop in the program if the y-intercept is very large. The program prints out its value, however, as well as the slope, and if these values seem unsuitable for a computer sketch, the operator can stop the program by pressing and releasing the BREAK key.

Study and run the following program for $x + y = 4$; that is, input 1,1,4. Compare this graph with the print-out of ordered pairs obtained on page 100.

```
10  PRINT "WHAT VALUES DO YOU WANT FOR A(<>0), B(<>0), C";
15  INPUT A,B,C
20  PRINT
25  LET X1=C/A
30  LET Y1=C/B
35  PRINT A;"X  +";B;"Y  =";C,"SLOPE  =";-A/B,"Y-INTERCEPT  =";Y1
40  PRINT
45  LET E=INT(ABS(X1))+5
50  IF E>13 THEN 170
55  LET D=INT(ABS(Y1))+5
60  LET M=2*E
65  LET N=2*M
70  PRINT TAB(M);"Y"
75  FOR Y=D TO -D STEP -1
80  LETX=(C-B*Y)/A
85  LET X2=2*X+M
90  IF X2>M THEN 145
95  IF X2=M THEN 130
100  IF X2<0 THEN 160
105  IF Y=0 THEN 120
110  PRINT TAB(X2);"*";TAB(M);"!";TAB(N+3);"(";X;",";Y;")"
115  GOTO 160
120  PRINT "+-+-+-+";TAB(X2);"*";TAB(N-6);"+-+-+-+X";
121  PRINT TAB(N+3);"(";X;",";Y;")"
125  GOTO 160
130  IF Y=0 THEN 120
135  PRINT TAB(X2);"*";TAB(N+3);"(";X;",";Y;")"
140  GOTO 160
145  IF X2>N THEN 160
150  IF Y=0 THEN 120
155  PRINT TAB(M);"!";TAB(X2);"*";TAB(N+3);"(";X;",";Y;")"
160  NEXT Y
165  STOP
170  PRINT "X-INTERCEPT = ";X1;"      TOO WIDE"
175  END
```

Exercises

1. Run the program above using as input 1,1,−4; −1,1,4; 1,−1,4 Then try 2,−1,4; 2,1,9; 1,1,9.
2. Run each of these programs by hand, and describe what each one does

```
10  LET A=.6666                    10  LET A=9876
20  PRINT INT(100*A+.5)/100        20  PRINT INT(A/100+.5)*100
30  END                            30  END
```

3. Write a program that will print a multiplication table from 1×1 to 10×10. Use TAB to space the columns across the page.

This solar sail to be propelled by the sun's rays is under consideration for a future Halley's Comet Mission.

4

Systems of Linear Equations or Inequalities

Systems of Equations in Two Variables

***OBJECTIVES** for Sections 4-1 through 4-4:*

1. ***Identify the solution set of a system of linear equations in two variables from the graphs of the equations.***
2. ***Solve a system of two linear equations in two variables by the linear-combination method.***
3. ***Use determinants to solve a system of two linear equations in two variables.***
4. ***Solve problems by translating stated relationships into a system of equations.***

4-1 Graphing a System of Equations in the Plane

Figures 1, 2, and 3 on page 108 show the three possible configurations when two linear equations are graphed on the same plane. The lines representing a given pair of equations must do just one of the following:

A. Intersect in exactly *one* point (Figure 1).
B. Coincide, and thus have *every* point in common (Figure 2).
C. Parallel one another, and thus have *no* point in common (Figure 3).

The **system of linear equations** graphed in Figure 1 can be regarded as a conjunction of the two open sentences

$$2x - y = 3 \qquad \text{and} \qquad x + 3y = 5 \qquad (x, y \in \mathbb{R})$$

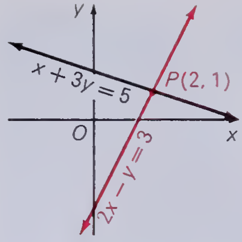


Figure 1

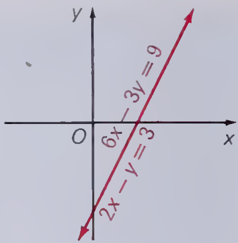


Figure 2

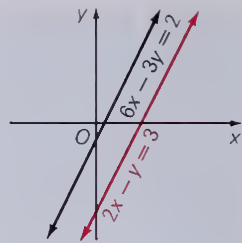


Figure 3

The **solution set** of a system of linear equations in two variables consists of all the ordered pairs of real numbers that satisfy *every* equation of the system. Thus the intersection point P in Figure 1 is the graph of the single solution of the system:

$$\begin{aligned} 2x - y &= 3 \\ x + 3y &= 5 \end{aligned}$$

The coordinates of P appear to be (2, 1). To verify that these coordinates actually satisfy both equations of the system, we have:

$$\begin{aligned} 2 \cdot 2 - 1 &= 3 \\ 2 + 3 \cdot 1 &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3 &= 3 \\ 5 &= 5 \end{aligned}$$

Therefore, the solution set of the system is $\{(2, 1)\}$.

In Figure 2 the graph of the solution set of the system

$$\begin{aligned} 2x - y &= 3 \\ 6x - 3y &= 9 \end{aligned}$$

consists of the infinite set of points comprising the single line that represents either one of the equations. Thus, the solution set is $\{(x, y)\colon 2x - y = 3\}$.

In Figure 3 the two parallel lines representing the system

$$\begin{aligned} 2x - y &= 3 \\ 6x - 3y &= 2 \end{aligned}$$

have no points in common. Hence, the solution set of the system is the empty set $\emptyset$. We describe such a system as **inconsistent**.

When a system of equations has at least one solution, we describe the system as **consistent**.

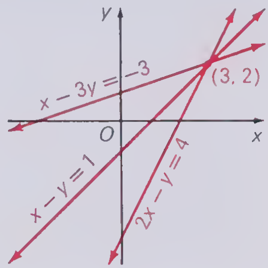
EXAMPLE 1 Graph the following system of linear equations, and then determine whether or not it is consistent.

$$\begin{aligned} 2x - y &= 4 \\ x - y &= 1 \\ x - 3y &= -3 \end{aligned}$$

SOLUTION Find the x- and y-intercept for each line. Then draw the three lines through their pairs of intercepts (diagram at right), and check to see if their apparent common point (3, 2) satisfies all three equations.

$$\begin{aligned} 2x - y &= 4 \\ 2 \cdot 3 - 2 &= 4 \\ 4 &= 4 \end{aligned} \qquad \begin{aligned} x - y &= 1 \\ 3 - 2 &= 1 \\ 1 &= 1 \end{aligned}$$

$$\begin{aligned} x - 3y &= -3 \\ 3 - 3 \cdot 2 &= -3 \\ -3 &= -3 \end{aligned}$$



Hence, (3, 2) is a solution of the system of equations, and it is therefore consistent. Answer.

Observe from Figures 1, 2, and 3 that the number of solutions for a system of two linear equations is related to the slopes of their graphs. The following theorem summarizes these relationships (see Exercises 35–39, page 115).

Theorem.
A system of two linear equations in two variables has:

1. exactly *one solution* if the two graphs have *different slopes,* or if just one of them has no slope;
2. an *infinite set of solutions* if both graphs have the *same slope and the same y-intercept,* or both have no slope and the same x-intercept;
3. *no solution* if both graphs have the *same slope but different y-intercepts,* or else no slope and different x-intercepts.

EXAMPLE 2 Tell how many solutions each system has.

a. $4x - 3y = 2$, $\frac{1}{3}x - \frac{1}{4}y = 1$

b. $2x + y = 3$, $x - 3y = 6$

c. $2y = -6$, $4y - 1 = -13$

SOLUTION

a. Both graphs have the same slope, $\frac{4}{3}$, but different y-intercepts: $-\frac{2}{3}$ and -4. Hence the system has *no solution.*

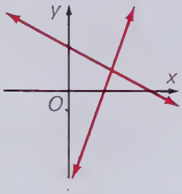
b. The two graphs have slopes -2 and $\frac{1}{3}$. Hence the system has exactly one solution.

c. Both graphs have slope $m = 0$ and y-intercept -3. Therefore the system has an infinite number of solutions.

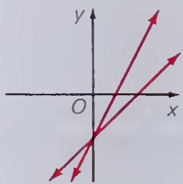
Oral Exercises

In Exercises 1–6 the diagram shows the graphs of the equations in the given system. State what appears to be the solution set of the system.

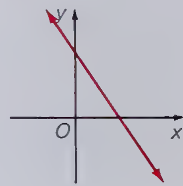
1. $x + 2y = 4$
$3x - y = -5$



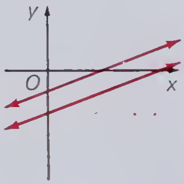
2. $2x - y = 2$
$x - y = 2$



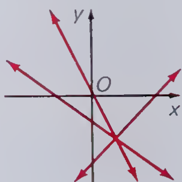
3. $\frac{1}{2}x + \frac{1}{3}y = 1$
$3x + 2y = 6$



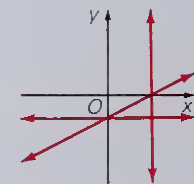
4. $4x - 10y = 20$
$-2x + 5y = -5$



5. $3x + 4y = -5$
$x - y = 3$
$2x + y = 0$



6. $x = 2$
$y = -1$
$x - 2y = 2$



7–10. For the linear systems in Exercises 1–4, find (a) the slope of each graph and (b) each y-intercept, and tell which case of the theorem on page 109 applies to the system.

Written Exercises

In Exercises 1–12, graph the given system of equations and determine the apparent solution set of the system. Verify that each system satisfies one of the cases of the theorem on page 109 by finding the slopes and (if necessary) the y-intercepts of the graphs.

A

1. $3y = x$
$3y = 2x - 6$

2. $2x - 5y = 10$
$\frac{5}{2}y - x = 5$

3. $y + 2x = 6$
$2y - x = 2$

4. $y - 3x = -2$
$3x - y = -2$

5. $2x + y = -7$
$x + 2y = -2$

6. $x + y = 4$
$x + 3y = 6$

7. $3y = 4x + 3$
$-6 = 8x - 6y$

8. $x + 4y = 2$
$x - 2y = -4$

9. $x - 3y = 7$
$2x + y = 7$

10. $y = -3x + 4$
$y = -3x - 3$

11. $x + 3y = -9$
$-3y - x = 9$

12. $2y + 3x = -8$
$3y - 2x = 1$

In Exercises 13–21, graph the given system of equations and determine the apparent solution set of the system. If the system has exactly one apparent solution, verify that the coordinates satisfy all three equations.

B

13. $x + y = 4$
 $x - y = 4$
 $x - 2y = 4$

14. $3x - y = 6$
 $y - 3x = 2$
 $y = 3x$

15. $3y - 2x = 3$
 $x + y = -4$
 $x + 3y = -6$

16. $2y - x = -4$
 $x - 2y = 4$
 $x + 2y = 0$

17. $y + 3x = 6$
 $y - 3x = 6$
 $x + y = 2$

18. $-2x - 5y = 10$
 $2x + 5y = 10$
 $x + \frac{5}{2}y = -5$

19. $x + y = 1$
 $x - y = 3$
 $3x + y = 6$

20. $y = 2x$
 $x + y = 3$
 $\frac{1}{2}x + y = 2$

21. $-4 = 2y + x$
 $y = -\frac{1}{2}x + 5$
 $y = x - 4$

C

22. Choose r and s so that $rs \neq 0$. Graph the following three equations to determine their apparent solution set. Verify that the apparent solution set is actually the solution set.

$$x + 2y = 5$$
$$x - y = -1$$
$$r(x + 2y) + s(x - y) = r(5) + s(-1)$$

23. Prove that any solution to the first two equations in Exercise 22 is a solution to the third.

4-2 Solving a System of Equations

Your aim in solving a system of two linear equations in two variables is the same as that in solving a linear equation in one variable. That is, you want to *transform the system into an equivalent one*, in this case of the form

$$x = a$$
$$y = b,$$

which gives the solution explicitly.

Consider the system:

$$2x + 3y = 4 \qquad (1)$$
$$x - 2y = 6 \qquad (2)$$

Using the properties of equality, you can *multiply* both members of Equation (1) by the same nonzero number, say, 2, and likewise multiply Equation (2) by 3 and thereby obtain the *equivalent system:*

$$4x + 6y = 8 \qquad 2 \times (1) = (1')$$
$$3x - 6y = 18 \qquad 3 \times (2) = (2')$$

Equations (*1*) and (*1′*) on page 111 are equivalent, and so are (*2*) and (*2′*). The *sum* of Equations (*1′*) and (*2′*),

$$7x = 26, \qquad ((1') + (2'))$$

obtained by adding the left members and right members of both equations, is called a *linear combination* of the two equations (*1*) and (*2*). Note that one of the variables was eliminated in forming Equation ((*1′*) + (*2′*)), enabling us to solve directly for the other one.

In general, when you multiply both members of an equation by the same nonzero constant, and add the resulting expressions to the corresponding members of another equation, you obtain a **linear combination** of the two equations.

Transformations Producing an Equivalent System of Linear Equations

1. Replace any equation of the system with an equivalent equation.
2. Replace any equation of the system with a linear combination of that equation and another one of the given equations.
3. Substitute for one variable in any equation either (a) its actual value, or (b) an equivalent expression for that variable obtained from another equation in the system.

EXAMPLE Solve the system:

$$\begin{aligned} x + 3y &= 6 \quad (1) \\ 2x - y &= 5 \quad (2) \end{aligned}$$

SOLUTION 1 *Linear-combination method*

1. By inspection you can see that the variable y can be eliminated from the system as follows: Replace Equation (*2*) with 3 × Equation (*2*). (Transformation 1). Replace (*2′*) with the linear combination (*1*) + (*2′*). (Transformation 2).

$$\begin{aligned} x + 3y &= 6 \quad (1) \\ 6x - 3y &= 15 \quad (2') \end{aligned}$$

$$\begin{aligned} x + 3y &= 6 \quad (1) \\ 7x &= 21 \quad (2'') \end{aligned}$$

2. Replace (*2″*) with an equation of the form $x = a$. (Transformation 1). We have now reduced the original system to:

$$\begin{aligned} x + 3y &= 6 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} x + 3y &= 6 \quad (1) \\ x &= 3 \quad (2''') \end{aligned}$$

3. Now we can solve for y by substituting the value 3 for x in Equation (*1*). (Transformation 3).

$$\begin{aligned} 3 + 3y &= 6 \quad (1') \\ x &= 3 \quad (2'') \end{aligned}$$

4. Replace Equation ($1'$) with an equation of the form $y = b$. The system has now been transformed to an equivalent one that states the solution explicitly.

$$y = 1$$
$$x = 3$$

5. Check by substituting $x = 3$ and $y = 1$ in both Equations (1) and (2).

$$x + 3y = 6 \quad (1) \qquad\qquad 2x - y = 5 \quad (2)$$
$$3 + 3(1) = 6 \qquad\qquad 2(3) - 1 = 5$$

$\therefore$ the solution set is $\{(3, 1)\}$. Answer.

Note: You could equally well have **eliminated** x instead of y in Step 1 by multiplying Equation (1) by -2 and adding the resulting equation to Equation (2). Also, in Step 3 you could have solved for y by substituting 3 for x in Equation (2) instead of in Equation (1).

SOLUTION 2 *Substitution method*

1. Replace Equation (1) with an expression for x in terms of y. (Transformation 1).

$$x = 6 - 3y \quad (1')$$
$$2x - y = 5 \quad (2)$$

2. Substitute in Equation (2) the expression for x in Equation ($1'$), and solve for y. (Transformation 3).

$$2(6 - 3y) - y = 5 \quad (2')$$
$$-7y = -7 \quad (2'')$$
$$y = 1 \quad (2''')$$

3. Solve for x in Equation ($1'$) by substituting the value for y in Equation ($2'''$). (Transformation 3).

$$x = 6 - 3 \cdot 1 = 3 \quad (1'')$$

4. The original system has been transformed to the equivalent one that gives the solution directly.

$$y = 1$$
$$x = 3$$

5. Check the answer as in Step 5 of Solution 1.

Figure 4 shows that the original system was finally transformed into a system of equations whose graphs are a pair of lines parallel to the two axes and having the same point of intersection (that is, the same solution) as the graphs of the original system. In general, the graph of an equation formed from a given pair of equations by using one or more of the transformations listed on page 112 is a line through the point of intersection of the graphs of the given pair if these lines intersect. (If the graphs of the given pair of equations are parallel lines, the graph of the new equation is parallel to them; if they are coincident, the new line is also coincident with them.)

Figure 4

Oral Exercises

State a value by which you would multiply both members of one of the equations in each system in order to eliminate the variable printed in red.

1. $3x - 7y = 24$
   $x + 3y = -8$
2. $3x + 4y = 9$
   $6x - 5y = -8$
3. $5x - 2y = 3$
   $8x + 4y = 3$
4. $x - 3y = 5$
   $2x + 9y = -10$
5. $2x - 7y = 4$
   $x - y = 2$
6. $\frac{1}{3}x - 9y = 8$
   $\frac{2}{3}x + 5y = -7$

State how to form a linear combination of the two equations to eliminate the variable printed in red.

EXAMPLE $3x - 2y = 8$
$5x + 3y = 9$

SOLUTION Multiply the first equation by 3, the second by 2, and add them. (Other linear combinations are also possible.)

7. $4x - 9y = -7$
   $5x + 5y = 1$
8. $3x - 5y = 1$
   $4x - 3y = 5$
9. $4x - 3y = 13$
   $6x - 7y = 32$
10. $\frac{1}{2}x - \frac{1}{4}y = -\frac{3}{2}$
    $3x + 3y = 9$
11. $2x - 3y = -11$
    $3x - 4y = 0$
12. $12x + 2y = 0$
    $16x - 3y = 17$

State how to transform one of the equations in each system so that one of the variables is expressed in terms of the other. Then state how you would use the transformed equation to solve the system by the substitution method.

13. $4x + 5y = -9$
    $x - 4y = 3$
14. $6x - 2y = -11$
    $2x + y = 8$
15. $x + 5y = 12$
    $3x - 4y = 17$
16. $2x - y = -2$
    $8x + 7y = 3$
17. $2x - 5y = 16$
    $2x + 3y = 0$
18. $-x + 5y = 4$
    $5x + 7y = -4$

Written Exercises

A **1–12.** Solve the systems of equations in Oral Exercises 1–12 by the linear combination method. Check your solutions.

13–18. Solve the systems of equations in Oral Exercises 13–18 by the substitution method. Check your solutions.

In Exercises 19–28 solve each system by either the linear combination method or the substitution method, whichever seems simpler.

19. $x - y = -7$
    $5x - 4y = 1$
20. $8x + 3y = 5$
    $5x + 4y = 18$
21. $6x + 5y = 2$
    $10x - 7y = 11$

22. $y = 4x + 3$
$3y - 5x = 19$

23. $\frac{3}{4}x - \frac{1}{2}y = 3$
$\frac{1}{2}x - \frac{3}{2}y = -5$

24. $-\frac{5}{6}x + \frac{2}{3}y = 7$
$\frac{1}{3}x - 2y = -8$

B 25. $3(x - 1) = 5(y - 4) + 2$
$2x = 3y - 9$

26. $2(x + 2) - 7(y - 1) = 16$
$2(x - 1) = 9 + 5(y - 2)$

27. $\frac{1}{2}(x + 3) - \frac{1}{4}(y - 2) = 3$
$5x - 3(y - 1) = 10$

28. $\frac{1}{3}(x + 1) - \frac{1}{2}y = 3$
$-\frac{1}{4}(x + 3) + \frac{1}{3}(y - 1) = -3$

In Exercises 29–32, solve first for $\frac{1}{x}$ and $\frac{1}{y}$; then determine values of x and y.

29. $\frac{2}{x} + \frac{1}{y} = 7$
$\frac{3}{x} - \frac{1}{y} = -2$

30. $\frac{3}{x} + \frac{4}{y} = 1$
$\frac{2}{x} - \frac{1}{y} = 8$

31. $\frac{8}{x} + \frac{3}{y} = 11$
$-\frac{7}{x} - \frac{2}{y} = -4$

32. $\frac{3}{x} - \frac{5}{y} = 1$
$-\frac{7}{x} + \frac{9}{y} = -1$

C 33. Determine A and B so that the graph of $Ax + By = 5$ will contain the points (3, 1) and (2, −1).

34. Determine A and C so that the graph of $Ax - 2y = C$ will contain the points (3, 5) and (−1, −3).

35. Prove that for all real numbers m_1, m_2, b_1, and b_2 such that $m_1 \neq m_2$, the following systems of equations are equivalent.

$$\begin{aligned} y &= m_1x + b_1 \\ y &= m_2x + b_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= \frac{b_1 - b_2}{m_2 - m_1} \\ y &= \frac{m_2b_1 - m_1b_2}{m_2 - m_1} \end{aligned}$$

36. Prove that for all real numbers m, b_1, *and* b_2, the solution set of each of the systems

$$\begin{aligned} y &= mx + b_1 \\ y &= mx + b_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= b_1 \\ x &= b_2 \end{aligned}$$

is an infinite set or the empty set according as $b_1 = b_2$ or $b_1 \neq b_2$.

37. Prove that for all real numbers m, b_1, and b_2, the solution set of the system

$$\begin{aligned} y &= mx + b_1 \\ x &= b_2 \end{aligned}$$

is $\{(b_2, mb_2 + b_1)\}$.

38. Use the results of Exercises 35–37 to prove the theorem stated on page 109.

39. Use the results of Exercises 35–37 to prove that two lines are parallel if and only if each line has no slope and different x-intercepts or both lines have the same slope and different y-intercepts.

4-3 Determinants

You can find general formulas for the solution of a system of two linear equations in two variables by solving the general system

$$\begin{aligned} a_1x + b_1y &= c_1 \qquad (1) \\ a_2x + b_2y &= c_2 \qquad (2) \end{aligned}$$

where a_1, b_1, c_1, a_2, b_2, and $c_2 \in \mathfrak{R}$. Multiplying Equation (*1*) by b_2 and Equation (*2*) by $-b_1$, you get:

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ -a_2b_1x - b_1b_2y &= -c_2b_1. \end{aligned}$$

Adding these equations gives

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

Similarly, multiplying Equation (*1*) by $-a_2$ and Equation (*2*) by a_1, and adding, you get

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1.$$

Therefore, if $a_1b_2 - a_2b_1 \neq 0$, you have

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \qquad (3)$$

You can check that the values for x and y given by the formulas (*3*) do in fact satisfy Equations (*1*) and (*2*).

EXAMPLE 1 Use the formulas (*3*) to solve the system:

$$\begin{aligned} x - 2y &= 5 \\ 3x + 4y &= -5 \end{aligned}$$

SOLUTION You have $a_1 = 1$, $b_1 = -2$, $c_1 = 5$ and $a_2 = 3$, $b_2 = 4$, $c_2 = -5$. Substituting in the formulas (*3*), you get:

$$x = \frac{5 \cdot 4 - (-5)(-2)}{1 \cdot 4 - 3(-2)} = \frac{20 - 10}{4 + 6} = \frac{10}{10} = \mathbf{1}$$

$$y = \frac{1(-5) - 3 \cdot 5}{1 \cdot 4 - 3(-2)} = \frac{-5 - 15}{4 + 6} = \frac{-20}{10} = \mathbf{-2}$$

Checking that **(1, −2)** is a solution is left to you.
$\therefore$ the solution set is $\{(1, -2)\}$. Answer.

There is a convenient way to denote the numerators and the denominator in the equations (*3*) for x and y.

> For any a_1, b_1, a_2, $b_2 \in \mathfrak{R}$, the **determinant** $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has the value $a_1b_2 - a_2b_1$.

Notice that the square array of numerals, set off with vertical bars (not absolute-value signs!), is just another numeral for "$a_1b_2 - a_2b_1$." The numerals a_1, b_1, a_2, b_2 in the array are called the **entries** (or **elements**) of the determinant.

Here is a convenient way to remember how to evaluate the determinant:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

You simply take the difference of products as indicated.

From Equations (3) and the definition of D, you can see that, for $D \neq 0$,

$$x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D} \qquad (4)$$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Notice that the entries a_1, b_1, a_2, b_2 of D are just the coefficients of x and y in Equations (1) and (2); D is called the **determinant of coefficients.** To obtain the entries for D_x, you replace the x-coefficients a_1, a_2 in D with the constants c_1, c_2. Similarly, to obtain the entries for D_y, you replace the y-coefficients b_1, b_2 in D with the constants c_1, c_2.

The solution of a linear system in determinant form (4) is called **Cramer's Rule.** If $D = 0$, then either the system is inconsistent or the system has an infinite solution set.

EXAMPLE 2 Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x + y &= 6 \\ 3x - 4y &= 9 \end{aligned}$$

SOLUTION By inspection, you have

$$D = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -8 - 3 = -11.$$

Then

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 6 & 1 \\ 9 & -4 \end{vmatrix}}{-11} = \frac{-24 - 9}{-11} = \frac{-33}{-11} = 3,$$

and

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix}}{-11} = \frac{18 - 18}{-11} = \frac{0}{-11} = 0.$$

You can check that (3, 0) is a solution.

$\therefore$ the solution set is $\{(3, 0)\}$. Answer.

Oral Exercises

State the value of the given determinant.

1. $\begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}$

4. $\begin{vmatrix} a & x \\ b & y \end{vmatrix}$

5. $\begin{vmatrix} -2 & m \\ -8 & n \end{vmatrix}$

6. $\begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 7 \\ -4 & -14 \end{vmatrix}$

8. $\begin{vmatrix} c & d \\ nc & nd \end{vmatrix}$

9. $\begin{vmatrix} a-3 & 2a-6 \\ 1 & 2 \end{vmatrix}$

10. On the basis of Exercises 6–9 above, what can you say about the value of any determinant of the form $\begin{vmatrix} a & b \\ ka & kb \end{vmatrix}$?

State the determinants D, D_x, and D_y for the given system of equations.

11. $3x + 5y = 2$
    $2x + 4y = 1$

12. $7x - 3y = 1$
    $-4x + 2y = -2$

13. $x - 4y = -3$
    $2x - 5y = 6$

14. $8x + 3y = 5$
    $-5x - 2y = 9$

15. $-x + 2y = 3$
    $4x + 3y = -17$

16. $5x - 3y = 11$
    $-2x - 4y = 6$

Written Exercises

Evaluate the given determinant.

A

1. $\begin{vmatrix} 6 & -5 \\ -2 & 3 \end{vmatrix}$

2. $\begin{vmatrix} 9 & -1 \\ 0 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 4 & 6 \\ -2 & -3 \end{vmatrix}$

4. $\begin{vmatrix} k & 3 \\ 5 & k \end{vmatrix}$

Solve each equation for k.

5. $\begin{vmatrix} k & 7 \\ 2 & 5 \end{vmatrix} = 1$

6. $\begin{vmatrix} -3 & k \\ 7 & -8 \end{vmatrix} = 10$

7. $\begin{vmatrix} k & 5 \\ 7 & k \end{vmatrix} = 1$

8–13. Use Cramer's Rule to solve the equations in Oral Exercises 11–16.

Use Cramer's Rule to solve the given system. If it is inconsistent or has an infinite solution set, so state.

14. $3x - 2y = 1$
    $5x - 3y = 4$

15. $2x + 3y = 1$
    $5x + 4y = -8$

16. $2x - 3y = 11$
    $3x + 5y = 7$

B

17. $2x - 8y = 6$
    $-x + 4y = 3$

18. $-2x + 4y = -4$
    $3x - 5y = 4$

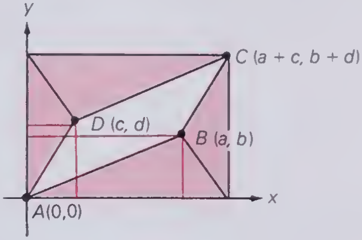
19. $-4x - 5y = -2$
    $8x + 10y = 4$

20. $5x - 3y = 4$
    $9x - 6y = 8$

21. $9x + 4y = 6$
    $\frac{1}{2}x - \frac{1}{3}y = 2$

22. $\frac{5}{4}x + 2y = 5$
    $\frac{1}{3}x - \frac{4}{5}y = -2$

C **23.** Use the diagram below to show that the area of parallelogram $ABCD = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. (*Hint:* Find the area of the rectangle and subtract the areas of the four triangles.)



24. Prove that if the graphs of the system of equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

are parallel (or coincident), then $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$.

25. Prove that for the system of equations given in Exercise 24 above, if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, then the graphs of the system are parallel or coincident.

26. Prove that if the system of equations

$$\begin{aligned} a_1x + b_1y &= 0 \\ a_2x + b_2y &= 0 \qquad a_1, b_1, a_2, b_2 \neq 0 \end{aligned}$$

has a solution other than $(0, 0)$, then $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. (*Hint:* Assume $x \neq 0$ first. Then start by dividing the first equation by b_1, the second by b_2.)

4-4 Using Two Variables to Solve Problems

The following examples illustrate how you can use systems of equations to solve practical problems.

EXAMPLE 1 Jack has \$10.00 to spend for entertainment. He found that he can spend all the money for two record albums and a ticket to the basketball game, or he can buy one album and two tickets to the game for himself and a friend and have \$2.75 left for refreshments. What is the price of a basketball ticket, and how much does an album cost?

SOLUTION 1. The problem asks for the price of a basketball ticket and for the price of an album.

(Solution continued on page 120)

2. Let the price of a ticket be x cents and the price of an album be y cents.

3. $\underbrace{\text{The price of one ticket}}_{x}\ \underbrace{\text{added to}}_{+}\ \underbrace{\text{the price of two albums}}_{2y}\ \underset{\downarrow}{\text{is}} \stackrel{?}{=}\ \underbrace{\text{\$10.00 (1000 cents).}}_{1000}$

$\underbrace{\text{The price of two tickets}}_{2x}\ \underbrace{\text{added to}}_{+}\ \underbrace{\text{the price of one album}}_{y}\ \underset{\downarrow}{\text{is}} \stackrel{?}{=}\ \text{\$10.00 less } \underbrace{\text{\$2.75}}_{725} \text{ (\$7.25).}$

4. Solve the system: $x + 2y = 1000$
$2x + y = 725$

Showing that the solution of this system is (150, 425) and checking this result in the words of the problem (Step 5) is left to you.

$\therefore$ a basketball ticket costs \$1.50 and an album costs \$4.25. **Answer.**

To solve motion problems about airplanes, as in Example 2, you must know the meanings of the following phrases:

Tail wind: a wind blowing in the same direction as the one in which the airplane is heading.
Head wind: a wind blowing in the direction opposite to the one in which the airplane is heading.
Wind speed: the speed of the wind.
Air speed: the speed of the airplane in still air.
Ground speed: the speed of the airplane relative to the ground.

With a tail wind, an airplane's ground speed is the sum of its air speed and the wind speed. With a head wind, the ground speed is the difference between the air speed and the wind speed.

EXAMPLE 2 With a given head wind, a certain airplane can travel 4800 km in 6 h. But flying in the opposite direction with the same wind blowing, the plane can fly that distance in 1 h less. Find the plane's air speed and the wind speed.

SOLUTION 1. The problem asks for the speed of the plane in still air and for the speed of the wind.

2. Let x = the air speed of the plane in kilometers per hour (km/h);
y = the wind speed in kilometers per hour.
The facts of the problem are listed below in the chart. (Recall the use of the relationship $d = rt$ on page 39.)

	Ground speed (km/h) r	Time (h) t	Distance (km) $rt = d$
With a head wind	$x - y$	6	$6(x - y)$
With a tail wind	$x + y$	5	$5(x + y)$

3. $\underbrace{\text{Distance with a head wind}}_{6(x - y)} \overset{\text{is}}{=} \underbrace{4800 \text{ km.}}_{4800}$

$\underbrace{\text{Distance with a tail wind}}_{5(x + y)} \overset{\text{is}}{=} \underbrace{4800 \text{ km.}}_{4800}$

4. Solve the system:

$$\begin{aligned} 6(x - y) &= 4800 \\ 5(x + y) &= 4800 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y &= 800 \\ x + y &= 960 \end{aligned}$$

Completing Step 4 and checking your results (Step 5) are left to you. You should find that the plane's air speed is 880 km/h and that the wind speed is 80 km/h. **Answer.**

Oral Exercises

In Exercises 1–6 let *d* represent the number of dimes and *q* represent the number of quarters collected from a parking meter. In each case, translate the sentence into an equation in *d* and *q*.

1. There are a total of 31 coins in the meter. (The meter accepts *only* dimes and quarters.)
2. The value of the coins is \$5.80.
3. If there were 3 fewer dimes and 2 more quarters the value of the coins would be \$6.00.
4. If there were half as many quarters and three times as many dimes, there would be a total of 48 coins in the meter.
5. If there were 5 fewer dimes and twice as many quarters, the value of the coins would be \$9.80.
6. If there were four fewer dimes, there would be twice as many quarters as dimes.

In Exercises 7–12, consider an airplane with an air speed of 400 km/h.

7. What is the plane's ground speed on a windless day?
8. What is the plane's ground speed if there is a head wind of 30 km/h?
9. What is the plane's ground speed if there is a tail wind of 40 km/h?
10. How long would it take the plane to make a 1000 km flight on a windless day?
11. How long would it take the plane to make a 1110 km flight against a head wind of 30 km/h?
12. How long would it take the plane to make a 1320 km flight with a tail wind of 40 km/h?

In Exercises 13–17, let x represent the rate of a boat in still water and y represent the rate of the current, both in kilometers per hour. Translate each sentence into an equation in x and y.

13. The rate of the boat traveling upstream is 4 km/h.
14. The rate of the boat traveling downstream is 10 km/h.
15. The boat can travel 18 km upstream in 4 h.
16. The boat can travel 19 km downstream in 2 h.
17. Suppose you know only that the boat can complete a round trip of 10 km upstream and 10 km downstream in 6 h. Could you find the rates of the boat and the current? If so, give the equation(s) you would use. If not, explain why not.

Problems

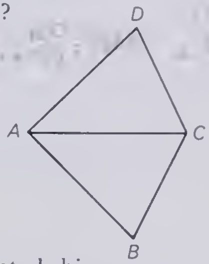
A

1. Use the relationships stated in Oral Exercises 1 and 2 to find the number of dimes and the number of quarters in the meter.
2. Use the relationships stated in Oral Exercises 13 and 14 to find the rate of the boat in still water and the rate of the current.
3. Use the relationships stated in Oral Exercises 15 and 16 to find the rate of the boat in still water and the rate of the current.
4. The difference of two numbers is 7, and the greater number is 3 greater than twice the smaller. Find the numbers.
5. Three times the sum of two numbers is 7 more than the greater number. One-half the difference of the numbers is 3. What are the two numbers?
6. The perimeter of a rectangle is 80 cm. Three times the length of the rectangle is equal to seven times its width. What are the dimensions of the rectangle?
7. The degree measure of one of two complementary angles is 30 less than twice that of the other. What are the degree measures of the angles?
8. The degree measure of one of two supplementary angles is 6 more than one-half that of the other. What are the degree measures of the angles?
9. A circular piece of metal is to be cut into 3 pieces of one size and 5 pieces of a larger size so that the central angle of each of the larger pieces has a degree measure 6 less than twice the degree measure of the central angle of each smaller piece. What is the degree measure of the central angle of the larger pieces?
10. At the Centerville Chess and Bridge Club one evening, 23 tables were needed to accommodate 62 bridge and chess players, one game per table. How many tables were used for chess and how many for bridge (a four-person game)?

11. With an 80 km/h head wind, a plane can fly a certain distance in 4 h. Flying in the opposite direction with the same wind blowing, it can fly that distance in 1 h less. What is the plane's air speed?

12. Traveling downstream, a boat can go 18 km in 2 h. Going upstream, it makes only two-thirds this distance in twice the time. What is the rate of the boat in still water, and what is the rate of the current?

13. Tickets to a concert cost \$4.50 for the general public and \$3.75 for students. If 400 people attended the concert and \$1680 was collected in ticket sales, how many student tickets were sold?



B 14. In the figure at the right $m(\overline{AB}) = m(\overline{AD})$ and $m(\overline{BC}) = m(\overline{DC})$; $m(\overline{AC})$ is $1\frac{1}{2}$ times as long as $m(\overline{AB})$. If the perimeter of quadrilateral $ABCD$ is 17 and the perimeter of ΔABC is 16, find the lengths $\overline{AB}$ and $\overline{BC}$.

15. If Herman's test grade and quiz grade are equally weighted, his average would be 85. If the test counts 3 times as much as the quiz, his average would be 83. What grade did he receive on the test?

16. The degree measure of each of the base angles of an isosceles triangle is 6 less than the degree measure of the vertex angle. Find the degree measures of the angles of the triangle.

In Exercises 17–20 find values of A and B so that the graph of the given equation will contain the given points.

17. $Ax + By = 4$; $(8, 4)$, $(13, 7)$

18. $y = Ax^2 + B$; $(-2, 5)$, $(3, 15)$

19. $y = x^2 + Ax + B$; $(1, 4)$, $(-2, 10)$

20. $y = Ax^2 - 3x + B$; $(-1, 10)$, $(2, 7)$

21. The height h (in meters) above the ground of an object propelled upward with an initial velocity v_0 (in meters per second) from an initial height h_0 (in meters) is given by the equation,

$$h = -10t^2 + v_0 t + h_0,$$

where t represents the time in seconds.

What are the initial velocity v_0 and height h_0 for an object that is 30 m from the ground after 1 s and that hits the ground after 3 s?

22. The equation $y = bx + ax^2$ describes the trajectory of a baseball thrown from a point considered to be the origin of a coordinate system. If the ball goes through the point $(20, 8)$ and lands at the point $(100, 0)$, find the value of a and b.

C **23.** Two railroad workers were working together in a 1.2 km mountain tunnel when a signal light flashed indicating the approach of a train, which was traveling at 60 km/h. Walking east, one worker reached the east end of the tunnel in 6 min, as the train entered the tunnel. The other worker reached the west end of the tunnel in 6 min and was passed by the train 0.24 km beyond the west end of the tunnel. At what rate did each worker walk?

24. After leaving the boathouse, a rower rowing upstream passes a log 2 km upstream from the boathouse. The rower rows upstream for one more hour and then rows back to the boathouse, arriving at the same time as the log. How fast was the current flowing?

25. A coin bank contains 27 coins, all nickels, dimes, or quarters. The combined value of the nickels and dimes is 90¢; the combined value of the dimes and quarters is $4.00. How many coins of each denomination are there in the bank?

ON THE CALCULATOR

These exercises are designed to help you familiarize yourself with your calculator. Your calculator may feature different buttons or a different order of operations. In this book we use [M+] to indicate "store in memory" as well as "add to memory."

EXAMPLE 1 Evaluate $(2x + 1)^5$ when $x = 6$.

SOLUTION 2 [×] 6 [+] 1 [=] [y^x] 5 [=] 371293. Answer.

EXAMPLE 2 Evaluate $\dfrac{13,257}{216} - 0.008(4819)$.

SOLUTION 13257 [÷] 216 [=] [M+] 0.008 [×] 4819 [=] [+/−] [+] [MR] [=] 22.823. Answer.

Exercises

Evaluate $\left(\frac{x}{2} - 3\right)^4$ for the given values of x.

1. $x = 10$ **2.** $x = 18$ **3.** $x = 7$ **4.** $x = 8.1$

5. Evaluate 0.00049 (28,153) ÷ 0.05

6. Evaluate $\dfrac{155}{70,845} - 632.9(-0.018)$

7. In one year there were 728,014 take-offs and landings at a certain airport. What was the average number of take-offs and landings per day?

Self-Test 1

VOCABULARY
system of linear equations (p. 107)
solution set of a system of equations (p. 108)
consistent equations (p. 108)
inconsistent equations (p. 108)
linear combination of equations (p. 112)
entry (element) of a determinant (p. 117)
determinant of coefficients (p. 117)
Cramer's Rule (p. 117)

1. Graph the system of equations *Obj. 1, p. 107*

$$\begin{aligned} x - y &= 4 \\ 5x + 3y &= 12 \end{aligned}$$

and give the apparent solution set.

2. Find the slope and the y-intercept of each graph (if necessary) to determine how many solutions each system has.

a. $\begin{aligned} 2x - y &= 3 \\ x - 2y &= -3 \end{aligned}$ **b.** $\begin{aligned} 2x - 10y &= 8 \\ -x + 5y &= -4 \end{aligned}$ **c.** $\begin{aligned} 2x + 3y &= 6 \\ -3y - 2x &= 6 \end{aligned}$

3. Solve by the linear combination method. *Obj. 2, p. 107*

$$\begin{aligned} 2x - 5y &= -18 \\ 3x + 4y &= 19 \end{aligned}$$

4. Solve by the substitution method.

$$\begin{aligned} 7x + 3y &= 6 \\ 2x - y &= 11 \end{aligned}$$

5. Use determinants to solve the system. *Obj. 3, p. 107*

$$\begin{aligned} 5x - 4y &= 8 \\ 6x - 5y &= 10 \end{aligned}$$

6. The regular fare for a certain route on Flyaway Airlines is \$45, but children pay $\frac{2}{3}$ of the regular fare. On a flight with 150 passengers, \$6150 was collected in ticket sales. How many adults and how many children were on the flight? *Obj. 4, p. 107*

Check your answers with those at the back of the book.

Systems of Inequalities in Two Variables

OBJECTIVES for Sections 4-5 and 4-6:
1. Graph the solution set of a system of linear inequalities in two variables.
2. Solve linear-programming problems in two variables.

4-5 Graphing a System of Linear Inequalities

How would you describe the graph of the solution set of the following system of linear inequalities?

$$x + y < 9$$
$$y \geq 2x$$

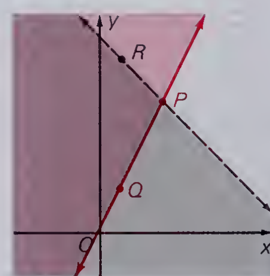


Figure 5

In Figure 5, gray shading is used to show (part of) the open half-plane that consists of the points lying below the line with equation $x + y = 9$. This is the graph of the inequality

$$x + y < 9$$

(recall Section 3-4). Red shading in Figure 5 shows the closed half-plane that is the graph of

$$y \geq 2x.$$

The region in Figure 5 where the two colors overlap is the *intersection* of the two half-planes; it consists of the points whose coordinates satisfy *both* inequalities in the given system. Thus, this region is the graph of the solution set of the given system. Note that the region includes the part of its boundary indicated by a solid half-line (the half-line from P through Q), but does not include the part shown by a dashed half-line (the half-line from P through R). P itself is not in the region.

In general:

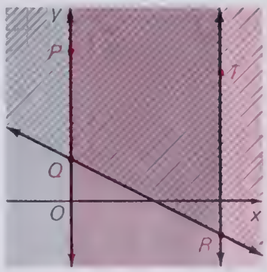
> To show the graph of the solution set of a system of inequalities, you take these steps:
>
> 1. Graph each inequality in the system.
> 2. Show by heavier shading the region that consists of those points that belong to all of the graphs drawn in Step 1.

EXAMPLE Graph the solution set of the system:

$$0 \leq x \leq 7$$
$$x + 2y \geq 4$$

SOLUTION The given system is equivalent to the following system of *three* inequalities, whose graphs are shown in the diagram at the right as indicated:

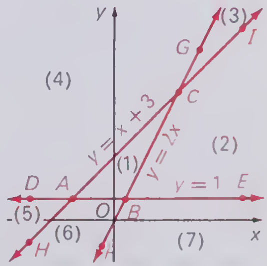
$0 \le x$ (red shading)
$x \le 7$ (gray shading)
$x + 2y \ge 4$ (diagonal hatching)



The region common to all graphs is the graph of the system; it is the region between the rays $\overrightarrow{QP}$ and $\overrightarrow{RT}$ and above the segment $\overline{QR}$, *including all boundary points.* Answer.

Oral Exercises

The lines whose equations are $y = 2x$, $y = x + 3$, and $y = 1$ separate the plane into seven regions numbered as shown at the right. In Exercises 1–9, name the region or regions that form the graph of the solution set of the given system, and identify the part(s), if any, of the boundary that belongs to the graph.



1. $y < 2x$
$y > 1$

2. $y > x + 3$
$y < 2x$

3. $y \ge 2x$
$y < 1$

4. $y \ge 1$
$y > 2x$
$y > x + 3$

5. $y < x + 3$
$y \le 1$
$y > 2x$

6. $y \ge x + 3$
$y \ge 2x$
$y \ge 1$

7. $y < x + 3$
$y < 2x$
$y < 1$

8. $y \ge x + 3$
$y < 1$

9. $y > 1$
$y < 2x$
$y < x + 3$

Written Exercises

In a coordinate plane graph the solution set of each system.

A

1. $x \ge 1$
$y \le 2$

2. $y \ge -x$
$y \le 0$

3. $y \ge x$
$y \ge -x$

4. $y \le 2 + x$
$y \le 2 - x$

5. $y \ge 2x$
$y \le x + 4$

6. $y \le -2x$
$y \ge 2x + 3$

7. $y < x + 3$
$y > 2x + 4$

8. $x - 2y \ge 5$
$2x + y \ge 5$

9. $x + 3y \ge 6$
$2x - y \le 3$

10. $2y + x \le 6$
$y - 3x \ge 4$

11. $x - 2y \le 10$
$2x + y \ge 0$

12. $-3x + 4y \le 12$
$y \ge 3$

13. $-1 < x < 2$

14. $2 \le y < 5$

15. $x - 1 < y < x + 1$

16. $-1 < x + y < 3$

17. $-1 \le x \le y$

18. $2x \le y + 1 \le 7$

B **19.** $y \geq 0$
$2y \leq x$
$y \leq -x + 9$

20. $x \geq 0$
$y \leq 0$
$y \leq x - 4$

21. $y \leq 4$
$y \geq 2x + 6$
$y \geq -x - 3$

22. $3 \geq x \geq 0$
$4 \geq y \geq 0$

23. $x \geq 0$
$y \geq 0$
$x + y \leq 8$
$2x + y \leq 10$

24. $y \geq -1$
$x \leq 2$
$2y - x \leq 4$
$y - x \leq 2$

25. $0 \leq x \leq 4$
$0 \leq y \leq 5$
$x + y \leq 6$

26. $-2 < x < 0$
$y < x + 6$
$x + y > -6$

27. $2 < y < 6$
$y > 2x$
$y > -x - 3$

28. $|x| < 3$
$x + y < 4$

29. $|y| \leq 2$
$|x + 3| \leq 1$

30. $-1 \leq x + y \leq 2$
$3 \leq x - y \leq 5$

C **31.** $|x| < 2$
$|x + y| < 1$

32. $|x + y| \geq 4$
$|x - y| \leq 4$

33. $|x| + |y| \leq 4$

4-6 Linear Programming

The following situation illustrates a type of problem in economics that involves a system of inequalities in its solution:

An assembler of dune buggies carries two models: the Sand Grabber and the Dune Dee. The company has equipment to assemble as many as 60 Sand Grabbers and as many as 45 Dune Dees per month. It takes 150 h of labor to assemble a Sand Grabber and 200 h of labor to assemble a Dune Dee, and the company has up to 12,000 h of labor available for dune-buggy assembly each month. If the profit gained on each Sand Grabber is \$120 and on each Dune Dee is \$180, find the number of each model the firm should assemble to gain the maximum (greatest) profit each month.

To solve this problem, let x represent the number of Sand Grabbers and y the number of Dune Dees that are assembled each month. Then

$120x$ = the monthly profit on Sand Grabbers,
$180y$ = the monthly profit on Dune Dees, and
$120x + 180y$ = the monthly profit on the two models together.

The values of x and y must satisfy the following system of inequalities, called **constraints:**

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ (The firm must assemble a nonnegative number of each model.)

$x \leq 60$ (At most 60 Sand Grabbers can be assembled each month.)

$y \leq 45$ (At most 45 Dune Dees can be assembled each month.)

$150x + 200y \leq 12000$ (Up to 12,000 h of labor are available for assembly each month.)

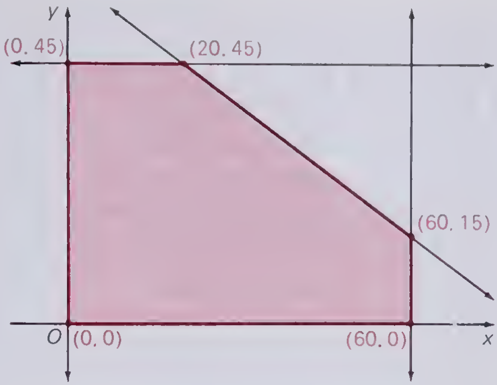


Figure 6

The shaded portion of Figure 6 is the solution set of the foregoing system of constraints. Notice that the graph of this solution set is the intersection of a finite number of closed half-planes (actually, five closed half-planes). The graph is called the **feasibility region**, and the points of the graph where the lines that form the boundary intersect are called **corner points** of the feasibility region. We can then find the maximum value of $120x + 180y$ over the feasibility region by using the following theorem that we accept without proof.

Theorem. If a and b are any real numbers, and if the linear expression $ax + by$ has a maximum (greatest) value over a feasibility region which is the intersection of a finite number of closed half-planes and which has corner points, then the maximum occurs for the coordinates of some corner point.

Similarly, if $ax + by$ has a minimum (least) value over the region, then the minimum occurs for the coordinates of some corner point.

Therefore, to **maximize** or minimize $120x + 180y$ (find its greatest or least value), we evaluate it at the five corner points, whose coordinates are found by solving simultaneously the equations of the boundary lines determining those points:

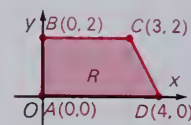
Corner Point	$120x + 180y$
(0, 0)	$120 \cdot 0 + 180 \cdot 0 = 0$
(60, 0)	$120 \cdot 60 + 180 \cdot 0 = 7200$
(60, 15)	$120 \cdot 60 + 180 \cdot 15 = 9900$
(20, 45)	$120 \cdot 20 + 180 \cdot 45 = 10\,500$
(0 45)	$120 \cdot 0 + 180 \cdot 45 = 8100$

Thus, over the feasibility region, the maximum value of $120x + 180y$ is **10,500** and this occurs at the point **(20, 45)**. The minimum value of $120x + 180y$ is 0 and occurs at the origin, that is, when no dune buggies are produced. Therefore, to maximize the monthly profit, the firm should assemble 20 Sand Grabbers and 45 Dune Dees, and thus obtain a profit of $10,500 each month.

The process illustrated in this example is called **linear programming** because it furnishes a means of finding maximum and minimum values of a linear expression over a feasibility region determined by linear inequalities.

Oral Exercises

State the value of the given expression for the coordinates of the given point. Refer to the graphs shown at right.

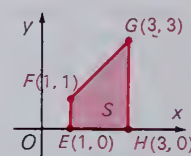


1. $x + y;\ A$ **2.** $x + y;\ B$ **3.** $x + y;\ C$

4. $x - y;\ D$ **5.** $x - y;\ E$ **6.** $x - y;\ F$

7. $x - 2y;\ G$ **8.** $x - 2y;\ H$ **9.** $2x - y;\ B$

State the requested value for the coordinates of the points in the given closed region.



10. minimum of $(x + y);\ R$ **11.** maximum of $(x + y);\ R$

12. minimum of $(x - y);\ R$ **13.** maximum of $(x + y);\ S$

14. minimum of $(2x - y);\ S$ 3 **15.** maximum of $(2x - y);\ S$

0, 6, 1, 1,

Written Exercises

In Exercises 1–6:

a. Graph the solution set of the system of inequalities.
b. Find the coordinates of the corner points of the graph.
c. Find the value of the linear expression printed in red at each of the corner points.
d. State the maximum and minimum values (if any) of the given linear expression under the given constraints.

A

1. $0 \le x \le 3$
$0 \le y \le -x + 5$
$x + 2y$

2. $1 \le y \le 3x + 1$
$2x + y \le 11$
$2x - y$

3. $1 \le y \le 7$
$x \ge 2$
$3x + y \le 16$
$5x + 2y$

4. $x \ge 0$
$y \le x + 1$
$y \le 4x - 8$
$4x - 3y$

5. $0 \le x \le 6$
$0 \le y \le 8$
$2x + y \le 14$
$3x - 2y$

6. $x \ge 1$
$2 \le y \le 8 - x$
$x + 2y \le 13$
$5x + y$

P 130 ⊕ 131 all orals
1, 5, 7

Exercises 7–10 refer to the following situation. A manufacturer of furniture can produce at most 20 tables and at most 30 chairs per day. Each table requires 3 h of labor, each chair 2 h of labor. The maximum total hours of labor that the manufacturer can assign is 96. 131 8-10

4-14

7. Letting x represent the number of tables produced daily and y represent the number of chairs produced daily, give three inequalities that express the conditions above.

8. Graph the system of inequalities in Exercise 7.

9. Give the coordinates of the corner points of the graph in Exercise 8. 5+ 133

10. Find the number of chairs and tables the company should produce for maximum profit, if the profit on each table is \$30, and the profit on each chair is \$40. $30x + 40y = P$

Exercises 11–14 refer to the following situation: Each can of Happy Hound dog food must contain at least 15 units of protein and at least 14 units of fiber, all to be derived from two ingredients, *A* and *B*. The table below gives the number of units of each nutrient per gram of ingredient.

	Ingredient A	*Ingredient B*
Protein	3	3
Fiber	4	2

B **11.** Graph the inequalities that express the conditions above.

12. How many grams of each ingredient should the company put in each can to minimize the cost if each gram of ingredient A costs 6¢ and each gram of ingredient B costs 5¢?

13. Answer question 12 if each gram of ingredient A costs 4¢.

14. Answer question 12 if each gram of ingredient A costs 10¢.

In Exercises 15, and 16, assume that *a*, *b*, and *c* are greater than 0.

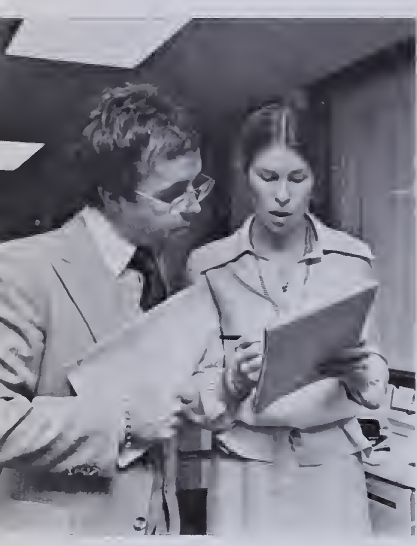
C **15.** Find the maximum of $ax + by$ over the rectangular region with vertices $(0, 0)$ $(0, c)$, $(1, c)$, and $(1, 0)$.

16. Find the minimum of $ax + by$ over the rectangular region with vertices $(-1, c)$, $(-1, -c)$, $(c, -c)$, and (c, c).

17. Graph the solution set R of the system of inequalities $x \geq 0$, $y \geq 0$, $x + 2y \leq 8$, $3x + 2y \leq 12$, and on the same coordinate plane graph the line with equation $x + y = p$ for $p = -2, 0, 3, 5$, and 8. Explain why the minimum value of $x + y$ over R is 0 and the maximum is 5.

Careers

in Business

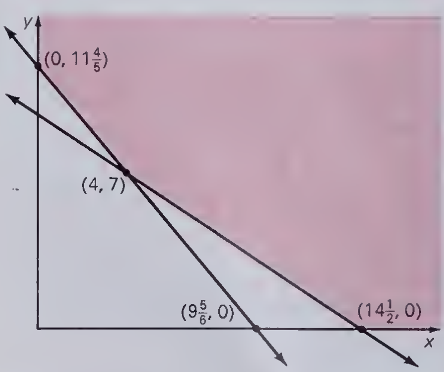


The close cooperation of many people is required to produce an advertisement that is attractive and that presents the essential features of the product accurately.

Linear programming is used in a great variety of industrial and business operations. Its techniques are not only applied to production, but to purchasing and distribution, to job assignments, to budgeting, and even to advertising.

It is important to note that often the relationships among the variables of the problem are only probable. For example, in the dune buggy example of Section 4-6, profits on each buggy are assumed to be constant while in fact such profits may vary depending on the sales volume. Linear programming is an approximation to a real-world situation. However, it is a sufficiently accurate mathematical model to be of great use to the world of business.

EXAMPLE An advertising manager of a company considers the alternatives in advertising various products in two weekly magazines. A half-page advertisement costs \$300 in magazine A and \$250 in magazine B per weekly issue. A recent advertising survey indicates that for every issue, 6000 readers will notice the advertisement in magazine A and 5000 will see it in B. In addition, 200 readers of A and 300 of B per week will usually complete attached questionnaire cards for additional information. In order to profit from the advertising campaign, it was determined that at least 59,000 readers should be reached and at least 2900 request cards for additional information must be received. How many weekly advertisements should be placed with each magazine in order to minimize the cost for these advertisements?



SOLUTION Let x be the number of weekly advertisements in magazine A and y the number in magazine B. Graph the following equations:

Readers} $\quad 6000x + 5000y \geq 59{,}000$
Cards} $\quad 200x + 300y \geq 2900$
$\qquad x \geq 0 \quad y \geq 0$

The feasibility region is shown at the right.

Determining the minimum value of $300x + 250y$, you find that 4 ads should be placed in magazine A, and 7 in magazine B.

Self-Test 2

VOCABULARY constraints (p. 128) corner point (p. 129) feasibility region (p. 129) maximize (p. 129) linear programming (p. 130)

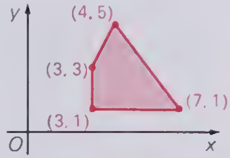
Graph the solution set of each system. *Obj. 1, p. 126*

1. $3x + 4y < 24$
$2x - y > -2$

2. $x \geq 0$
$3x \leq 2y \leq 6 - x$

3. $0 \leq x \leq 4$
$1 \leq y \leq 5$
$3x + 2y \leq 16$

Determine the maximum and minimum values of the given expression over the feasibility region shown at right. *Obj. 2, p. 126*



4. $2x + y$

5. $5x + 4y$

6. Find the maximum daily profit that can be realized on manufacturing x bureaus and y bookcases if the maximum numbers of each item are 15 and 20, respectively. Each bureau requires 4 h of labor and each bookcase 1 h; there are 68 h of labor available each day. The profit on each bureau is \$30 and on each bookcase \$10.

Check your answers with those at the back of the book.

Chapter Summary

1. For a *system of two linear equations* in two variables, the graphs of the equations either intersect in a single point or are coincident or parallel lines. Correspondingly, the system has a single solution, an infinite number of solutions, or no solution. A system that has at least one solution is said to be *consistent;* otherwise, it is *inconsistent.*

2. You can solve a system of linear equations in two variables by making transformations that yield equivalent systems, using the *linear-combination method* or the *substitution method.* Transformations that produce an equivalent system of linear equations are the following:
   1. Replace any equation of the system with an equivalent equation in the same variables.
   2. Replace any equation of the system with the sum of that equation and an equation obtained by multiplying both members of another equation of the system by a real number.
   3. In any equation substitute for one variable (a) its value, if known, or (b) an equivalent expression for that variable obtained from another equation of the system.

3. You can also use *determinants* to solve such a system by applying *Cramer's Rule.*
4. Systems of linear equations can be applied to solve problems by translating relationships into a system of equations.
5. The solution set of a system of linear inequalities in two variables is the intersection of open or closed half-planes representing the inequalities.
6. If a linear expression has a *maximum* or *minimum value* over a *feasibility region* which is the intersection of a finite number of closed half-planes and which has *corner points,* then that value occurs for the coordinates of some corner point.

Chapter Review

Tell whether the system of equations has: a. exactly one solution, b. an infinite set of solutions, or c. no solutions. *4-1*

1. $2x + 3y = 8$
   $4x - 6y = 9$

2. $3x + 5y = 10$
   $-6x - 10y = 20$

Solve by the linear-combination method. *4-2*

3. $x - y = 3$
   $4x + 2y = 18$

   a. $\{(1, 1)\}$ b. $\{(4, 1)\}$ c. $\{(1, 4)\}$ d. $\{(2, 2)\}$

Solve by the substitution method.

4. $y = 3x + 2$
   $2y - 3x = 7$

   a. $\{(2, 3)\}$ b. $\{(1, 5)\}$ c. $\{(6, 2)\}$ d. $\{(2, 6)\}$

Test Items 5–6 refer to the following system of equations. *4-3*

$$x - 2y = 3 \qquad 2x + 3y = -8$$

5. Find the determinant of coefficients, D, for the system.

   a. $\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$ b. $\begin{vmatrix} 3 & -2 \\ -8 & 3 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 3 \\ 2 & -8 \end{vmatrix}$

6. Solve the system using Cramer's Rule.

   a. $\{(2, 3)\}$ b. $\{(2, -1)\}$ c. $\{(-1, -2)\}$ d. $\{(-2, 2)\}$

A parking meter which accepts only nickels and dimes contains coins with a combined value of $1.35. Let *d* stand for the number of dimes and *n* the number of nickels in answering Test Items 7–8. *4-4*

7. Which equation expresses the combined value of the coins correctly?

   a. $5d + 5n = 135$ b. $10d + 5n = 135$ c. $10d + 5n = 1.35$

8. Given the additional information that there are three more nickels than dimes in the parking meter, find the number of dimes.
a. 3 b. 4 c. 8 d. 6

Graph the following system of inequalities. Use the graph to answer Test Items 9–10. 4-5

$$y \leq 2x \qquad y > -3x + 4$$

9. Which one of the following points does *not* lie on the graph of the system?
a. (3, 2) b. (4, −2) c. (2, −2) d. (5, 0)

10. Which one of the following points lies on the graph of the system?
a. (0, 4) b. (0, 0) c. (2, 4) d. (−1, −2)

In Test Items 11–12, use the following system of constraints. 4-6

$$x \geq 0 \qquad y \geq 0 \qquad y \leq 4 - 2x$$

11. Find the minimum value of $y - 2x$.
a. −2 b. 4 c. 0 d. −4

12. Find the maximum value of $2x + y$.
a. 8 b. 2 c. 4 d. 6

Chapter Test

Graph the given system of equations. Identify the apparent solution set. 4-1

1. $3x + 2y = 8$
$2x - y = 3$

2. $x - 3y = 6$
$-x + 3y = -3$

3. Solve by the linear-combination method: $4x + 2y = 0$, $3x + 4y = 5$ 4-2

4. Solve by the substitution method: $3x + y = 1$, $5x - 2y = 9$

5. Solve for k: $\begin{vmatrix} 4k & 3 \\ k & 2 \end{vmatrix} = -15$ 4-3

6. Use Cramer's Rule to solve: $2x - 3y = 7$, $-3x + 4y = -10$

7. A rectangle with a perimeter of 24 cm is twice as long as it is wide. Find its dimensions. 4-4

8. Find the values of A and B so that the line with equation $Ax + By = 3$ passes through (−2, −1) and (1, 5).

9. Graph the solution set of the following system. 4-5

$$0 \le x \le 4$$
$$6y + 4x \le 24$$

10. Find the minimum value of $3x - 2y$ subject to the following constraints. 4-6

$$0 \le x \le 4$$
$$0 \le y$$
$$4y \le 3x + 8$$

11. Find the maximum value of $2x + y$ subject to the following constraints.

$$1 \le x \le 4$$
$$-2 \le y \le -x + 2$$

Cumulative Review (*Chapters 1–4*)

In Review Items 1–5, simplify the given expression.

1. $(-\frac{1}{2})(8 - 10) - 4$

   a. -5 b. 3 c. -3 d. -18

2. $(-48) \div [\frac{1}{4}(-12)4]$

   a. 576 b. -576 c. -4 d. 4

3. $x - (y - 2x) - (3x - y)$

   a. 0 b. $-4x - 2y$ c. $-4x$ d. $-2y$

4. $\dfrac{|14 - 3| - |7 - 16|}{3|(-2) + 1|}$

   a. 0 b. $\frac{2}{3}$ c. $-\frac{20}{3}$ d. $\frac{20}{3}$

5. $-2[3a + 3a(b - a)] - a(-3a + 2b)$

   a. $-6a - 6ab$ b. $-9a^2 - 6a - 4ab$ c. $9a^2 - 6a - 8ab$ d. $15a^2 - 18ab$

In Review Items 6–10, solve over $\mathcal{R}$.

6. $2x - 1 = 13$

   a. $\{6\}$ b. $\{7\}$ c. $\{8\}$ d. $\{25\}$

7. $7a - (a + 11) = 7$

   a. $\{3\}$ b. $\{-\frac{4}{3}\}$ c. $\{-\frac{3}{4}\}$ d. $\{-1\}$

8. $\dfrac{3x}{2} - \dfrac{5x}{2} = 10$

   a. $\{-10\}$ b. $\{-5\}$ c. $\{10\}$ d. $\{5\}$

9. $8 - y \leq 2(2y - 1)$

a. $\{y: y \geq \frac{10}{3}\}$ b. $\{y: y \geq 2\}$ c. $\{y: y \leq -4\}$ d. $\{y: y \leq -2\}$

10. $|x - 3| < 7$

a. $\{x: -4 < x < 10\}$ b. $\{x: x < -4 \text{ or } x > 10\}$ c. $\{x: x < 10\}$

11. Each of the two congruent sides of an isosceles triangle is 2 cm longer than twice the length of the base. If the perimeter of the triangle is 29 cm, what is the length of the base?

a. 9 cm b. 12 cm c. 5 cm d. 8 cm

12. What is the slope of the line $2x - 3y = 18$?

a. $\frac{2}{3}$ b. $\frac{3}{2}$ c. -6 d. 2

13. Which is an equation of the line with slope -3 and y-intercept 5?

a. $3x - y = 5$ b. $3x + y = -5$ c. $3x + y = 5$ d. $x - 3y = 5$

14. What is the slope of the line containing the points $(-1, 4)$ and $(3, -8)$?

a. -3 b. $-\frac{1}{3}$ c. 3 d. $\frac{1}{3}$

15. If y varies directly as x and $y = 15$ when $x = 3$, what does y equal when $x = 15$?

a. 3 b. 75 c. 225 d. 45

In Review Items 16-17, solve the given system.

16. $2x + 3y = 7$
$x - 4y = -13$

a. $\{(2, 1)\}$ b. $\{(-5, 2)\}$ c. $\{(-1, 3)\}$ d. $\{(\frac{9}{5}, \frac{19}{5})\}$

17. $2y - 5x = 10$
$10x - 4y = 6$

a. $\{(5, 0)\}$ b. $\{(1, 1)\}$ c. $\{(0, -2)\}$ d. $\emptyset$

18. The sum of 2 numbers is 130. One number is 46 more than half the other number. What are the numbers?

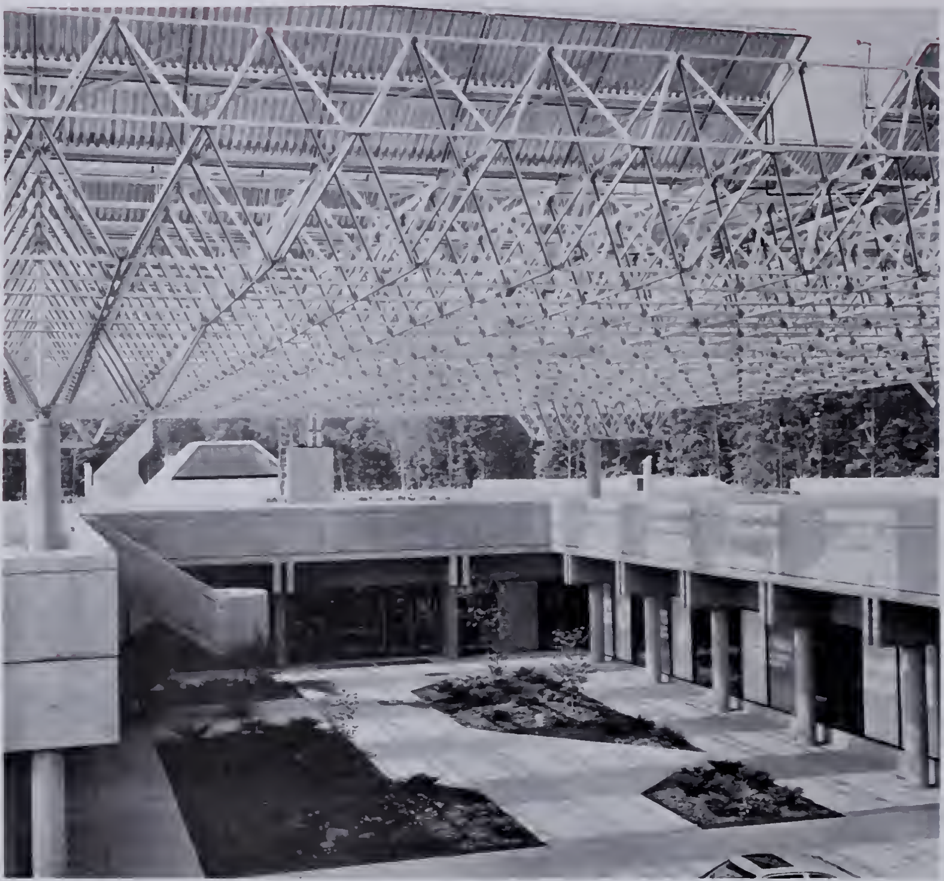
a. 56, 74 b. 52, 78 c. 58, 72 d. 60, 70

19. Which point does not lie on the graph of the system: $y < 3x$
$2 \leq y \leq 4$?

a. $(3, 3)$ b. $(5, 4)$ c. $(1, 4)$ d. $(2, 2)$

20. Find the minimum of the expression $-2x + y$ over the triangular region with vertices $(-1, -1)$, $(1, -3)$, and $(-3, -2)$.

a. -3 b. -5 c. -4 d. -6



Terraset School in Reston, Virginia, is an earth-covered building designed to save energy. The solar collectors shown provide energy for heating the school.

5 Graphs in Space; Determinants

Systems of Equations in Three Variables

***OBJECTIVES** for Sections 5-1 through 5-3:*

1. ***Draw graphs of ordered triples and linear equations in space.***
2. ***Determine the x-, y-, and z-intercepts of a given plane and the traces of the plane in the coordinate planes.***
3. ***Draw the space-graph of a given linear equation in three variables.***
4. ***Solve a system of three linear equations in three variables by transforming it into a simple equivalent system.***

5-1 Coordinates in Space

Just as a coordinate system in the two-dimensional plane establishes a one-to-one correspondence between the set of points in the plane and the set of ordered pairs of real numbers, so does a rectangular coordinate system in three-dimensional space establish a one-to-one correspondence between the set of points in space and the set of *ordered triples* of real numbers.

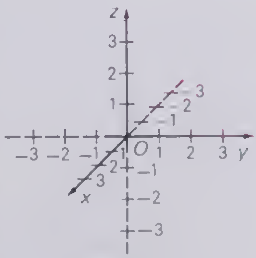


Figure 1

To set up a rectangular coordinate system in space, draw three mutually perpendicular number lines, or **axes**, passing through a common point O, the **origin**, of each. The **axes** are usually labeled x, y, and z, as in Figure 1, with an arrowhead on each to indicate the positive direction, and the same scale ordinarily is used on each.

In order to give spatial perspective to the figure, the angle between the positive x- and y-axes is drawn as a 135°-angle instead of a 90°-angle. When equal units are used on all three axes, the units of length on the y- and z-axes are drawn the same, while the unit on the x-axis is *drawn* as if it were two-thirds of the unit on each of the other axes; this "foreshortening" helps give the appearance of depth to the drawing. Also, the negative portion of each axis is often shown as a dashed line.

The coordinate axes determine three **coordinate planes** (Figure 2), each passing through the origin and each containing two of the axes:

1. the **xy-plane,** which contains the x- and y-axes and is perpendicular to the z-axis;
2. the **yz-plane,** which contains the y- and z-axes and is perpendicular to the x-axis;
3. the **xz-plane,** which contains the x- and z-axes and is perpendicular to the y-axis.

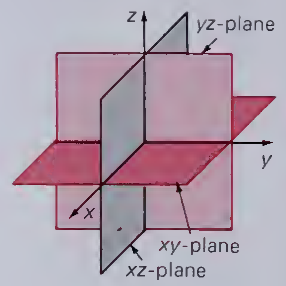


Figure 2

The coordinate planes separate space into eight regions, called **octants,** each determined by the positive part or the negative part of each of the three axes. Each octant is designated by a succession of three plus or minus signs, according as the octant is determined by the positive or negative part of the x-axis, the y-axis, and the z-axis. Thus the $(-, -, +)$-octant (read "the minus minus plus octant") is bounded in part by the negative part of the x-axis, the negative part of the y-axis, and the positive part of the z-axis. The $(+, +, +)$-octant is also called the **first octant;** the other octants are not numbered.

EXAMPLE In which octant does the given point lie?

a. $(3, -1, 2)$ b. $(-2, -4, -1)$ c. $(-1, 2, 2)$

SOLUTION a. $(+, -, +)$ b. $(-, -, -)$ c. $(-, +, +)$

To assign an ordered triple of numbers, or **coordinates,** to a point such as P in Figure 3, draw three planes through P, the first (represented by $ABCP$) perpendicular to the x-axis, the second ($EDCP$) perpendicular to the y-axis, and the third ($AFEP$) perpendicular to the z-axis. The numbers paired with the points in which these planes intersect the respective axes are, in order, the **x-coordinate,** the **y-coordinate,** and the **z-coordinate** of P. For example, in Figure 3, P has coordinates $(3, 4, 5)$.

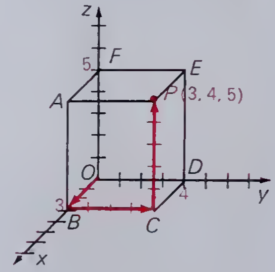


Figure 3

Together with the coordinate planes, the three planes drawn through P form a *rectangular parallelepiped,* or *box,* which we shall call the **coordinate box** of P.

Notice that, starting at the origin, you can arrive at P by moving on this box along edges parallel to each axis in succession. One such path is shown by red arrows in Figure 3: **O to B, B to C, C to P.** This suggests how to locate a point whose coordinates are given. For example, Figure 4 shows the plotting of the point $R(2, -4, -1)$:

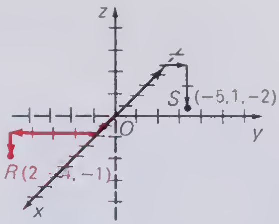


Figure 4

1. From O move 2 units in the positive x-direction along the x-axis.
2. Then move -4 units (4 units in the negative direction) parallel to the y-axis.
3. Then move -1 unit parallel to the z-axis.

The point $S(-5, 1, -2)$ has also been plotted in Figure 4.

For better visualization, it is helpful in drawing space figures to show "hidden edges" by dashed segments and visible edges by darkened segments, as illustrated in Figure 5 for the coordinate box of $P(2, -3, -5)$.

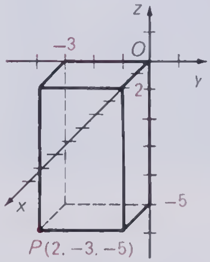
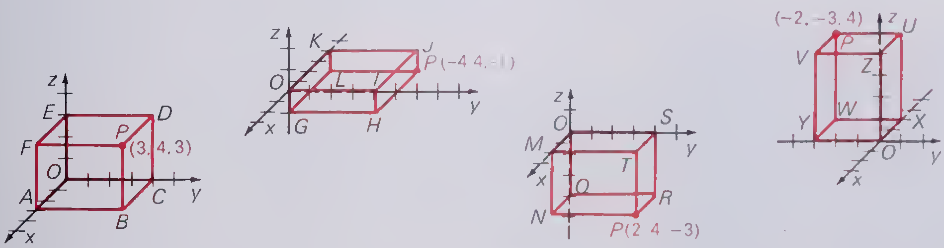


Figure 5

Oral Exercises

In Exercises 1–24 state the coordinates of the given vertex of a coordinate box shown below.



1. A 2. B 3. C 4. D 5. E 6. F
7. G 8. H 9. I 10. J 11. K 12. L
13. M 14. N 15. Q 16. R 17. S 18. T
19. U 20. V 21. W 22. X 23. Y 24. Z

In Exercises 25–32 name the coordinate plane that contains the two given points.

25. A and F
26. E and C
27. S and T
28. S and R
29. I and H
30. K and J
31. U and Z
32. W and Y

In which octant does the given point lie?

33. (5, −2, −1) 34. (−3, 2, −8) 35. (1, 1, −1)

Name the coordinate plane or axis containing all points of the given form.

36. $(0, a, b)$ 37. $(a, b, 0)$ 38. $(0, 0, a)$

In Exercises 39–42, name a fourth point that is in the same plane (*not* necessarily a coordinate plane) as the three given points in the coordinate boxes on page 141.

39. P, D, and C 40. M, N, and Q 41. J, K, and H 42. O, P, and Z

Written Exercises

A 1–4. Copy the diagrams shown in the Oral Exercises on page 141, but show the hidden edges as dashed segments.

Draw a diagram of a coordinate system in space. In the diagram show the coordinate box of the given point. Show the hidden edges as dashed segments and be sure to indicate units along the coordinate axes.

5. (2, 3, 5) 6. (−2, 4, 3) 7. (4, − 3, 6) 8. (−2, −5, 3)
9. (7, 5, −4) 10. (3, −3, 6) 11. (−2, −5, −3) 12. (−2, 4, −3)

In Exercises 13–20 draw a diagram of a coordinate system in space and locate the given point using a diagram like Figure 4, page 141. Give the coordinates of each point at which the arrow changes direction.

13. (3, −1, 4) 14. (2, 4, −1) 15. (−3, −5, 3) 16. (6, −2, −4)
17. (−4, 1, 5) 18. (−1, 7, −6) 19. (−3, −4, −4) 20. (2, 3, −5)

Sketch the triangle in space whose vertices have the given coordinates.

B 21. (3, 0, 0), (0, −4, 0), (0, 0, 5) 22. (0, 0, 4), (−6, 0, 0), (0, 2, 0)
23. (0, 0, 0), (3, 0, −3), (0, 5, −3) 24. (0, 0, 0), (−4, 0, 3), (−4, −2, 3)

On a coordinate system in space, graph each of the following sets of points. In each set connect the four points that all lie in one plane.

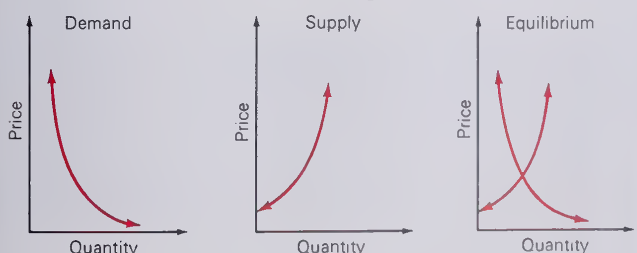
C 25. (0, 0, 3), (0, 0, 0), (0, 4, 0), (−5, 0, 0), (−5, 4, 0)
26. (0, −3, 0), (0, 4, 0), (−4, 0, 0), (0, −3, 5), (−4, 0, 5)

Careers

in Economics

Economists deal with a wide variety of problems in business, government, and international affairs. They study the production, distribution, and consumption of goods. They are concerned with the relationship between supply and demand, and its effect on prices. In government, economists are involved in decisions regarding budgets, taxes, inflation, unemployment, and wage and price guidelines. In international affairs, the issues of trade, tariffs, the balance of payments, currency valuation, and aid to developing nations require knowledge of economics.

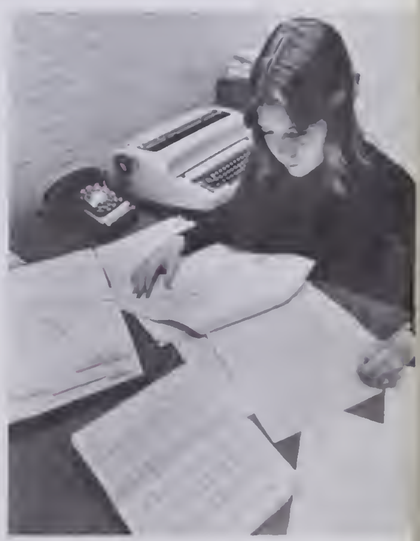
EXAMPLE One of the most basic principles in the study of economics is the relationship between supply and demand and its effect on prices. If the price of a certain commodity goes up, then there will be less demand for that item, or consumers will buy smaller quantities of that item. A graph of this relationship looks something like the one on the left. Now look at the producer's point of view.



If the price of an item increases, then the manufacturing of that item becomes more attractive and producers will supply greater quantities of that item. This is shown in the graph at the center.

You can see that these forces work to balance each other. As the price of an item becomes too high, the supply will become greater than the demand. Then the price will be lowered to attract more buyers for the product. If a price becomes too low, the demand for the item will exceed the supply and the producer can then begin to charge higher prices. The price for which supply and demand are equal is called the equilibrium price, which you can see from the graph at the right is at the point of intersection of the curves.

An economist presents a variety of information graphically (above). Using standard references, an economist analyzes new data (below).



5-2 Graphs of Linear Equations in Three Variables

An equation such as $3x + 2y + 4z = 6$ is called a *linear equation in three variables.* In general, any equation of the form

$$Ax + By + Cz = D,$$

where A, B, C, and D are real constants such that A, B, and C are *not* all 0, is a **linear equation** in the variables x, y, and z. In our work, *we shall assume that the replacement set of each variable is* $\mathcal{R}$.

An equation in one or two variables, such as

$$2y = 3 \qquad \text{or} \qquad 4x - 3z = 5,$$

can be regarded as an equation in three variables for which one or two of the coefficients of the three variables are zero:

$$0x + 2y + 0z = 3 \qquad \text{or} \qquad 4x + 0y - 3z = 5.$$

The ordered triple

$$(4,\ 5,\ -4)$$

is called a *solution* of the linear equation

$$3x + 2y + 4z = 6$$

because the assertion

$$3 \cdot 4 + 2 \cdot 5 + 4 \cdot (-4) = 6$$

is a true statement. In general, a **solution** of an open sentence in three variables is an ordered triple of values of the variables for which the open sentence is true. Such an ordered triple of numbers is said to **satisfy** the sentence. The set of *all* ordered triples of real numbers that are solutions of the open sentence is the **solution set** of the sentence over $\mathcal{R}$. To denote the solution set of the equation $3x + 2y + 4z = 6$ over $\mathcal{R}$, we shall use the notation

$$\{(x, y, z)\colon\ 3x + 2y + 4z = 6\}.$$

In Section 5-1 you saw that an ordered triple of real numbers gives the coordinates of a point in space. The set consisting of those points and only those points whose coordinates satisfy a given open sentence in three variables is the graph of the sentence. We accept the following without proof.

> **Theorem.** In space, the graph of a linear equation in three variables is a plane. Conversely, every plane is the graph of some linear equation in three variables, called an **equation of the plane.**

For example, $x = 0$ for all points in the yz-plane and for no other points. Accordingly, the linear equation

$$x = 0$$

is an equation of the yz-plane. Similarly,

$$y = 0 \qquad \text{and} \qquad z = 0$$

are equations of the xz-plane and the xy-plane, respectively.

Since three noncollinear points determine a plane, you can use the foregoing theorem to graph a linear equation in three variables by finding three noncollinear points whose coordinates satisfy the equation. When possible, it often is easiest to choose the points where the plane cuts the coordinate axes.

EXAMPLE 1 Sketch (part of) the graph of the equation

$$3x + y + 2z = 6.$$

SOLUTION

1. To find the coordinates of the point where the graph cuts the x-axis, replace y and z with 0 in the given equation and solve for x:

$$\begin{aligned} 3x + y + 2z &= 6 \\ 3x + 0 + 2 \cdot 0 &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

$\therefore$ the plane cuts the x-axis at the point $A(2, 0, 0)$.

2. Next, replace x and z with 0 in the given equation and solve for y.

$$\begin{aligned} 3x + y + 2z &= 6 \\ 3 \cdot 0 + y + 2 \cdot 0 &= 6 \\ y &= 6 \end{aligned}$$

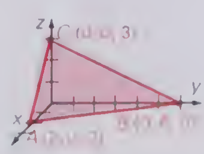
$\therefore$ the plane cuts the y-axis at the point $B(0, 6, 0)$.

3. Now replace x and y with 0 in the given equation and solve for z:

$$\begin{aligned} 3x + y + 2z &= 6 \\ 3 \cdot 0 + 0 + 2z &= 6 \\ 2z &= 6 \\ z &= 3 \end{aligned}$$

$\therefore$ the plane cuts the z-axis at the point $C(0, 0, 3)$.

4. Draw a sketch showing the three points where the plane cuts the three axes, draw the line segments connecting the three points by pairs, and shade the space triangle as shown. This triangle is part of the plane graph of the given equation. Answer.



The x-coordinate, **2**, of the point where the plane of Example 1 cuts the x-axis is called the *x-intercept* of the plane. Similarly, the *y-intercept* of the plane is **6** and the *z-intercept* is **3**. In general, if a plane intersects the x-axis in a *single point*, then the x-coordinate of that point is called the **x-intercept** of the plane. The **y-intercept** and **z-intercept** are defined similarly.

In the graph of Example 1, the points A and B lie in the xy-plane and also in the plane that is the graph of

$$3x + y + 2z = 6.$$

Therefore the line $\overleftrightarrow{AB}$ is the line of intersection of the xy-plane and the graph.

You call a line in which a plane intersects a coordinate plane the **trace** of the given plane in that coordinate plane. Thus, the trace of the graph of Example 1 in the xy-plane is $\overleftrightarrow{AB}$, in the yz-plane is $\overleftrightarrow{BC}$, and in the xz-plane is $\overleftrightarrow{AC}$.

Since $\overleftrightarrow{AB}$ lies in the xy-plane, the coordinates of its points must satisfy the equation $z = 0$. But $\overleftrightarrow{AB}$ also lies in the graph of $3x + y + 2z = 6$, so the coordinates of its points must satisfy this equation, too. Since there are no other points on both of these planes, you can conclude that $\overleftrightarrow{AB}$ is the solution set of the system of equations:

$$\begin{aligned} 3x + y + 2z &= 6 \\ z &= 0 \end{aligned}$$

If you replace z with "0" in the first equation of the system, you obtain the equivalent system

$$\begin{aligned} 3x + y &= 6 \\ z &= 0 \end{aligned}$$

whose solution set is $\overleftrightarrow{AB}$.

Similarly, the trace $\overleftrightarrow{BC}$ in the yz-plane is the solution set of the system

$$\begin{aligned} y + 2z &= 6 \\ x &= 0 \end{aligned}$$

and, the trace $\overleftrightarrow{AC}$ in the xz-plane is the solution set of the system

$$\begin{aligned} 3x + 2z &= 6 \\ y &= 0. \end{aligned}$$

EXAMPLE 2 a. Find the x-, y-, and z-intercepts of the graph of $4x + 3z = 12$.

b. Write a linear system of equations whose solution set is the trace of the given graph in each coordinate plane.

c. Sketch the traces, and in your diagram shade part of the given graph.

SOLUTION a. $4x + 3 \cdot 0 = 12$; $x = 3$. The x-intercept is 3. Answer.

$4 \cdot 0 + 3 \cdot 0 = 12$; $0 = 12$. This equation is *false* for every value of y.
$\therefore$ there is *no* y-intercept, and the graph is parallel to the y-axis. Answer.

$4 \cdot 0 + 3z = 12$; $z = 4$. The z-intercept is 4. Answer.

b. 1. Trace in the xy-plane ($\overleftrightarrow{AB}$ in the diagram):

$$z = 0$$
$$x = 3$$

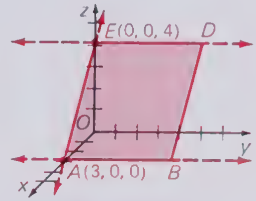
2. Trace in the yz-plane ($\overleftrightarrow{ED}$ in the diagram):

$$x = 0$$
$$z = 4$$

3. Trace in the xz-plane ($\overleftrightarrow{AE}$ in the diagram):

$$y = 0$$
$$4x + 3z = 12$$

c. The space-graph of $4x + 3z = 12$ and its traces are shown at the right.



EXAMPLE 3 a. Explain why the space-graph of $4x + 3z = 0$ contains the y-axis.

b. Sketch part of the graph.

SOLUTION a. For every point on the y-axis, you have $x = 0$ and $z = 0$. Substituting 0 for x and for z in the given equation, you obtain the statement

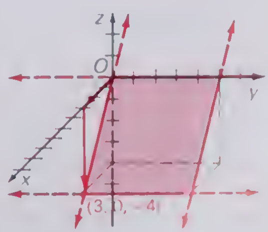
$$4 \cdot 0 + 3 \cdot 0 = 0, \qquad \text{or} \qquad 0 = 0,$$

which is true for every value of y. Thus, the coordinates of every point on the y-axis satisfy the equation $4x + 3z = 0$, so that the graph of the equation contains the y-axis.

b. Since the graph contains the y-axis, you need to know the coordinates of only one point not on that axis in order to sketch part of the graph. Substituting 3 for x and letting y have the value 0, you obtain

$$4 \cdot 3 + 3z = 0, \qquad \text{or} \qquad z = -4.$$

Therefore, the point with coordinates $(3, 0, -4)$ is on the graph. Sketch a part of the plane containing the y-axis and the point with coordinates $(3, 0, -4)$, as shown in red. Answer.



Examples 2 and 3 on pages 146–147 illustrate the following facts:

> If the coefficient of a variable in an equation of a plane is zero, then:
>
> 1. The plane is parallel to the axis of that variable if the constant term is not zero;
> 2. The plane contains the axis of that variable if the constant term is zero.

You can apply the foregoing facts in particular to graph a linear equation in three variables when the coefficients of *two* of the variables are zero. For example, in $y = 2.5$ both the coefficient of x and the coefficient of z are zero, but the constant term is not zero. Therefore, the graph is parallel to the x-axis and also to the z-axis; that is, the graph is parallel to the xz-plane. You can now sketch the graph (Figure 6) of the equation when you note that the ordered triple (0, 2.5, 0) satisfies the equation.

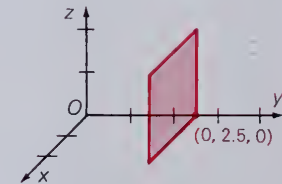
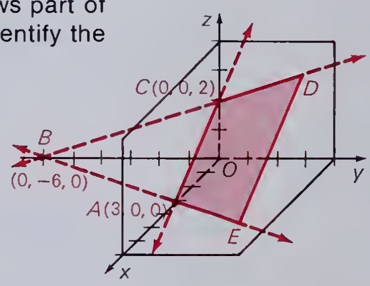


Figure 6

Oral Exercises

Exercises 1–8 refer to the diagram, which shows part of the graph of the equation $2x - y + 3z = 6$. Identify the requested part of the graph.



1. The x-intercept.
2. The y-intercept.
3. The z-intercept.
4. The trace in the xy-plane.
5. The trace in the yz-plane.
6. The trace in the xz-plane.
7. The part of the graph in the $(+, -, +)$-octant.
8. The part of the graph in the $(+, +, +)$-octant.

In Exercises 9-26:

a. Name the x-, y-, and z-intercepts.

b. Tell whether or not the graph is parallel to or contains any of the coordinate axes, and if so which one(s).

c. Tell whether or not the graph is parallel to or coincides with one of the coordinate planes, and if so which one.

9. $2x + 2y + 3z = 12$	10. $2x + y = 8$	11. $-6x + 3y + 2z = 18$
12. $x = 3$	13. $2y - 5z = 0$	14. $x - 4y - 2z = 8$

15. $3x + 5z = 15$
16. $-8x + 3y - 4z = 24$
17. $z = 0$
18. $4x - 4y + 10z = 20$
19. $4x - 2y = 0$
20. $y = -4$
21. $4x + 4y - 3z = -12$
22. $5x - 2y + 4z = -20$
23. $2x = -14$
24. $2y - 5z = -10$
25. $-5x - 6y - 10z = 30$
26. $7y = 0$

For each of the following equations of a plane state a system of equations whose solution set is the trace of the plane in the given coordinate plane.

27. $x - 8y - 2z = 8$; xy-plane
28. $2x + 5y = 10$; xz-plane
29. $4y + 3z = 0$; yz-plane
30. $7x + 7y - 2z = 14$; xy-plane
31. $2x - y + 6z = 12$; xz-plane
32. $3x - z = 9$; yz-plane
33. Give a necessary and sufficient condition on a linear equation $ax + by + cz = d$ in three variables for the graph of that equation to be a plane through the origin of the coordinate system.

Written Exercises

A 1–18. For each of the equations in Oral Exercises 9–26:
   a. Sketch part of the graph of the equation.
   b. Give three systems of equations whose solution sets are the traces of the graph in each of the coordinate planes. If the equation has no trace in a coordinate plane, so state.

Graph part of the plane that has the given traces and that satisfies any additional condition given.

B 19. xy-trace: $6x + 5y = 30$, $z = 0$;
yz-trace: $y + 3z = 6$, $x = 0$;
xz-trace: $2x + 5z = 10$, $z = 0$.

20. xz-trace: $3x + 4z = 12$, $y = 0$;
parallel to the y-axis.

21. xy-trace: $4x - 5y = 20$, $z = 0$;
yz-trace: $y - 2z = -4$, $x = 0$;
xz-trace: $2x + 5z = 10$, $y = 0$.

22. xy-trace: $x + 3y = 0$, $z = 0$;
contains the z-axis.

23. xz-trace: $z = 6$, $y = 0$;
parallel to the xy-plane.

24. xy-trace: $x + 2y = 4$, $z = 0$;
yz-trace: $7y - 2z = 14$, $x = 0$;
xz-trace: $7x - 4z = 28$, $y = 0$.

25. xy-trace: $2x + 3y = -12$, $z = 0$;
yz-trace: $9y - 4z = -36$, $x = 0$;
xz-trace: $3x - 2z = -18$, $y = 0$.

26. xy-trace: $y = 2$, $z = 0$;
yz-trace: $5y + 2z = 10$, $x = 0$;
xz-trace: $z = 5$, $y = 0$.

C 27. Write a system of equations for the xy-trace of a plane with
yz-trace: $5y + 2z = 10$, $x = 0$;
xz-trace: $5x + 4z = 20$, $y = 0$.

28. Write a system of equations for the xz-trace of a plane with
xy-trace: $-3x + 7y = 21$, $z = 0$;
yz-trace: $y = 3$, $x = 0$.

On one set of coordinate axes sketch parts of the planes defined by the following pairs of equations. Give a system of equations whose solution set is the line of intersection of the pair of planes.

29. $5x + 3y + 15z = 15;\ -5x + 4y + 20z = 20$

30. $2x + z = 4;\ 4x - y + 2z = 8$

5-3 Systems of Linear Equations in Three Variables

A **solution** of a system of linear equations in three variables over $\mathcal{R}$ is an ordered triple of real numbers that satisfies all equations of the system. The **solution set** of the system is the set of all its solutions.

The geometric interpretation of the solution of a system of two equations in two variables (Section 4-1) can be extended to systems of three equations in three variables.

A system such as

$$\begin{aligned} x &= 5 \\ y &= 4 \\ z &= 6 \end{aligned}$$

can readily be solved, because you can see by inspection that its one and only solution is (5, 4, 6). Thus, the solution set is $\{(5, 4, 6)\}$. Figure 7 pictures this fact by showing $P(5, 4, 6)$ as the single point on the graphs of all three of the equations.

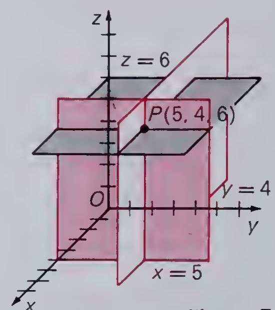


Figure 7

As Figure 7 suggests, two nonparallel planes intersect in a line, and three such planes *can* intersect in a single point. In that case, a system of three linear equations in three variables represented by the three planes is *consistent* and the solution is unique.

But such a system is also consistent if the graphs of the equations in such a system consist of:

1. Three different planes intersecting in a single line (Figure 8).
2. Two coincident planes intersecting a third plane in a line (Figure 9).
3. Three coincident planes (Figure 10).

In each of these cases the system has an infinite set of solutions.

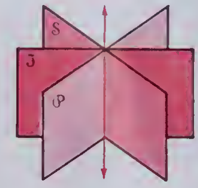


Figure 8

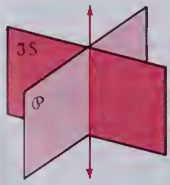


Figure 9

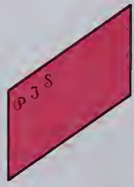


Figure 10

The solution set is the empty set $\emptyset$, and the system is *inconsistent,* if the graphs of the equations consist of:

1. Three parallel planes (Figure 11).
2. Two coincident planes parallel to a third plane (Figure 12).
3. Three planes intersecting in three parallel lines (Figure 13).
4. Two parallel planes intersecting a third plane in two parallel lines (Figure 14).

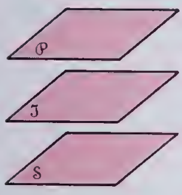


Figure 11

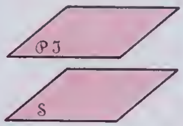


Figure 12

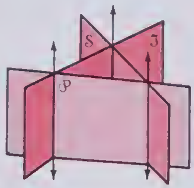


Figure 13

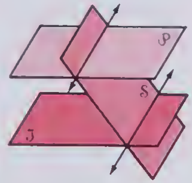


Figure 14

Because it often is difficult to obtain accurate information about coordinates of points of intersection of space figures from flat drawings, you should not ordinarily attempt to solve systems of linear equations in three variables by graphing the equations. Instead, to solve such a system you might transform it into an equivalent system which can be solved by inspection.

The transformations used in solving systems of linear equations in two variables (page 112) are also applicable in solving three-variable systems. In applying transformations, you might use the linear-combination method, the substitution method, or a combination of these. If the transformations yield a *false* statement such as $0 = 2$, then the system is inconsistent. If they do not yield a false statement but do yield a true statement such as $0 = 0$, then the solution set is an infinite set.

EXAMPLE Determine the solution set of the system:

$$\begin{aligned} x + 2y + z &= 5 \quad (1) \\ 2x - y + z &= 4 \quad (2) \\ 3x + y + 4z &= 1 \quad (3) \end{aligned}$$

SOLUTION 1. To obtain an equation in which the coefficient of y is 0, multiply each member of the second equation by 2 and add the resulting equation to the first equation.

$$\begin{aligned} x + 2y + z &= 5 \quad (1) \\ 4x - 2y + 2z &= 8 \quad (2) \times 2 \\ \hline 5x \qquad + 3z &= 13 \quad (4) \end{aligned}$$

(Solution continued on page 152.)

2. To obtain a second equation in which the coefficient of y is 0, add the third equation to the second.

$$\begin{array}{rl} 2x - y + z = 4 & (2) \\ 3x + y + 4z = 1 & (3) \\ \hline 5x + 5z = 5 & (5) \end{array}$$

3. The equations obtained in Steps 1 and 2 involve only x and z. You can use the method of transforming two equations in two variables to replace that pair of equations by the equivalent pair shown at the right.

$$\begin{array}{ll} 5x + 3z = 13 & (4) \\ 5x + 5z = 5 & (5) \end{array} \quad \text{is equivalent to} \quad \begin{array}{l} x = 5 \\ z = -4 \end{array}$$

4. Replace x with 5 and z with -4 in the second of the given equations, and solve for y.

$$\begin{aligned} 2x - y + z &= 4 \quad (2) \\ 2 \cdot 5 - y + (-4) &= 4 \\ 6 - y &= 4 \\ y &= 2 \end{aligned}$$

5. Thus you find that the given system is equivalent to the system

$$\begin{aligned} x &= 5 \\ y &= 2 \\ z &= -4 \end{aligned}$$

whose solution set is $\{(5, 2, -4)\}$.

Checking in the given system, you have:

$$\begin{aligned} 5 + 2 \cdot 2 + (-4) &= 5 \\ 2 \cdot 5 - 2 + (-4) &= 4 \\ 3 \cdot 5 + 2 + 4(-4) &= 1 \end{aligned}$$

$\therefore$ solution set is $\{(5, 2, -4)\}$. **Answer.**

Oral Exercises

In Exercises 1–4, describe how you would use the linear-combination method to obtain from the given system two new equations in which the coefficient of the given variable is 0.

1. x: $\begin{aligned} x + y + 2z &= 1 \\ -x - y + 3z &= 4 \\ 2x - 3y - z &= 12 \end{aligned}$

2. z: $\begin{aligned} 2x + 5y + 3z &= -6 \\ x - y + z &= 2 \\ -4x + 3y - 6z &= -1 \end{aligned}$

3. y: $\begin{aligned} 4x - 3y + z &= -9 \\ x + 6y - z &= 5 \\ 2x - 2y + z &= -3 \end{aligned}$

4. x: $\begin{aligned} 3x - y - z &= 4 \\ 5x + 2y - 4z &= 1 \\ 6x - 3y - 2z &= 7 \end{aligned}$

In Exercises 5–8, describe how you would use the substitution method to obtain from the given system two new equations in which the coefficient of the given variable is 0.

5. y: $\begin{aligned} 3x - y + z &= 3 \\ x + 5y &= 6 \\ 4x - z &= 5 \end{aligned}$

6. x: $\begin{aligned} 3x - 2y + z &= 9 \\ x + 4y &= -2 \\ -y + 4z &= 5 \end{aligned}$

7. z: $\begin{aligned} x + y + z &= 5 \\ x - y + 2z &= 2 \\ x + y &= 6 \end{aligned}$

8. y: $\begin{aligned} 3x - 2y + z &= 4 \\ 2x + y - z &= 3 \\ x - 2z &= 7 \end{aligned}$

In Exercises 9–12:

a. State whether the given system of equations in x, y, and z is consistent or inconsistent.

b. If it is consistent, state whether the solution is unique or there is an infinite set of solutions. Give reasons for your statements.

c. Describe the relationships between the graphs of the equations using Figures 7–14 on pages 150 and 151 as models.

9. $\begin{aligned} x + y + z &= 2 \\ x + y + z &= 3 \\ 2x + 2y + 2z &= 9 \end{aligned}$

10. $\begin{aligned} x + y &= 0 \\ x - y &= 0 \\ z &= 2 \end{aligned}$

11. $\begin{aligned} x + y &= 0 \\ x - y &= 0 \\ y &= 0 \end{aligned}$

12. $\begin{aligned} x + y &= 4 \\ x + y &= 2 \\ x &= 0 \end{aligned}$

13. Given a system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

what relations must hold between the equations in order for the graphs of the system to: (a) resemble Figure 10 on page 150? (b) resemble Figure 11 on page 151?

Written Exercises

A **1–7.** Solve the systems in Oral Exercises 1–7.

In Exercises 8–16 the given system has a single solution. Find it.

8. $\begin{aligned} 5x - y + z &= 5 \\ 3x + y - z &= 3 \\ x + 2y - z &= 3 \end{aligned}$

9. $\begin{aligned} 4x - y + z &= 7 \\ x - 2y - 3z &= 0 \\ x + z &= 6 \end{aligned}$

10. $\begin{aligned} 4x + y - z &= -2 \\ x + 3y - 4z &= 1 \\ 2x - y + 3z &= 4 \end{aligned}$

11. $\begin{aligned} -x + y - 3z &= 2 \\ 2x + y + z &= 0 \\ 5x - 3y + 5z &= 6 \end{aligned}$

12. $\begin{aligned} -x + z &= 3 \\ y + z &= 1 \\ 2x + 5y - 3z &= -1 \end{aligned}$

13. $\begin{aligned} 2x + y + z &= 1 \\ -3x + y + 2z &= 9 \\ y + z &= 3 \end{aligned}$

14. $\begin{aligned} 5a - 5b + 2c &= 13 \\ 2a - 4b + 3c &= 8 \\ 3a + 2b - 4c &= 2 \end{aligned}$

15. $\begin{aligned} -4x - 3y &= 9 \\ 2x + 2y + 7z &= 15 \\ 4y + 5z &= 15 \end{aligned}$

16. $\begin{aligned} 3x + 2y &= 1 \\ 6y + 5z &= 4 \\ -9x + 4y - 10z &= -24 \end{aligned}$

In Exercises 17–19 the given system has a single solution. Find it.

B **17.** $\frac{1}{4}x - \frac{2}{3}y + z = -5$
$2x - z = 17$
$x + \frac{1}{3}y + 2z = 9$

18. $\frac{1}{4}x + \frac{1}{2}y + 3z = 2$
$\frac{3}{4}x - \frac{3}{2}y - z = 0$
$\frac{1}{2}x + y - 2z = 4$

19. $x - \frac{4}{3}y - \frac{1}{3}z = 1$
$y + z = 6$
$-2x - \frac{5}{3}y = 5$

In Exercises 20–22, tell whether the given system has a single solution, no solution, or an infinite solution set. If the system has a single solution, find it.

20. $x - 2y + z = 4$
$y - z = 0$
$-2x + 4y - 2z = 8$

21. $x - y + 2z = 2$
$x + 2y - z = 1$
$2x + y + z = 4$

22. $x + z = 2$
$y + z = 0$
$x + y = 2$

C **23.** Write an equation for a plane, parallel to the xy-plane, that contains point (a, b, c) where $c \neq 0$.

24. Write an equation for a plane, perpendicular to the y-axis, that contains point (a, b, c).

Self-Test 1

VOCABULARY

coordinate plane (p. 140)
z-coordinate (p. 140)
coordinate box of a point (p. 140)
x-, y-, z-intercept of a plane (p. 146)
trace of a plane in a coordinate plane (p. 146)

1. Draw a diagram of a coordinate system in space and in the diagram show the coordinate box of the point $(-2, 5, 4)$. Show the hidden edges as dashed segments and give the coordinates of all vertices of the box. *Obj. 1, p. 139*
2. Determine the x-, y-, and z-intercepts of the plane with equation $3x - 2y - 6z = -12$, and graph part of the plane. *Obj. 2, p. 139*
3. Give three systems of equations whose solution sets are the traces in the three coordinate planes of the plane defined by $4x - 6y + 2z = 20$.
4. Draw the part of the plane $3x + 5y + 15z = 15$ that lies in the first octant. *Obj. 3, p. 139*
5. Solve the system: *Obj. 4, p. 139*
$$\begin{aligned} x - 2y - z &= 2 \\ 2x + y + 3z &= 4 \\ -x + 4y - 2z &= 3 \end{aligned}$$

Check your answers with those at the back of the book.

ON THE CALCULATOR

One of the advantages of the calculator is that you can use it to evaluate complicated algebraic expressions quickly. The memory functions of your calculator can be useful here. Remember to clear the memory [MC] before you begin each exercise.

I Don't Care!

EXAMPLE Evaluate $b = 12a - 23$ when $a = \frac{2}{3}(87 - 54)$.

SOLUTION Evaluate a: 87 [−] 54 [=] [×] 2 [÷] 3 [=] [M+]

To find b, use these steps

12 [×] [MR] [−] 23 [=] 241. Answer.

Exercises

Evaluate the expression for the given value of a.

1. $a^2 - 12$; $a = 0.05(276 - 54)$
2. $\frac{1}{a}(17142 + 4583)$; $a = 13.2^2 - \frac{11}{25}$
3. $\frac{a}{2} - 518 \cdot 5 \cdot 4$; $a = (43.6 - 21.8)^3$
4. $\frac{1}{a} + \frac{1}{a^2}$; $a = 3 + 10 - 2 \cdot 21 \div 7$
5. $5^3 - 30a$; $a = \frac{3}{5}(21 + 9)^2$
6. πa^2; $a = \frac{1}{2}(17.4 + 3.2)$

Determinants

***OBJECTIVES** for Sections 5-4 through 5-6:*

1. *Use determinants to solve a system of three linear equations in three variables.*
2. *Use three variables to solve problems.*
3. *Use properties of determinants to simplify the expansion of a determinant by minors.*

5-4 Third-order Determinants

The determinants introduced in Section 4-3, such as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, have two (horizontal) rows, $a_1 \quad b_1$ and $a_2 \quad b_2$, and two (vertical) columns, $\begin{matrix} a_1 \\ a_2 \end{matrix}$ and $\begin{matrix} b_1 \\ b_2 \end{matrix}$; such a determinant is called a second-order determinant, or a determinant of order 2.

You can use *third-order determinants* in the solution of three linear equations in three variables over $\mathcal{R}$.

> For any $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 \in \mathcal{R}$, the **determinant**
>
> $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
>
> has the value
>
> $$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

A convenient way to remember how to evaluate a *third-order* determinant is to copy the 3×3 (read "three by three") array and repeat the first two columns after the third column as shown at the right. Compute the product of the entries along each diagonal arrow as shown below. Add the products found from the descending arrows to the negatives of the products found from the ascending arrows. (This does not work for higher-order determinants.)

$$\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

$$\begin{array}{ccccccc} a_1 & b_1 & c_1 & a_1 & b_1 & & \\ a_2 & b_2 & c_2 & a_2 & b_2 & & \\ a_3 & b_3 & c_3 & a_3 & b_3 & & \\ & & & + & + & + & \end{array} \qquad \begin{array}{ccccccc} & & & - & - & - \\ a_1 & b_1 & c_1 & a_1 & b_1 & \\ a_2 & b_2 & c_2 & a_2 & b_2 & \\ a_3 & b_3 & c_3 & a_3 & b_3 & \end{array}$$

$$\begin{aligned} &a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1 \\ &\qquad = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{aligned}$$

EXAMPLE 1 Evaluate the determinant:
$$D = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & 0 \\ -1 & 3 & -4 \end{vmatrix}$$

SOLUTION

$$\begin{array}{cccccccc} & & & - & - & - \\ 3 & 1 & 2 & 3 & 1 & \\ 4 & 3 & 0 & 4 & 3 & \\ -1 & 3 & -4 & -1 & 3 & \\ & & & + & + & + \end{array}$$

$D = -36 + 0 + 24 - (-6) - 0 - (-16) = 10.$ Answer.

If you use transformations, as in Section 5-3, to solve the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

over $\mathcal{R}$, and let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

you find that, if $D \neq 0$, the system has the unique solution

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D}.$$

These equations are called **Cramer's Rule** for systems of three linear equations in three variables. The determinant D is the **determinant of coefficients.** If $D = 0$, then either the system is inconsistent or the system has an infinite solution set.

EXAMPLE 2 Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x + y - z &= 3 \\ 4x - y + 4z &= 0 \\ -3y + 2z &= 6 \end{aligned}$$

SOLUTION

$$D = \left|\begin{matrix} 2 & 1 & -1 \\ 4 & -1 & 4 \\ 0 & -3 & 2 \end{matrix}\right|\begin{matrix} 2 & 1 \\ 4 & -1 \\ 0 & -3 \end{matrix} = -4 + 0 + 12 - 0 - (-24) - 8 = 24$$

$$D_x = \left|\begin{matrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 6 & -3 & 2 \end{matrix}\right|\begin{matrix} 3 & 1 \\ 0 & -1 \\ 6 & -3 \end{matrix} = -6 + 24 + 0 - 6 - (-36) - 0 = 48$$

$$D_y = \left|\begin{matrix} 2 & 3 & -1 \\ 4 & 0 & 4 \\ 0 & 6 & 2 \end{matrix}\right|\begin{matrix} 2 & 3 \\ 4 & 0 \\ 0 & 6 \end{matrix} = 0 + 0 - 24 - 0 - 48 - 24 = -96$$

$$D_z = \left|\begin{matrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ 0 & -3 & 6 \end{matrix}\right|\begin{matrix} 2 & 1 \\ 4 & -1 \\ 0 & -3 \end{matrix} = -12 + 0 + (-36) - 0 - 0 - 24 = -72$$

Thus, you have

$$x = \frac{D_x}{D} = \frac{48}{24} = 2,\ y = \frac{D_y}{D} = \frac{-96}{24} = -4,\ z = \frac{D_z}{D} = \frac{-72}{24} = -3$$

$\therefore$ (as you can check) the solution set is $\{(2, -4, -3)\}$. Answer.

Written Exercises

Evaluate the given determinant.

A 1. $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 5 & 3 \end{vmatrix}$ 2. $\begin{vmatrix} -2 & 1 & 0 \\ 1 & 3 & 4 \\ -1 & 2 & 5 \end{vmatrix}$ 3. $\begin{vmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \\ 2 & 0 & -2 \end{vmatrix}$ 4. $\begin{vmatrix} 4 & 0 & 5 \\ -3 & 5 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

Use Cramer's Rule to find the solution set of the given system.

5. $\begin{aligned} 2x + y - 2z &= 3 \\ x - y + z &= 0 \\ x - 2y + z &= -1 \end{aligned}$

6. $\begin{aligned} x - y - 3z &= 1 \\ 2x + 2y + z &= 3 \\ x - y - 4z &= 6 \end{aligned}$

7. $\begin{aligned} x - 3y + z &= 5 \\ y - z &= 2 \\ x - y &= 6 \end{aligned}$

8. $\begin{aligned} 3x + y + 2z &= 4 \\ -x + 2y + z &= 3 \\ y + z &= 1 \end{aligned}$

9. $\begin{aligned} x + 2y &= 10 \\ -x + z &= 7 \\ y - z &= 1 \end{aligned}$

10. $\begin{aligned} x + 2y &= 1 \\ 3y + 2z &= -1 \\ 2x - 5z &= -6 \end{aligned}$

In Exercises 11–13, evaluate D. If $D = 0$, state whether the system is inconsistent or the system has an infinite solution set. If the system has a single solution, find it.

B 11. $\begin{aligned} x + y - z &= 2 \\ 6x + y + z &= 4 \\ 4x - y + 3z &= 0 \end{aligned}$

12. $\begin{aligned} 2x - y + 4z &= 2 \\ -x + 6y - 9z &= 0 \\ 3x + 4y - z &= 1 \end{aligned}$

13. $\begin{aligned} 2x - 3y + z &= 3 \\ 10x + y - z &= 4 \\ 4x + 2y - z &= -5 \end{aligned}$

C 14. Prove that for all real numbers a, b, c, d, e, f, and k,

$$\begin{vmatrix} a & b & c \\ ka & kb & kc \\ d & e & f \end{vmatrix} = 0.$$

5-5 Solving Problems With Three Variables

Systems of linear equations in three variables sometimes appear in the solutions of practical problems.

EXAMPLE The sum of the length, width, and height of a rectangular box is 16 cm, the width is twice the height, and twice the length exceeds the sum of the width and height by 5. Find the length, width, and height of the box.

SOLUTION 1. The problem asks for the length, width, and height of the box.

2. Let the length be l cm, the width w cm, and the height h cm.

3. The sum of the length, width, and height is 16 cm.

$$l + w + h = 16. \qquad (1)$$

The width is twice the height: $w = 2h$. (2)

Twice the length is 5 more than the sum of the width and height.

$$2l = w + h + 5. \qquad (3)$$

4. The system of equations (*1*), (*2*), and (*3*) is equivalent to:

$$\begin{aligned} l + w + \ \ h &= 16 \\ w - 2h &= 0 \\ 2l - w - \ \ h &= 5 \end{aligned}$$

Transforming this system into the equivalent system

$$\begin{aligned} l &= 7 \\ w &= 6 \\ h &= 3 \end{aligned}$$

or solving the system by Cramer's Rule is left to you.

5. The check to show that 7 cm, 6 cm, and 3 cm satisfy the requirements for the length, width, and height of the box as described in the problem is also left to you.

$\therefore$ the length is 7 cm, the width is 6 cm, and the height is 3 cm. Answer.

Oral Exercises

In Oral Exercises 1–6, let *n* represent the number of nickels, *d* the number of dimes, and *q* the number of quarters collected on the Walnut Street bus. Translate each sentence into an equation in *n*, *d*, and *q*.

n + d + q = 105
3n - 2d + 0q = 0
5n + 10d + 25q = 2200

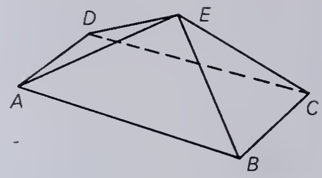
1. The total number of coins is 105. n + d + q = 105
2. Three times the number of nickels is the same as twice the number of dimes. 3n = 2d
3. The value of the coins collected is \$22. 5n + 10d + 25q = 2200
4. There are 95 coins altogether.
5. If the number of nickels and the number of dimes are subtracted from the number of quarters, the difference is 25.
6. The value of the coins collected is \$17.

Problems

A
1. Use the relationships stated in Oral Exercises 1–3 to find the number of nickels, dimes, and quarters deposited in the bus's coin box during its first inbound trip.
2. Use the relationships stated in Oral Exercises 4–6 to find the number of nickels, dimes, and quarters deposited in the bus's coin box during its first outbound trip.

3. The sum of three numbers is 42. The third number is twice the sum of the first two and is 4 more than three times their difference. What are the numbers?

4. A hipped roof is in the shape of a pyramid with a rectangular base. It has edges AE, BE, CE, and DE all of equal length. If the perimeter of the base is 36 m, the perimeter of $\triangle AEB$ is 30 m, and the perimeter of $\triangle BEC$ is 24 m, find AB, BC, and AE.



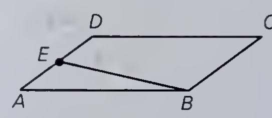
5. The proceeds from the school benefit car wash, totaling \$355, were all in one dollar, five dollar and ten dollar bills. If there were 120 bills altogether, and twice as many one dollar bills as five and ten dollar bills combined, how many bills of each denomination were there?

6. Jennifer Hartmann is examining some gold, silver, and bronze coins to determine the mass of each type of coin. She has, however, only 10 g pieces for her balance. She finds that 10 g will balance with one coin of each type together, 20 g will balance with two gold and three silver coins together, and 30 g will balance with four silver and two bronze coins together. What is the amount of metal in each type of coin?

7. Jim, Bob, and Hal run 1000 m, 3000 m, and 5000 m, respectively, in a medley. Their combined time is 29 min. Bob's time is 1 min more than three times Jim's time. The sum of Bob's time and twice Jim's time is equal to Hal's time. How much time did it take each runner to run his leg of the race?

8. After two tests, Maria's average in math was 78. After three tests, it was 82. If the teacher ignored Maria's lowest test score, her average would be 86. What were Maria's test scores?

B 9. The perimeter of parallelogram $ABCD$ is 66. E is the midpoint of side AD. If the perimeter of triangle ABE is 36, and the perimeter of trapezoid $BCDE$ is 54, find AB, BC, and BE.



10. Three kinds of tickets were sold for a school student-parent dinner: a \$1.00 ticket for one adult, a \$1.50 ticket for one parent and one student, and a \$2.00 ticket for two parents and one student. If 62 adults and 32 students attended the dinner and \$69 was collected, how many of each kind of ticket were sold?

11. A sporting goods store sells three types of sweat shirts: a pullover for \$6, a hooded sweat shirt for \$7, and a zipper-fronted sweater shirt for \$8. During a week in which the store sold 137 sweat shirts, the sales of hooded sweat shirts exceeded those of pullovers by \$40, and the sales of zipper-fronted sweat shirts exceeded those of hooded sweat shirts by \$30. How many of each type of sweat shirt were sold that week?

C 12. A train's route from city A to city D includes stops at cities B and C. The table gives the time (in hours) the trip will take if the given rates of speed (in kilometers per hour) are maintained between stops. What are the distances between the cities?

A to B	B to C	C to D	Total time
90	60	60	3
60	80	40	4
40	80	50	4

13. The expression $\frac{1}{2}at^2 + v_0t + c_0$ gives the distance an object is from an observer at time t, where a is the acceleration of the object in meters per second squared, v_0 is the initial velocity of the object in meters per second, and c_0 is the initial distance of the observer from the object. If an object is 20 m from the observer after 1 s, 55 m from the observer after 2 s, and 110 m from the observer after 3 s, find its acceleration, initial velocity, and initial distance from the observer.

5-6 Properties of Determinants

In the defining equations $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

the sums of products in the right-hand members are called **expansions** of the determinants. Notice that the first of these expansions shows all possible arrangements of the subscripts 1 and 2 of the letters a and b, and that the second expansion shows all possible arrangements of the subscripts 1, 2, and 3 of the letters a, b, and c.

The fact that a determinant of order 2 or 3 is a sum of terms involving all possible arrangements of the subscripts of the elements suggests that a determinant of order $n(n > 3)$, that is, a determinant having n rows and n columns, can similarly be defined as a sum of terms involving all possible arrangements of the subscripts $1, 2, \ldots, n$. This is indeed true, but there are no simple arrow diagrams to help us make the rather lengthy computations (120 terms, for instance, in the sum for a determinant of order 5, and 720 for one of order 6!). We therefore omit this definition and give an alternative (but equivalent) definition of a determinant of higher order in terms of determinants of next lower order by using *minors*. The **minor** of an element in a determinant is defined to be the determinant obtained when you delete the row and column containing the element.

For example,

$$\text{the minor of 4 in } \begin{vmatrix} 4 & 3 & -9 \\ 2 & 5 & -2 \\ 7 & 8 & 0 \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} 5 & -2 \\ 8 & 0 \end{vmatrix}.$$

Similarly, the minor of 2 is $\begin{vmatrix} 3 & -9 \\ 8 & 0 \end{vmatrix}$ and of 5 is $\begin{vmatrix} 4 & -9 \\ 7 & 0 \end{vmatrix}$.

Now if you rewrite the right-hand member of the expression defining a third-order determinant on page 156 as

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1,$$

you can factor it to obtain

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

It follows that if you let A_1, A_2, and A_3 represent the minors of a_1, a_2, and a_3, respectively, you can then write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3.$$

The right-hand member of this latter equation is called the **expansion** of the determinant by minors of the elements in the first column.

By suitably arranging the terms in the definition of a third-order determinant, you can show that such a determinant can be expanded by minors about any row or any column as follows:

1. Choose a row or column and form the product of each element in the row or column with its minor.
2. Use the product obtained or its negative according as the sum of the number of the row and number of the column containing the element is even or odd.
3. The sum of the resulting numbers is the value of the determinant.

EXAMPLE 1 Expand by minors of elements of the second row and evaluate:

$$\begin{vmatrix} 5 & -1 & -2 \\ 3 & 6 & -7 \\ 2 & -3 & 4 \end{vmatrix}$$

SOLUTION The elements of the second row are 3, 6, and -7. The element 3 is in the second row, first column; since $2 + 1 = 3$ (odd), we use the negative of

its product with its minor. Similarly, we use the product of 6 and its minor and the negative of the product of -7 and its minor. Thus,

$$\begin{vmatrix} 5 & -1 & -2 \\ 3 & 6 & -7 \\ 2 & -3 & 4 \end{vmatrix} = -3\begin{vmatrix} -1 & -2 \\ -3 & 4 \end{vmatrix} + 6\begin{vmatrix} 5 & -2 \\ 2 & 4 \end{vmatrix} - (-7)\begin{vmatrix} 5 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= (-3)(-10) + 6(24) + 7(-13) = 83. \quad \text{Answer.}$$

We can extend this idea to fourth-order (or higher-order) determinants. For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4,$$

where

$$A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \; A_2 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \text{ etc.}$$

Cramer's Rule, in addition, extends to the solution of four linear equations in four variables, etc. Thus, the solution of

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= e_1 \\ a_2x + b_2y + c_2z + d_2w &= e_2 \\ a_3x + b_3y + c_3z + d_3w &= e_3 \\ a_4x + b_4y + c_4z + d_4w &= e_4 \end{aligned}$$

is $\quad x = \dfrac{D_x}{D},\; y = \dfrac{D_y}{D},\; z = \dfrac{D_z}{D},\; w = \dfrac{D_w}{D}$, provided $D \neq 0$ where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \; D_x = \begin{vmatrix} e_1 & b_1 & c_1 & d_1 \\ e_2 & b_2 & c_2 & d_2 \\ e_3 & b_3 & c_3 & d_3 \\ e_4 & b_4 & c_4 & d_4 \end{vmatrix}, \; D_y = \begin{vmatrix} a_1 & e_1 & c_1 & d_1 \\ a_2 & e_2 & c_2 & d_2 \\ a_3 & e_3 & c_3 & d_3 \\ a_4 & e_4 & c_4 & d_4 \end{vmatrix}, \text{ etc.}$$

EXAMPLE 2 Expand by minors and evaluate: $\begin{vmatrix} 4 & -1 & 1 & 2 \\ 3 & -1 & 0 & 2 \\ 0 & 4 & 0 & 1 \\ 3 & 1 & 1 & 2 \end{vmatrix}$

SOLUTION Expand by minors of the third column since two elements are 0.

$$1\begin{vmatrix} 3 & -1 & 2 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & -1 & 2 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & -1 & 2 \\ 3 & -1 & 2 \\ 3 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & -1 & 2 \\ 3 & -1 & 2 \\ 0 & 4 & 1 \end{vmatrix}$$

$$= [3(8-1) + 3(-1-8)] - [4(-1-8) - 3(-1-8)]$$

$$= -6 - (-9) = 3. \quad \text{Answer.}$$

Determinants have some properties that are useful in simplifying their expansion by minors. The properties are presented here without proof. While third-order determinants are used in illustrating them, the properties are valid for determinants of any order.

Property 1: If each element in any row (or each element in any column) is 0, then the determinant is equal to 0.

$$\begin{vmatrix} 2 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 3 & -1 \end{vmatrix} = -0\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$
$$= 0 + 0 + 0 = 0$$

Property 2: If any two rows (or any two columns) of a determinant are interchanged, the resulting determinant is the negative of the original determinant.

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 17$$

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = -17$$

Property 3: If two rows (or two columns) of a determinant have corresponding elements that are equal, then the determinant is equal to 0.

$$\begin{vmatrix} 1 & 3 & -2 \\ 3 & 2 & 4 \\ 1 & 3 & -2 \end{vmatrix} = -3\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$$
$$= -3 \cdot 0 + 2 \cdot 0 - 4 \cdot 0 = 0$$

Property 4: If each element in one row (or one column) of a determinant is multiplied by a real number k, then the determinant is multiplied by k.

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3(2) & 3(-1) & 3(5) \end{vmatrix} = 3\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 2 & -1 & 5 \end{vmatrix}$$

Verify this by expanding both determinants by minors of the elements in the third row.

Property 5: If each element of one row is multiplied by a real number k and the resulting products are added to the corresponding elements of another row, or each element of one column is multiplied by a real

number k and the resulting products are added to the corresponding elements of another column, then the resulting determinant is equal to the original determinant.

$$\begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 1+3(1) & -2+3(3) & 5+3(-2) \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 4 & 7 & -1 \end{vmatrix}$$

EXAMPLE 3 Evaluate: $\begin{vmatrix} 2 & -1 & -6 \\ 3 & 4 & 2 \\ 5 & -2 & 3 \end{vmatrix}$

SOLUTION Use the above properties to obtain zeros in the second column. Multiply the first row by 4 and add to the second row.

$$\begin{vmatrix} 2 & -1 & -6 \\ 3+4(2) & 4+4(-1) & 2+4(-6) \\ 5 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -6 \\ 11 & 0 & -22 \\ 5 & -2 & 3 \end{vmatrix}$$

Multiply the first row by -2 and add to the third row.

$$\begin{vmatrix} 2 & -1 & -6 \\ 11 & 0 & -22 \\ 5+(-2)(2) & -2+(-2)(-1) & 3+(-2)(-6) \end{vmatrix} = \begin{vmatrix} 2 & -1 & -6 \\ 11 & 0 & -22 \\ 1 & 0 & 15 \end{vmatrix}$$

Expand the last determinant by minors of the second column.

$$\begin{vmatrix} 2 & -1 & -6 \\ 11 & 0 & -22 \\ 1 & 0 & 15 \end{vmatrix} = -(-1)\begin{vmatrix} 11 & -22 \\ 1 & 15 \end{vmatrix} + 0\begin{vmatrix} 2 & -6 \\ 1 & 15 \end{vmatrix} - 0\begin{vmatrix} 2 & -6 \\ 11 & -22 \end{vmatrix}$$

$$= 187 + 0 - 0 = 187. \quad \text{Answer.}$$

Oral Exercises

Give the requested element or determinant using this expansion by minors. You need not evaluate the determinants.

$$\begin{vmatrix} 1 & 4 & 5 & -7 \\ 2 & 3 & 6 & -6 \\ 9 & 8 & 7 & -5 \\ -1 & -2 & -3 & -4 \end{vmatrix} = a_1\begin{vmatrix} 3 & 6 & -6 \\ 8 & 7 & -5 \\ -2 & -3 & -4 \end{vmatrix} - a_2\begin{vmatrix} 4 & 5 & -7 \\ 8 & 7 & -5 \\ -2 & -3 & -4 \end{vmatrix} + 9A_3 - (-1)A_4$$

1. a_1
2. a_2
3. (the determinant) A_3
4. A_4

Written Exercises

Expand the given determinant by minors of the given row or column and then evaluate.

A

1. Column 3

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ -2 & 6 & 2 \end{vmatrix}$$

2. Row 3

$$\begin{vmatrix} 3 & 5 & 29 \\ -1 & 2 & 35 \\ 0 & 0 & 4 \end{vmatrix}$$

3. Column 1

$$\begin{vmatrix} 1 & 3 & 6 \\ -2 & 2 & -4 \\ -1 & 5 & 2 \end{vmatrix}$$

Evaluate.

4. $\begin{vmatrix} 3 & 2 & 4 & 2 \\ 0 & 0 & -1 & 1 \\ -2 & 1 & -3 & 2 \\ 1 & 3 & 2 & 0 \end{vmatrix}$

5. $\begin{vmatrix} -1 & 3 & 2 & 1 \\ 1 & 0 & -1 & -2 \\ 3 & 0 & 1 & 1 \\ -2 & 4 & 0 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 4 & 3 & -2 & 1 \\ 2 & -2 & 0 & 4 \\ 0 & 7 & 5 & 4 \\ -1 & 0 & 0 & 2 \end{vmatrix}$

Use Cramer's Rule to solve the system.

B

7. $$\begin{aligned} x - y + z &= 0 \\ 2x - y - w &= 0 \\ 3x + y + w &= 5 \\ 2y + z + w &= -1 \end{aligned}$$

8. $$\begin{aligned} x + y + z - w &= 4 \\ 2x + y - 3w &= 1 \\ y + z + 5w &= 0 \\ x - y + z &= -1 \end{aligned}$$

9. Use expansion by minors to prove:

a. $$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = a_1b_2c_3d_4$$

b. $$\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$$

In Exercises 10–12 refer to properties 1–5 on pages 164–165.

C

10. Use Properties 1 and 5 of this section to prove Property 3 for 3×3 determinants.

11. Use Property 2 *only* to prove Property 3 for 4×4 determinants. (*Hint:* First prove Property 3 for 3×3 determinants.)

12. Prove that, for any real numbers r and t,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ra_1 + ta_2 & rb_1 + tb_2 & rc_1 + tc_2 \end{vmatrix} = 0.$$

Self-Test 2

VOCABULARY order of a determinant (p. 155)
Cramer's Rule (p. 157)
expansion of a determinant by minors (p. 161)

1. Use determinants to solve the system: *Obj. 1, p. 155*

$$\begin{aligned} 2x + y &= 3 \\ 3x + y - z &= 2 \\ 2y - z &= -5 \end{aligned}$$

2. The three angles of triangle ABC have the properties that the degree measure of $\angle C$ is three times that of $\angle B$, and the sum of the degree measures of $\angle A$ and the supplement of $\angle C$ is twice the degree measure of $\angle B$. Find the degree measures of the three angles. *Obj. 2, p. 155*

3. Evaluate: $\begin{vmatrix} 3 & -2 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & 3 & 6 & 1 \\ -1 & 1 & 2 & 2 \end{vmatrix}$ *Obj. 3, p. 155*

Check your answers with those at the back of the book.

Chapter Summary

1. A rectangular coordinate system in three-dimensional space assigns an ordered triple of numbers to each point. The three *coordinate axes* determine three *coordinate planes,* and the coordinate planes separate space into eight *octants.*
2. The graph of a linear equation in three variables is a plane. The x-, y-, and *z-intercepts* of a plane are the x-, y-, and z-coordinates, respectively, of the points where the plane intersects the x-, y-, and z-axes. A line in which a plane cuts a coordinate plane is called the *trace* of the given plane in that coordinate plane.
3. The transformations used in solving systems of linear equations in two variables are also applicable in solving three-variable systems. See page 112.
4. Third-order determinants can be used to solve a system of three linear equations in three variables by a method called *Cramer's Rule.*
5. A determinant can be *expanded by minors* of the elements in any row or column.
6. Cramer's Rule can be extended to the solution of n linear equations in n variables.
7. You can use properties of determinants in simplifying their expansion by minors. See properties 1–5 on pages 164–165.

1324 1-12

1-16

Chapter Review

1. Which point lies in the first octant? 5-1

   a. $(1, 2, -3)$ b. $(-2, -2, -2)$ c. $(4, 5, 6)$ d. $(0, 1, -3)$

2. Which point lies on the xz-plane?

   a. $(1, 0, 3)$ b. $(0, 1, 5)$ c. $(0, 4, 0)$ d. $(3, 1, 8)$

3. Find the x-intercept of the plane with equation $3x + 4y + 2z = 12$. 5-2

   a. 2 b. 4 c. 3 d. 6

4. Which trace of the plane with equation $4x - 2y - z = 8$ is defined by the equations below?

$$\begin{aligned} 4x - 2y &= 8 \\ z &= 0 \end{aligned}$$

   a. xy-trace b. yz-trace c. xz-trace

5. The following system of equations has a single solution. Find it. 5-3

$$\begin{aligned} y - z &= 1 \\ 3x - 2y + z &= 0 \\ 2x + 2y - z &= 5 \end{aligned}$$

   a. $\{(2, 1, 1)\}$ b. $\{(1, 2, 1)\}$ c. $\{(-2, 1, 2)\}$ d. $\{(2, 1, 1)\}$

Review Items 6 and 7 refer to the system of equations below.

$$\begin{aligned} 2x + y \phantom{{}+2z} &= 0 \\ x - y + 2z &= 4 \\ -2x + 4y - z &= -2 \end{aligned}$$

6. Which determinant is D_x? 5-4

   a. $\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ -2 & 4 & -1 \end{vmatrix}$ b. $\begin{vmatrix} 0 & 1 & 0 \\ 4 & -1 & 2 \\ -2 & 4 & -1 \end{vmatrix}$ c. $\begin{vmatrix} 2 & 0 & 0 \\ 1 & 4 & 2 \\ -2 & -2 & -1 \end{vmatrix}$

7. Evaluate D_y.

   a. 8 b. -4 c. 0 d. 4

There are x nickels, y dimes, and z quarters. There are two more nickels than dimes. The combined value of all the coins is \$1.15.

8. Which equation expresses the value of the coins? 5-5

   a. $x + y + z = 115$ b. $5x + 10y + 25z = 115$ c. $x - y = 2$

9. Which equation expresses the relation of the number of dimes to the number of nickels?

   a. $5x - 10y = 2$ b. $y - x = 2$ c. $x - y = 2$

Review Items 10 and 11 refer to the determinant below. 1, 2, 7, 6 (not by Cramer's rule)

$$\begin{vmatrix} 0 & 6 & 0 \\ 1 & 5 & 4 \\ 3 & -2 & 1 \end{vmatrix}$$

10. Find the minor of 3. 5-6

a. $\begin{vmatrix} 0 & 6 \\ 1 & 5 \end{vmatrix}$ b. $\begin{vmatrix} 6 & 0 \\ 5 & 4 \end{vmatrix}$ c. $\begin{vmatrix} 0 & 0 \\ 1 & 4 \end{vmatrix}$

11. If A_2 is the minor of 6, what is the value of the determinant?

a. A_2 b. $6A_2$ c. $-6A_2$ d. not given

Chapter Test

1. Sketch the coordinate box of the point (3, −2, 3). 5-1
2. Sketch the triangle in space whose vertices have the coordinates (2, 0, 0), (0, −3, 0), and (0, 0, 4).
3. Draw a sketch showing the trace of the graph $2x + 4y + z = 8$ in each coordinate plane, and shade the part of the graph in the first octant. 5-2
4. Solve by transforming into a simple system. 5-3

$$\begin{aligned} 2x + y - z &= 1 \\ 4x - 3y + 2z &= 8 \\ 3x + 2y + z &= 0 \end{aligned}$$

Solve each system by Cramer's Rule. 5-4

5. $\begin{aligned} x + 2y + 3z &= -7 \\ 2y - 5z &= 8 \\ 2x - 3y + z &= 3 \end{aligned}$

6. $\begin{aligned} 3x - 2y - z &= -5 \\ x + 3y + 4z &= 7 \\ 2x + 3y - 2z &= -6 \end{aligned}$

7. Ron, Hank, and Louise have a total of 52 tropical fish. If Louise had 6 more, then she would have as many as both boys together. Hank has 3 more fish than Ron. How many fish does each of the three have? 5-5

8. Expand by minors of the elements of the second column, and then evaluate. 5-6

$$\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & -3 \\ 2 & 6 & 7 \end{vmatrix}$$

9. Evaluate: $\begin{vmatrix} 4 & 8 & 3 & 8 \\ 0 & 0 & 2 & 1 \\ 5 & 6 & -1 & -2 \\ -1 & 3 & 5 & 0 \end{vmatrix}$

The Israeli sculptor Agam created this movable sculpture for Lincoln Center in New York. Individual elements within it may be moved to create an infinity of different sculptures.

6 Polynomials and Rational Expressions

Polynomials and Their Factors

***OBJECTIVES** for Sections 6-1 through 6-3:*

1. *Apply the laws of exponents to simplify products and quotients of monomials.*
2. *Write a product of polynomials in simple form.*
3. *Write a polynomial in factored form.*

6-1 Laws of Exponents

You will recall from page 31 that b^n, the nth power of b, where n is a positive integer, denotes a product of n equal factors. That is:

$$b^n = \overbrace{(b \times b \times \ldots \times b)}^{n \text{ factors}}$$

Each factor is b, the *base*, and the number of such factors is n, the *exponent*. The laws for working with positive-integral exponents are summarized in the following theorem.

Theorem. If a and $b \in \mathcal{R}$, and m and n are positive integers, then:

1. $b^m b^n = b^{m+n}$
2. $(b^m)^n = b^{mn}$
3. $(ab)^m = a^m b^m$
4. If $m > n$ and $b \neq 0$, $\dfrac{b^m}{b^n} = b^{m-n}$.
5. If $m < n$ and $b \neq 0$, $\dfrac{b^m}{b^n} = \dfrac{1}{b^{n-m}}$.
6. If $b \neq 0$, $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.
7. If $b \notin \{-1, 0, 1\}$, then $b^m = b^n$ if and only if $m = n$.

Knowing the definition of a power and the properties of real numbers, you can see why each of the laws stated in the preceding theorem holds. Consider, for example, the reasoning for Law 1.

$$
\begin{aligned}
b^m b^n &= \overbrace{(b \times \ldots \times b)}^{m \text{ factors}}\overbrace{(b \times \ldots \times b)}^{n \text{ factors}} && \text{Definition of a power} \\
&= \overbrace{(b \times \quad \ldots \quad \times b)}^{(m+n) \text{ factors}} && \text{Associative axiom of multiplication} \\
&= b^{m+n} && \text{Definition of a power} \\
b^m b^n &= b^{m+n} && \text{Transitive property of equality}
\end{aligned}
$$

Laws 1–3 together with the properties of multiplication enable you to simplify a *product* of two or more monomials.

EXAMPLE 1 Simplify each product.

a. $(2x^2y^3)(-5x^4y^2)$ b. $(-4p^2s^4)^3$ c. $(-5m^2q)^2(-3m^3q)^3$

SOLUTION
a. $(2x^2y^3)(-5x^4y^2) = 2(-5)(x^2x^4)(y^3y^2) = -10x^6y^5$.
b. $(-4p^2s^4)^3 = (-4)^3(p^2)^3(s^4)^3 = -64p^6s^{12}$.
c. $(-5m^2q)^2(-3m^3q)^3 = (-5)^2(-3)^3(m^2)^2(m^3)^3(q^2q^3) = -675m^{13}q^5$.

You can deduce Laws 4 and 5 from Law 1 with the help of a corollary of the following *basic property of quotients,* which you may recall from an earlier algebra course.

Basic Property of Quotients

For all r, s, t, and $u \in \mathcal{R}$, and t and $u \neq 0$,

$$\frac{rs}{tu} = \frac{r}{t} \cdot \frac{s}{u}.$$

This theorem says that a quotient of products can be written as a product of quotients. If you let (1) $t = 1$, or (2) $r = 1$, you obtain the required corollary.

Corollary. For all r, s, t, and $u \in \mathcal{R}$, and t and $u \neq 0$: ✓

1. $\frac{rs}{u} = r \cdot \frac{s}{u}$
2. $\frac{s}{tu} = \frac{1}{t} \cdot \frac{s}{u}$.

PROOF OF LAW 4

First, note that if $m > n$, then $m - n$ is positive (Definition of "$>$").

$\frac{b^m}{b^n} = \frac{b^{(m-n)+n}}{b^n}$	$(m - n) + n = m$
$= \frac{b^{m-n} \cdot b^n}{b^n}$	Law 1 for positive exponents
$= b^{m-n} \cdot \frac{b^n}{b^n}$	Corollary above
$= b^{m-n} \cdot 1$	$\frac{b^n}{b^n} = 1$
$= b^{m-n}$	Axiom of 1

EXAMPLE 2 Simplify each quotient, assuming that no variable equals 0.

a. $\frac{-16r^3s^5}{2r^7s^4}$ b. $\left(\frac{-4x^2}{xz^2}\right)^3$

SOLUTION a. $\frac{-16r^3s^5}{2r^7s^4} = \left(-\frac{16}{2}\right)\left(\frac{1}{r^{7-3}}\right)(s^{5-4}) = -8\frac{s}{r^4}$.

b. $\left(\frac{-4x^2}{xz^2}\right)^3 = \frac{(-4)^3(x^2)^3}{(x^3)(z^2)^3} = \frac{-64x^6}{x^3z^6} = \frac{-64x^3}{z^6}$.

Thus far we have used only *positive* integers as exponents. The laws for exponents will hold for *any* integral exponent with the addition of these definitions: ✓

For all nonzero $b \in \mathcal{R}$ and all positive integers n,

$$b^0 = 1 \quad \text{and} \quad b^{-n} = \frac{1}{b^n}.$$

EXAMPLE 3 Show that Law 1 holds for $b^{-m}b^{-n}$ when $m = 5$, $n = 2$.

SOLUTION In order to show that $b^{-5}b^{-2} = b^{-5+(-2)} = b^{-7}$, we have:

$b^{-5}b^{-2} = \frac{1}{b^5} \cdot \frac{1}{b^2}$	Definition of b^{-n}
$= \frac{1}{b^5 \cdot b^2}$	Basic Property of Quotients
$= \frac{1}{b^7}$	Law 1
$= b^{-7}$	Definition of b^{-n}

Oral Exercises

Give an equivalent numeral without exponents.

EXAMPLE $\frac{3^{-2}}{2^3}$ **SOLUTION** $\frac{3^{-2}}{2^3} = \frac{1}{8 \cdot 9} = \frac{1}{72}$

174 oral
-3 odd
writ x3

1. 10^4
2. 10^{-4}
3. $(-2)^3$
4. 7^{-2}
5. $2 \cdot 5^{-3}$
6. $(-3)^{-2}$
7. $2 \cdot 2^4$
8. $3^5 \cdot 3^{-4}$
9. $\frac{2^5}{2^7}$
10. $\frac{2^2}{2^{-3}}$
11. $\frac{3}{5^{-2}}$
12. $\left(\frac{3}{2}\right)^3$
13. $(2 + 3)^2$
14. $2(1 - 4)^{-2}$
15. $\frac{1}{(3 + 4)^2}$
16. $(3 - 2)^{-5}$

Give an equivalent expression in which each variable appears no more than once and only positive exponents are used. Assume that no variable equals 0.

17. $(4x^3y)(-3x^2)$
18. $(5x^2y)(x^5y^{-3})$
19. $(-3x^{-2})(xy^3)$
20. $(-2a^3b^4)(-6a^{-3}b^2)$

Written Exercises

Give an equivalent expression containing no negative exponents. Assume that any variable in a denominator or with a nonpositive exponent does not equal zero.

A

1. $(3r^4)(-5r^2)$
2. $(5a^6)(-6a^{-5})$
3. $(4x^3y^{-4})(7xy^2)$
4. $(2p^5q^{-2})(6p^{-3}q^{-1})$
5. $(2^{-1}cd^{-4})(8c^{-3}d^4)$
6. $\left(\frac{2}{3}x^0y^5\right)(-3^4x^2y^{-6})$
7. $(-2^4u^6v^{-2}w)(2u^{-3}vw^{-5})$
8. $(-3r^2s)^2(2rs^3)$
9. $(-x^3y)^2(-xy)^{-3}$ ok
10. $(2e^2f^{-1})^3(2ef)^{-4}$
11. $(5a^4b^{-2})^{-1}(5a^{-3}b^4)^{-2}$
12. $(-3yz^2)^{-2}(3y^{-2}z)^2$
13. $(x^2)^2(x^{-2})^2$
14. $(a^3b^3)(ab)^{-2}$
15. $\frac{p^2q^{-3}}{p^{-1}q^2}$

16. $\dfrac{(2r^2s)}{4r^3s^{-3}}$

17. $\dfrac{2a^2b}{(-2ab^3)^{-2}}$

18. $\left(\dfrac{5a^3b}{10a^2b^2}\right)^4$

B 19. $\dfrac{xy}{x^{-1} + y^{-1}}$

20. $\dfrac{(x + y)^2}{(x + y)^{-1}}$

21. $(c^{-2} + d^{-2})^{-1}$

22. $\dfrac{u^{-1} + v^{-1}}{u^{-1} - v^{-1}}$

23. $(p + q)(p^{-1} + q^{-1})^{-1}$

24. $(a^{-1} - b^{-1})(a - b)^{-1}$

25. Deduce Law 5 from Law 1.

26. Deduce Law 6 from Law 3.

27. Prove that for any positive integer n and any real number $b \neq 0$, $\dfrac{1}{b^{-n}} = b^n$.

28. Prove that for any positive integer n and any nonzero real numbers a and b, $(ab)^{-n} = a^{-n}b^{-n}$.

C 29. Show that if Law 1 is extended to the case in which m or n equals 0, then we must define b^0 to be 1.

30. Show that if Law 1 is extended to the case in which m or n is negative, then we must define b^{-n} to be $\dfrac{1}{b^n}$. (*Hint:* Let $m = -n$ in Law 1.)

31. Prove the Basic Property of Quotients (page 172). $\left[\textit{Hint: } \dfrac{rs}{tu} = rs \cdot \dfrac{1}{tu} = (r \cdot s) \cdot \left(\dfrac{1}{t} \cdot \dfrac{1}{u}\right).\right]$

programming in BASIC

You can use INT to test whether or not a number is a factor of another number. For example:

```
10  PRINT "INPUT N";
20  INPUT N
30  FOR F=1 TO N
40  LET Q=N/F
50  IF Q<> INT(Q) THEN 70
60  PRINT F;
70  NEXT F
80  END
```

RUN this for N=60. Notice that there are no factors between N/2 and N. Now change the program to include:

```
25  PRINT 1;
30  FOR F=2 TO N/2
75  PRINT N
```

RUN this revised program for values of N from 61 to 72. You will see that the factors occur in pairs except when N is a perfect square. Then the square root is printed only once.

BASIC has a special built-in function for finding the positive square root of a positive number, SQR(X). Try the program at the right.

```
10  FOR N=2 TO 10
20  PRINT N,SQR(N)
30  NEXT N
40  END
```

Now change line 30 in the program on page 175 to

```
30  FOR F=2 TO SQR(N)
```

and observe the results.

Exercises

1. Write a program that will print out all the pairs of factors of a given number N, including 1 and N.
2. Write a program that will store the pairs of factors of N in subscripted variables L(I) and M(I).
3. Write a program that will factor $Ax^2 + Bx + C$, $A > 0$, by finding and storing pairs of factors of A in L(H) and M(H) and pairs of positive and negative factors of C in R(K) and S(K). Test

   B = L(I)*S(J) + M(I)*R(J)

   to make factors (L(I)X + R(J))(M(I)X + S(J)).
4. a. Write a program that will print out primes less than 100, beginning with 3.

   Hint:
   ```
   10  FOR N=3 TO 99 STEP 2
   20  FOR F=3 TO SQR(N) STEP 2
   ```

   b. Put a counter (LET K=K+1) in the program and count the number of primes from 3 to 99, from 101 to 199, from 201 to 299, and so on up to the set from 1901 to 1999, by changing line 10.

6-2 Multiplying Polynomials

You can find the product of two polynomials by using the familiar axioms of addition and multiplication and the first law of exponents. For example, to find the product of the *binomial* $3x - 2$ and the *trinomial* $5x^4 - x^3 + 4x$, you can proceed as follows:

$$\begin{aligned}(3x-2)(5x^4-x^3+4x) &= 3x(5x^4-x^3+4x) - 2(5x^4-x^3+4x) && \text{Distributive axiom}\\ &= 15x^5 - 3x^4 + 12x^2 - 10x^4 + 2x^3 - 8x && \text{Law 1 of exponents}\\ &= 15x^5 - 13x^4 + 2x^3 + 12x^2 - 8x && \text{Simplification}\end{aligned}$$

You write the product as a polynomial in simple form (page 32).

You are less likely to make errors in adding like terms, if you use a vertical arrangement in multiplying:

$$
\begin{array}{rrrrr}
5x^4 & -\ \ x^3 & & +\ 4x & \\
 & & & 3x & -\ \ 2 \\
\hline
15x^5 & -\ \ 3x^4 & & +\ 12x^2 & \\
 & -\ 10x^4 & +\ 2x^3 & & -\ 8x \\
\hline
15x^5 & -\ 13x^4 & +\ 2x^3 & +\ 12x^2 & -\ 8x
\end{array}
$$

> To obtain the product of two polynomials, multiply each term of one of the polynomials by each term of the other, and then add all the products.

Three special cases of binomial products that are useful to know are given here:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

EXAMPLE Write $(2x^2 - 5y)^2$ as a polynomial in simple form.

SOLUTION $(2x^2 - 5y)^2 = (2x^2)^2 - 2(2x^2)(5y) + (5y)^2 = 4x^4 - 20x^2y + 25y^2$.

Oral Exercises

Express each product as a polynomial in simple form.

1. $(5x)(3x^2)$
2. $(-2y)^2(2y)$
3. $(3t + 7)(3t - 7)$
4. $(z + 5)^2$
5. $(-2n)^3(3n^2)$
6. $(p - 4)^2$
7. $(-3x)(-2x)^2$
8. $(1 - a)(1 + a)$

Written Exercises

Write each product in simple form.

A

1. $3x^2(x^2 - 4)$
2. $(a^2 - 3)(a^2 + 3)$
3. $(2c + 7d)^2$
4. $(x^2 + 3x)5x^2$
5. $(0.6z - 9)(0.6z + 9)$
6. $(b^2 - r^3)(b^2 + r^3)$
7. $(n^5 + 4)^2$
8. $(v^2 - 5)^2$
9. $(3m^3 - 2p^2)^2$
10. $(t^n + 8)^2$
11. $(x^n - y)^2$
12. $(v^n - w)(v^n + w)$

Write each product in simple form.

B 13. $(3x + 2)(4x^2 - 5x + 3)$
14. $(y - 5)(y^2 + 5y + 25)$
15. $(k - 3)(k^2 + 3k + 9)$
16. $(a + 4)^3$
17. $(3b - c)^3$
18. $(r - t)^2(r^2 + t^2)$
19. $(2x - y)^2(2x + y)^2$
20. $[(2x + y)(2x - y)]^2$
21. $(c - 2d)^4$

22. On the basis of the answers to Exercises 14 and 15 give a formula for writing $(a - b)(a^2 + ab + b^2)$ as a polynomial in simple form.

23. Explain the relationship between the answers to Exercises 19 and 20.

Express each product as a sum, leaving negative exponents where necessary.

24. $(5a^{-2} - 2)^2$
25. $(3b^{-5} + 7)(3b^{-5} - 7)$

C 26. $(p^{3n} + 6p^{2n})^2$
27. $(r^{5n} + 4r^{-3n})(r^{5n} - 4r^{-3n})$
28. $(a^n + b^{2n})^3$
29. $(c^n + d^n)(c^{2n} - c^nd^n + d^{2n})$
30. $(x^n - x^{-n})^2$
31. $(y^n + y^{-n})^3$

32. Use the fact that $(a - b)^2 \geq 0$ to prove that the average of the squares of two real numbers is always at least as large as their product.

programming in BASIC

Up to now, we have used separate, unrelated variables. BASIC also provides for *lists* of variables. These variables are called *subscripted variables,* and their use often corresponds to that of subscripted variables in algebra. For example, we can write a general fourth-degree polynomial in x as:

$$a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$$

In BASIC, these coefficients may be represented by:

A(1), A(2), A(3), A(4), A(5)

Moreover, the subscript may itself be a variable. This makes it easy to READ in a list of DATA:

```
10  FOR I = 1 TO 5
20  READ A(I)
30  NEXT I
40  DATA 1, −2, 3, 5, 1
. . . . . . . . . . . . . . . . . . . . . . . .
```

This makes A(1) = 1, A(2) = −2, and so on. DATA statements may go anywhere in the program. The READ statement takes values from the DATA list in the order in which the variables appear in the program.

Study the following program that will find the coefficients of the sum of two polynomials. (Recall Section 2-1.) By changing the DATA statements, you can use it for different problems. Notice that N is the degree of the polynomial of higher degree and hence the degree of the sum.

```
10  READ N
20  FOR I=1 TO N+1
30  READ A(I)
40  NEXT I
50  FOR I=1 TO N+1
60  READ B(I)
70  NEXT I
80  FOR I=1 TO N+1
90  PRINT A(I)+B(I);" ";
100  NEXT I
110  DATA (Higher degree)
120  DATA (Coef. of 1st polynomial)
130  DATA (Coef. of 2d polynomial)
140  END
```

Two examples:

$$\begin{array}{l} 3x^3 + 4x^2 - 7x + 5 \\ 6x^3 \qquad\qquad + 5x \\ \hline \end{array}$$

```
110  DATA 3
120  DATA 3,4,-7,5
130  DATA 6,0,5,0
```

$$\begin{array}{l} x^4 \qquad\quad + 3x^2 + 5x \\ \qquad x^3 + 2x^2 \qquad\quad + 5 \\ \hline \end{array}$$

```
110  DATA 4
120  DATA 1,0,3,5,0
130  DATA 0,1,2,0,5
```

Notice that the data must have a coefficient for each of N + 1 terms in each polynomial, including zeros where necessary.

Subscripted variables as described here can be used for I = 1 to 10. If more than 10 are needed, a DIMension statement must be used as described on page 180.

You probably don't need to use a computer just to add polynomials, but an interesting program can be made for multiplying polynomials. First notice that if M and N are the degrees of the factor polynomials, the product will have the degree M + N and may have up to M + N + 1 terms. For example, consider the product of $(a_1x^3 + a_2x^2 + a_3x + a_4)$ and $(b_1x^3 + b_2x^2 + b_3x + b_4)$. The coefficients of the product can be written as shown below:

No. of term		Sum of subscripts
1	a_1b_1	2
2	$a_1b_2 + a_2b_1$	3
3	$a_1b_3 + a_2b_2 + a_3b_1$	4
4	$a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1$	5
5	$\cancel{a_1b_5} + a_2b_4 + a_3b_3 + a_4b_2 + \cancel{a_5b_1}$	6
6	$\cancel{a_1b_6} + \cancel{a_2b_5} + a_3b_4 + a_4b_3 + \cancel{a_5b_2} + \cancel{a_6b_1}$	7
7	$\cancel{a_1b_7} + \cancel{a_2b_6} + \cancel{a_3b_5} + a_4b_4 + \cancel{a_5b_3} + \cancel{a_6b_2} + \cancel{a_7b_1}$	8

The terms that are crossed out are zero because $a_5 = b_5 = a_6 = b_6 = a_7 = b_7 = 0$. Keeping the terms in the pattern suggests, however, a way of using nested loops to find the coefficients of the product. Notice that the sum of the subscripts in the terms of each coefficient is one more than the number of that term in the product.

To allow for polynomial factors up to degree 9 (10 terms), we must provide for up to 19 terms in the product. To use the pattern of coefficients shown earlier, we must allow for up to 19 terms for A(I) and B(I) by including the DIMension statement as shown in the following program. We must use DIM when there are more than 10 values for a subscripted variable. The statement reserves enough space in the computer for these values.

```
10  DIM A[19],B[19]
20  READ M,N
30  FOR I=N+1 TO M+N+1
40  LET A[I]=0
50  LET B[I]=0
60  NEXT I
70  FOR I=1 TO M+1
80  READ A[I]
90  NEXT I
100  FOR I=1 TO N+1
110  READ B[I]
120  NEXT I
130  DATA 1 (Deg. of 1st poly., M)
140  DATA 1 (Deg. of 2d poly., N≤M)
150  DATA 1,1 (Coef. of 1st poly.)
160  DATA 1,1 (Coef. of 2d poly.)
170  FOR I=1 TO M+N+1
180  LET S=0
190  FOR J=1 TO I
200  LET S=S+A[J]*B[I+1-J]
210  NEXT J
220  PRINT S;" ";
230  NEXT I
240  END
```

In line 170, I is the number of the term in the product; and in line 200, I + 1 is the sum of the subscripts.

Exercises

1. If you RUN the multiplication program above as listed, the output will be

   1 2 1

   which represents $x^2 + 2x + 1$. Make that polynomial the first factor by making these changes and RUN the program again:

   ```
   130  DATA  2
   150  DATA  1, 2, 1
   ```

   Continue changing lines 130 and 150 in this way up to

   ```
   130  DATA  9
   ```

   and observe the pattern of your results.

2. Find the product shown at the top of page 177 by using these DATA statements in the previous program:

```
130 DATA 4
140 DATA 1
150 DATA 5, -1, 0, 4, 0
160 DATA 3, -2
```

3. Rewrite the multiplication program to use INPUT statements instead of READ and DATA statements.

4. Write a program that will use subscripted variables to store the prices of four items and will allow you to compute bills for orders for numbers of one or more of the items.

6-3 Factoring a Polynomial

In Section 6-2 you saw that:

I. $a^2 - b^2 = (a - b)(a + b)$ **Differences of squares**

II. $a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$

III. $a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$

A polynomial, such as any of those in I–III, that can be expressed as a product of two or more *polynomials of lower positive degree* is said to be **reducible.** Each of the latter polynomials is called a **factor** of the given polynomial. To **factor a polynomial** over a designated set (the **factor set**), you express it as a product of polynomials belonging to the factor set. Unless stated otherwise, we assume only integral coefficients for the factors of a polynomial with integral coefficients.

EXAMPLE 1 Factor the polynomial $6xy + 21y^2$.

SOLUTION You can easily observe that $3y$ is the monomial of greatest coefficient and degree that is a factor of each term in the given polynomial. Then by the distributive law, we have

$$6xy + 21y^2 = 3y(2x + 7y). \quad \text{Answer.}$$

In Example 1, $3y$ is called the **greatest monomial factor** of the given polynomial because it is the monomial with the *greatest* numerical coefficient and the *greatest* degree that is a factor of each term of the polynomial. The other factor, $2x + 7y$, cannot be reduced to a product of factors of lower positive degree, and is hence **irreducible.** Moreover, its greatest monomial factor is 1.

A polynomial is said to be **factored completely** when it is expressed as a product of a constant and one or more irreducible polynomials each of which has 1 as its greatest monomial factor. Of course, powers of the irreducible polynomials may be used as necessary, and we omit the constant when the polynomial has no integral factor. In Example 1 the constant is 3, and the irreducible polynomials are y and $2x + 7y$.

EXAMPLE 2 Factor each polynomial completely.

a. $6x^4 - 150x^2$
b. $3a^5 - 6a^4b + 3a^3b^2$

SOLUTION a. $6x^4 - 150x^2 = 6x^2(x^2 - 25) = 6x^2(x - 5)(x + 5)$.
b. $3a^5 - 6a^4b + 3a^3b^2 = 3a^3(a^2 - 2ab + b^2) = 3a^3(a - b)^2$.

Every reducible *quadratic trinomial* of the form $ax^2 + bx + c$ has two binomial factors, of the form $Ax + B$ and $Cx + D$. Since

$$(Ax + B)(Cx + D) = ACx^2 + (BC + AD)x + BD,$$

the problem of factoring such a trinomial is to find values of the coefficients A, B, C, and D such that

$$AC = a, \qquad AD + BC = b, \qquad \text{and} \qquad BD = c.$$

EXAMPLE 3 Factor $6x^2 + 7x + 2$.

SOLUTION First, you might analyze the coefficients as follows: Since the coefficient a (AC) of x^2 is positive, we know that A and C must both be of the same sign, and likewise since the constant term c (BD) is positive, B and D must be of the same sign. (Two numbers are of the "same sign" if both are positive or both are negative; they are of "opposite signs" if one is positive and the other negative.) Since the coefficient $b(AD + BC)$ of x is also positive, A and C cannot have the opposite sign of D and B or else AD and BC would both be negative. Hence A, B, C, and D must all be of the same sign, which we can take to be positive. Since $AC = 6$ and $BD = 2$, the possible factors are:

$$\begin{array}{ccccccccccc}
A & & B & & C & & D & & AC & AD + BC & BD \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \underbrace{\qquad\qquad} & \downarrow \\
(6x & + & 2)(& & x & + & 1) & = & 6x^2 + & 8x & + \ 2 \\
(6x & + & 1)(& & x & + & 2) & = & 6x^2 + & 13x & + \ 2 \\
(3x & + & 2)(& & 2x & + & 1) & = & 6x^2 + & 7x & + \ 2 \\
(3x & + & 1)(& & 2x & + & 2) & = & 6x^2 + & 8x & + \ 2
\end{array}$$

The combination of factors shown in red makes $AD + BC = b$, or **7**.

$\therefore$ in factored form, $6x^2 + 7x + 2 = (3x + 2)(2x + 1)$. Answer.

Notice that in Example 3, if we had chosen A, B, C, and D as all negative instead of positive, the result would have been equivalent. That is,

$$[-3x + (-2)][-2x + (-1)] = 6x^2 + 7x + 2.$$

EXAMPLE 4 Factor $8y^2 - 2y - 3$.

SOLUTION Let us abbreviate the possible factors by writing only the coefficients.

$\overbrace{A \times C}^{8}$		$\overbrace{B \times D}^{-3}$		$\overbrace{AD + BC}^{-2}$
4	2	1	-3	$-12 + 2 = -10$
4	2	-1	3	$12 - 2 = 10$
4	2	3	-1	$-4 + 6 = 2$
4	2	-3	1	$4 - 6 = -2$

Hence the factors are $(4y - 3)$ and $(2y + 1)$, and

$$8y^2 - 2y - 3 = (4y - 3)(2y + 1). \quad \text{Answer.}$$

Of course, as soon as you find the correct factors to make

$$AC = a, \ AD + BC = b, \text{ and } BD = c,$$

there is no point in going through the other possibilities.

EXAMPLE 5 Factor $x^2 - 3x + 1$.

SOLUTION

$\overbrace{A \times C}^{1}$		$\overbrace{B \times D}^{1}$		$\overbrace{AD + BC}^{3}$
1	1	1	1	2
1	1	-1	-1	-2

Since there are no other different factorizations possible, $x^2 - 3x + 1$ is irreducible. Answer.

Two other factor patterns that are useful to know in addition to the three at the beginning of this section are:

IV. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Sum of cubes

V. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Differences of cubes

EXAMPLE 6 Factor $64x^3 + 27$.

SOLUTION Using formula IV with $a = 4x$ and $b = 3$, we have

$$64x^3 + 27 = (4x + 3)(16x^2 - 12x + 9). \quad \text{Answer.}$$

Oral Exercises

State the greatest monomial factor of the given polynomial.

1. $3ab + 6a^2$
2. $32x^2 - 48y^2$
3. $x^4y^2 - x^2y^3$
4. $8r^2s + 12r^3s$
5. $15a^2b^3 - 25ab^2c^2$
6. $8x^2yz - 6xy^2z + 10xyz^2$

Identify the given polynomial as a difference of squares, a difference of cubes, a sum of cubes, or the square of a binomial.

7. $n^2 - 36$
8. $y^3 - 27$
9. $x^4 - 25$
10. $v^3 + 125$
11. $x^2 + 10x + 25$
12. $9a^2 - 49b^2$
13. $x^2 - 8x + 16$
14. $36y^4 - 1$
15. $t^3 - 64$
16. $c^2 - 6cd + 9d^2$
17. $25d^2 - 10d + 1$
18. $8 - 27k^3$

Written Exercises

Factor the given polynomials completely. Write "irreducible" for any that cannot be factored over the set of polynomials with integral coefficients.

A

1. $4x^2 - 12x + 9$
2. $9r^2 - 25t^2$
3. $2k^2 + 11k + 15$
4. $49n^2 + 14n + 1$
5. $27y^3 - 1$
6. $z^2 + 16$
7. $3x^2 + 10xy - 8y^2$
8. $c^3 + 125d^3$
9. $6a^2 - 7ab - 20b^2$
10. $25p^2q^2 - 30pq + 9$
11. $10t^2 - 19t + 6$
12. $64 - m^3$
13. $x^2 + 3x + 1$
14. $x^2y^2 + 13xy - 48$
15. $6x^2 + xy - 12y^2$

B

16. $z^4 - 25$
17. $3x^4 - 8x^2 - 35$
18. $r^6 + 27$
19. $y^6 - 10y^3z^2 + 25z^4$

In Exercises 20–28 more than one factorization may be necessary. Factor out any monomial factors first.

20. $t^6 - 64$
21. $b^4 - 13b^2 + 36$
22. $16a^4 - 54a$
23. $12x^3y^2 + 12x^2y + 3x$
24. $40p^3q^3 - 250pq$
25. $(x - 1)^2 - (y + 1)^2$
26. $(a^2 + 2a + 1) - b^2$
27. $a(a + 2) - 3(a + 2)$
28. $b^3 - 3b^2 + 4b - 12$

Factor completely.

C

29. $x^{3n} + y^{9n}$
30. $49p^{2n} + 14p^n + 1$
31. $a^{2n}b^{6n} - 25$
32. $r^{6n} - t^{12n}$
33. $3x^{4y} - 10x^{2y} + 3$
34. $z^{4k+1} - z^{2k+1} - 6z$
35. $9x^4 - 7x^2y^2 + y^4$ [*Hint:* $9x^4 - 7x^2y^2 + y^4 = (9x^4 - 6x^2y^2 + y^4) - x^2y^2$]

Self-Test 1

VOCABULARY
reducible (p. 181)
factor a polynomial (p. 181)
greatest monomial factor of a polynomial (p. 181)
irreducible (p. 181)
factor a polynomial completely (p. 182)

Give an equivalent expression in which each variable occurs at most once and in which only positive exponents appear. Assume that no variable equals 0.

1. $\dfrac{6x^3y^{-4}}{2x^5y^{-7}}$ **2.** $(3x^3y^{-2})(2x^{-2}y^{-1})^2$ *Obj. 1, p. 171*

3. Write $(2x - 3)^3$ as a polynomial in simple form. *Obj. 2, p. 171*

Factor completely.

4. $2x^2 + 13x - 7$ **5.** $27a^3 - b^3$ *Obj. 3, p. 171*

Check your answers with those printed at the back of the book.

Applications of Factoring

OBJECTIVES for Sections 6-4 and 6-5:
1. ***Solve polynomial equations by factoring.***
2. ***Solve problems involving factorable polynomial equations.***
3. ***Solve polynomial inequalities by factoring.***

6-4 Solving Equations by Factoring

The quadratic polynomial in the equation

$$2x^2 + 5x - 3 = 0$$

can be factored as

$$(2x - 1)(x + 3).$$

Then it is a simple matter to find the solution set of the given equation by using the theorem proved on page 186 which states that *a product of real numbers is zero if and only if at least one of the factors is zero.*

Theorem. For all a and $b \in \mathcal{R}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

PROOF

I. The "if" part says that if either a or b is zero, then $ab = 0$. This follows directly from the multiplication property of zero, that is: $a \cdot 0 = 0 \cdot a = 0$

II. The "only if" part says that if $ab = 0$, then at least one of a and b is zero. The reasoning goes as follows: Suppose $b \neq 0$. We want to show that a must then be zero.

1. $ab = 0$	Hypothesis
2. $ab\left(\frac{1}{b}\right) = 0 \cdot \frac{1}{b}$	Multiplication property of equality
3. $a\left(b \cdot \frac{1}{b}\right) = 0$	Associative axiom and multiplication property of zero
4. $a \cdot 1 = 0$	Axiom of multiplicative inverses
5. $a = 0$	Axiom of 1

From the theorem above we know that the equation at the beginning of the section,

$$2x^2 + 5x - 3 = 0, \qquad \text{or} \qquad (2x - 1)(x + 3) = 0,$$

is equivalent to the statement that either $2x - 1 = 0$, or $x + 3 = 0$. Solving these two linear equations, we obtain

$$x = \tfrac{1}{2} \qquad \text{or} \qquad x = -3.$$

Checking, you find that each of these values satisfies the original equation. Hence the solution set is $\{\frac{1}{2}\} \cup \{-3\} = \{\frac{1}{2}, -3\}$.

EXAMPLE 1 Solve $y^3 - y^2 = 6y$.

SOLUTION

1. First rewrite the equation into an equivalent equation with one member 0.

$$y^3 - y^2 - 6y = 0$$

2. Factor completely.

$$y(y^2 - y - 6) = 0$$
$$y(y - 3)(y + 2) = 0$$

3. Solve the compound sentence:

$$\begin{array}{rcrcr} y = 0 & \text{or} & y - 3 = 0 & \text{or} & y + 2 = 0 \\ y = 0 & \text{or} & y = 3 & \text{or} & y = -2 \end{array}$$

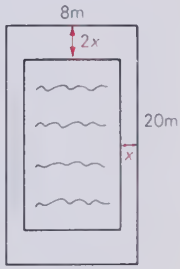
4. Check each solution in the *original* equation, $y^3 - y^2 = 6y$.

$$0 - 0 = 0 \qquad 3^3 - 3^2 = 6 \cdot 3 \qquad (-2)^3 - (-2)^2 = 6(-2)$$
$$0 = 0 \qquad 27 - 9 = 18 \qquad -8 - 4 = -12$$

$\therefore$ the solution set is $\{0, 3, -2\}$. Answer.

EXAMPLE 2 In a community park a rectangular swimming pool and walk are to be built on a piece of ground 20 m long and 8 m wide. The pool is to be surrounded by a paved walk that is twice as wide at the ends as at the sides of the pool. If the area of the pool is $\frac{3}{5}$ that of pool-plus-paving, how wide is the walk **(a)** at either side and **(b)** at either end?

SOLUTION



1. The problem asks for the widths of the walk at either side and at either end of the pool.
2. Let x = width of walk along either side, and
   $2x$ = width of walk at either end.
   Then the dimensions of the pool (in meters) are $(8 - 2x)$ and $(20 - 4x)$.
3. $\underbrace{\text{The area of the pool}} \text{ is } \tfrac{3}{5} \text{ } \underbrace{\text{that of the pool-plus-paving.}}$

$$(8 - 2x)(20 - 4x) = \tfrac{3}{5}(8 \times 20)$$

4.
$$160 - 72x + 8x^2 = 96$$
$$8x^2 - 72x + 64 = 0$$
$$x^2 - 9x + 8 = 0$$
$$(x - 8)(x - 1) = 0$$
$$x - 8 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 8 \quad \text{or} \quad x = 1$$

Since the entire plot is only 8 m wide, the solution $x = 8$ is not possible. Hence, $x = 1$ is the only possibility.

5. The width of the pool is $(8 - 2x)$, or 6 m; the length is $(20 - 4x)$, or 16 m. The area of the pool is $(6 \times 12) = 96\text{ m}^2$ which is $\frac{3}{5}$ that of pool-plus-paving, $\frac{3}{5}(8 \times 20)$.

$\therefore$ the walk is 1 m wide at each side and 2 m wide at each end. Answer.

Oral Exercises

Give the solution set for these equations in factored form.

1. $(2x - 1)(x - 3) = 0$
2. $(3x - 1)(x + 5) = 0$
3. $2(y - 2)(2y + 1) = 0$
4. $z(z + 3) = 0$
5. $a(a - 3)(4a + 1) = 0$
6. $(x - 15)(x + 15) = 0$
7. $2y(y + 8)(y - 3) = 0$
8. $(5v + 8)(7v - 1) = 0$

Written Exercises

Determine the solution set of the given equation.

A

1. $x^2 + 2x - 15 = 0$
2. $x^2 - 81 = 0$
3. $169 - x^2 = 0$
4. $9x^2 - 6x + 1 = 0$
5. $y^2 - 12y + 36 = 0$
6. $2x^2 + 15x + 25 = 0$
7. $5x^2 - 80 = 0$
8. $6a^2 + 7a - 10 = 0$
9. $7z^2 + 14z = 0$
10. $8b^2 - 10b + 3 = 0$
11. $x^2 - 4x = 12$
12. $6x^2 = 150x$
13. $10c^2 = 19c + 15$
14. $(x - 3)^2 = 16$
15. $n^2 + 56 = 15n$

B

16. $(2x + 5)(x - 9) = 11 - 4x$
17. $(2y - 3)^2 = y$
18. $2x^2 - (x - 3)^2 = -2$
19. $2x(x - 3) - (x - 4)(x - 5) = 8$
20. $z^3 + 8z^2 - 20z = 0$
21. $3a^3 - 48a = 0$
22. $x^4 - 29x^2 + 100 = 0$
23. $x^4 - 5x^2 - 36 = 0$
24. $x^5 - 17x^3 + 16x = 0$
25. $x^2(x^2 - 25) - 4(x^2 - 25) = 0$

Give a quadratic equation whose solution set is the given set.

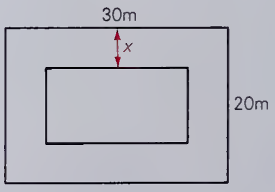
C

26. $\{5, 9\}$
27. $\{0, 15\}$
28. $\{-7, 11\}$
29. $\{-4, -8\}$
30. $\left\{\frac{1}{2}, 6\right\}$
31. $\{-2\}$
32. $\left\{\frac{3}{4}, \frac{-3}{4}\right\}$
33. $\left\{\frac{-3}{5}, \frac{4}{7}\right\}$

Problems

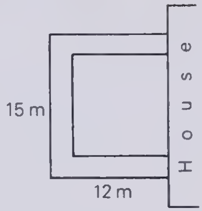
A

1. Find the positive integer whose square is 45 more than 4 times the integer.
2. Find the negative integer whose square is 10 more than 3 times the integer.
3. If the square of a number exceeds 9 times the number by 360, what is the number?
4. The floor of a room 20 m wide and 30 m long is being waxed, starting in the center. If $\frac{1}{3}$ of the floor has been waxed, leaving an unwaxed strip of uniform width along the edges of the room, how wide is the strip?

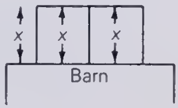


5. The area of a rectangle is 120 m², and it is 7 m longer than it is wide. What are the dimensions of the rectangle?
6. A rectangular box with a square base has a height of 35 cm. What is the length of a side of the base if the surface area of the box is 3600 cm²?

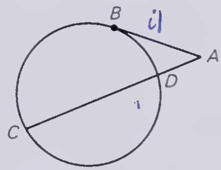
7. In a right triangle with area 30 m^2, one leg is 7 m longer than the other. Find the length of the shorter leg.



B 8. Nora Rowley wants to build a brick walk of uniform width around the outer three edges of her garden which adjoins her house as shown. If the area available for the walk and garden is 12 m by 15 m and there are enough bricks to cover 54 m^2 of ground, how wide should the walk be?



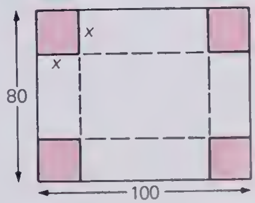
9. A farmer wants to build a divided enclosure against the side of a barn as shown. If the farmer has 31 m of fencing and wants to enclose a total area of 80 m^2, what should the width of the enclosure be?



10. In the figure, $m(\overline{AB})$ (read "the measure of segment A, B") is 4, and $m(\overline{AC}) - m(\overline{AD}) = 6$. Find $m(\overline{AC})$, given that $m(\overline{AC}) \cdot m(\overline{AD}) = (m(\overline{AB}))^2$.

(x + 6) · x 16

x^2) = x



C 11. An open rectangular box is to be made by cutting out squares from the corners of a piece of cardboard measuring 80 cm by 100 cm and folding along the dotted lines. If the original length of a side of the cutout is doubled, it is found that the volume of the box remains the same. What is the original length x?

12. A truck and a sedan leave a filling station at the same time and travel south and east, respectively. After an hour they are 75 km apart. If the sedan travels at an average speed that is three fourths that of the truck, find the speed of each.

6-5 Solving Inequalities by Factoring

You know that for a and $b \in \mathcal{R}$, if $ab > 0$ then a and b are of the same sign, while if $ab < 0$, then a and b are of opposite signs. You can use these facts to solve an inequality in which one member consists of a reducible quadratic polynomial and the other is 0.

EXAMPLE 1 Find and graph the solution set of $x^2 - 2x > 8$ over $\mathcal{R}$.

SOLUTION

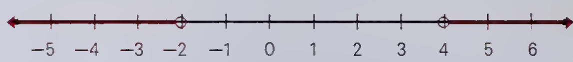
$$x^2 - 2x > 8$$
$$x^2 - 2x - 8 > 0$$
$$(x - 4)(x + 2) > 0$$

The inequality is satisfied if and only if $x - 4$ and $x + 2$ both have the same sign.

Both factors positive	or	*Both factors negative*
$x - 4 > 0$ *and* $x + 2 > 0$		$x - 4 < 0$ *and* $x + 2 < 0$
$x > 4$ *and* $x > -2$		$x < 4$ *and* $x < -2$
The intersection of these two solution sets is $\{x: x > 4\}$.		The intersection of these two solution sets is $\{x: x < -2\}$.

$\therefore$ the solution set of the given inequality is the union

$\{x: x > 4 \quad \text{or} \quad x < -2\}$ Answer.



EXAMPLE 2 Find and graph the solution set of $y^2 - 2y < 3$ over $\mathcal{R}$.

SOLUTION

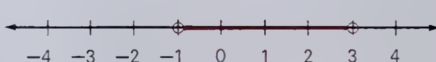
$$y^2 - 2y < 3$$
$$y^2 - 2y - 3 < 0$$
$$(y - 3)(y + 1) < 0$$

The inequality is satisfied if and only if $y - 3$ and $y + 1$ have opposite signs:

	or	
$y - 3 > 0$ *and* $y + 1 < 0$		$y - 3 < 0$ *and* $y + 1 > 0$
$y > 3$ *and* $y < -1$		$y < 3$ *and* $y > -1$
$\{y: y > 3\} \cap \{y: y < -1\} = \emptyset$		$\{y: y < 3\} \cap \{y: y > -1\}$ $= \{y: -1 < y < 3\}$

$\therefore$ the solution set of the given inequality is the union

$\emptyset \cup \{y: -1 < y < 3\} = \{y: -1 < y < 3\}$. Answer.



Oral Exercises

State the solution set of each inequality.

1. $x(x - 2) > 0$
2. $x(x - 3) < 0$
3. $x(x + 5) \leq 0$
4. $x(x - 6) > 0$
5. $(x - 2)(x - 3) > 0$
6. $(x + 1)(x - 5) < 0$
7. $(x - 2)^2 > 0$
8. $8(x - 5)(x - 4) < 0$
9. $(x + 2)^2(x - 3) < 0$

Written Exercises

Find the solution set of each inequality over $\mathcal{R}$ and draw its graph.

A

1. $(x - 2)(x + 3) \geq 0$
2. $(y - 3)(y - 7) \leq 0$
3. $a^2 + 5a + 4 < 0$
4. $x^2 - 6x \geq 0$
5. $x^2 - 3x - 28 < 0$
6. $3b^2 - 75 > 0$
7. $32 \geq 2c^2$
8. $2x^2 + 9x - 5 < 0$
9. $x^2 - 2x < 15$
10. $x^2 - 10x + 28 \geq 4$
11. $(k - 2)^2 > 0$
12. $r^2 + 6r + 9 \leq 0$

B

13. $y^2(y - 4) \geq 0$
14. $2x^3 + 10x^2 < 0$
15. $2x^3 < x^2$
16. $t(t - 5)^2 \geq 0$
17. $(z - 2)(z^2 + 1) > 0$
18. $12y^3 - 3y \leq 0$

C

19. $x^3 + 10x^2 + 25x \leq 0$
20. $4x^3 + 9x > 12x^2$
21. $x^4 - 16 < 0$
22. $x^{4n} + x^{2n} - 2 < 0$, for integral $n > 0$

Self-Test 2

Solve by factoring.

1. $2x^2 + 18x = 0$
2. $5x^2 + 11x - 12 = 0$ *Obj. 1, p. 185*
3. A rectangle that is 5 cm longer than it is wide has an area of 84 cm^2. Find its width. *Obj. 2, p. 185*
4. Solve the inequality $x^2 - 3x < 28$ and graph its solution set. *Obj. 3, p. 185*

Check your answers with those printed at the back of the book.

Rational Algebraic Expressions

***OBJECTIVES** for Sections 6-6 through 6-11:*

1. *Simplify a rational expression by factoring its numerator and denominator.*
2. *Find the quotient and remainder when one polynomial is divided by another.*
3. *Express a product or quotient of rational expressions as a rational expression in lowest terms.*
4. *Transform a sum or difference of rational expressions into an equivalent rational expression in lowest terms.*
5. *Solve problems involving equations with rational coefficients and fractional equations.*

6-6 Simplifying Rational Expressions

Just as any number which is the quotient of two integers is called a **rational number,** so the quotient of two polynomials is called a **rational expression** or, more fully, a **rational algebraic expression.** In neither case can the divisor be zero.

The following theorem enables you to **reduce a fraction to lowest terms,** that is, to express it as an equivalent fraction whose numerator and denominator have no common factors except 1 and -1.

Theorem. For all r, s, and $t \in \mathcal{R}$, s and $t \neq 0$,

$$\frac{r}{s} = \frac{r \div t}{s \div t}, \quad \text{and} \quad \frac{r}{s} = \frac{r \cdot t}{s \cdot t}.$$

For example, $\frac{56}{42} = \frac{56 \div 14}{42 \div 14} = \frac{4}{3}$, and $\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{\frac{2}{3} \cdot 6}{\frac{5}{6} \cdot 6} = \frac{4}{5}$.

Likewise you can simplify a rational expression by factoring the numerator and denominator completely and then dividing both by all their common prime factors. The rational expression is said to be **simplified,** or in **lowest terms.**

EXAMPLE 1 Simplify $\dfrac{y^5 - y^4 - 6y^3}{y^3 - 2y^2 - 3y}$.

SOLUTION $$\frac{y^5 - y^4 - 6y^3}{y^3 - 2y^2 - 3y} = \frac{y^3(y^2 - y - 6)}{y(y^2 - 2y - 3)} = \frac{y \cdot y^2(y+2)(y-3)}{y(y+1)(y-3)}$$

Dividing numerator and denominator by the product $y(y - 3)$ of all their common prime factors (that is, their greatest common factor), you obtain

$$\frac{y^2(y+2)}{y+1} \quad (y \notin \{0, 3, -1\}). \quad \text{Answer.}$$

EXAMPLE 2 Simplify $(28 - 7a)^{-1}(64 - a^3)$.

SOLUTION $$(28 - 7a)^{-1}(64 - a^3) = \frac{64 - a^3}{28 - 7a} = \frac{(4-a)(16 + 4a + a^2)}{7(4-a)}$$
$$= \frac{a^2 + 4a + 16}{7} \quad (a \neq 4). \quad \text{Answer.}$$

Hereafter in this book it will be assumed, usually without comment, that the replacement sets of the variables in a fraction include no numbers for which the denominator is zero.

Oral Exercises

Simplify the given expression.

1. $\frac{x-5}{5-x}$
2. $\frac{(y-3)^2}{3-y}$
3. $\frac{(a-2)(a+3)}{(a-2)^2}$
4. $\frac{x^2-4}{(x-2)^2}$
5. $(x^2+6x+5)(x+5)^{-2}$
6. $(x^3+5x^2-14x)x^{-3}$
7. $\frac{(a+b)^2}{(a+b)^5}$
8. $\frac{4x^5+3x^2}{x^4}$
9. $\frac{x^2-8x+16}{x^2-16}$

Written Exercises

Simplify the given expression.

A

1. $6(3x-15)^{-1}$
2. $5a^2(a^2-7a)$
3. $(2b)^{-3}(12b^2+20b^3)$
4. $(20a^2-15a)(25a)^{-1}$
5. $(c^3-7c^2)(2c-14)^{-1}$
6. $(4y^2-18y)^{-1}(9-2y)$
7. $(12n^3+15n^2)(-8n-10)^{-1}$
8. $(k-2)^3(2-k)^{-1}$
9. $(r-4)^2(4-r)^{-3}$
10. $(x^3y^2-3x^2y)(x^2y^2-3xy)^{-1}$
11. $(5c+10d)(3c^2d+6cd^2)^{-1}$
12. $(a^2-b^2)(a-b)^{-2}$
13. $\frac{p^3-1}{2p^2-2}$
14. $\frac{m^3+m}{(m+1)^2}$
15. $\frac{a^2-6a+9}{a^3-9a}$
16. $\frac{x^2-5xy+4y^2}{x^2-3xy-4y^2}$
17. $\frac{27y^3-1}{3y^2+5y-2}$
18. $\frac{z^3-2z^2-15z}{4z^2-100}$
19. $\frac{t^2-2t+4}{t^4+8t}$
20. $\frac{4x^2+17xy-15y^2}{16x^2-9y^2}$

B

21. $\frac{(x-y)^{-3}}{(x-y)^{-5}}$
22. $\frac{v^4-w^4}{(w-v)(w^2+v^2)}$
23. $\frac{(2x^2-98)(x^3+4x^2-21x)}{2(x+7)^2(x^3-10x^2+21x)}$
24. $\frac{(p^3-q^3)(p^3-pq^2)}{p^3+p^2q+pq^2}$
25. $\frac{r^4-10r^2+9}{r^2-4r+3}$
26. $\frac{(250-2a^3)(a^2+25)}{a^4-625}$
27. $\frac{b^6-64}{(b^2-4)(b^2-2b+4)}$
28. $\frac{(c^2+1)(c-1)^2}{c^4-1}$

C **29.** Use the fact that $\dfrac{(a^2)^3 - (b^2)^3}{(a^3)^2 - (b^3)^2} = 1$ to prove that

$$a^4 + a^2b^2 + b^4 = (a^2 - ab + b^2)(a^2 + ab + b^2).$$

30. Prove that if $\dfrac{a + c}{b + c} = \dfrac{a}{b}$ for some nonzero real number c, then $a = b$.

6-7 Dividing One Polynomial by Another

The following theorem enables you to replace a rational number named by an improper fraction with an equivalent mixed numeral which names the sum of an integer and a proper fraction.

> **Theorem.** For all a, b, and $c \in \mathcal{R}$, $c \neq 0$,
>
> $$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}.$$

Thus,

$$\frac{29}{8} = \frac{24 + 5}{8} = \frac{\overbrace{3 \cdot 8}^{a} + \overset{b}{5}}{\underset{c}{8}} = \frac{3 \cdot 8}{8} + \frac{5}{8} = 3 + \frac{5}{8}, \text{ or } 3\frac{5}{8}.$$

Likewise, using the above theorem along with the division algorithm, you obtain

$$\frac{4233}{28} = \frac{151 \cdot 28 + 5}{28} = \frac{151 \cdot 28}{28} + \frac{5}{28} = 151 + \frac{5}{28}, \text{ or } 151\frac{5}{28}.$$

In more advanced mathematics it often becomes necessary to transform a rational expression, by similar means, into the *sum of a polynomial and another rational expression.* The process, called **division,** consists of successively subtracting a monomial multiple of the divisor from the dividend until you finally obtain *either the remainder zero or a polynomial of lower degree than that of the divisor.*

```
     151
28)4233
   28     subtract 1 × 28
   143
   140    subtract 5 × 28
     33
     28   subtract 1 × 28
      5
```

EXAMPLE 1 Transform $\dfrac{2x - 5x^2 + 6x^3 - 5}{2x + 1}$ into a sum by division.

SOLUTION Before dividing, first arrange the terms of both dividend and divisor in order of decreasing degree.

$$
\begin{array}{rl}
 & 3x^2 - 4x \quad + 3 \\
2x + 1 & \overline{)\,6x^3 - 5x^2 + 2x - 5} \\
 & \underline{6x^3 + 3x^2} \longleftarrow \text{subtract } 3x^2(2x + 1) \\
 & \qquad -8x^2 + 2x \\
 & \qquad \underline{-8x^2 - 4x} \longleftarrow \text{subtract } -4x(2x + 1) \\
 & \qquad\qquad\qquad 6x - 5 \\
 & \qquad\qquad\qquad \underline{6x + 3} \longleftarrow \text{subtract } 3(2x + 1) \\
 & \qquad\qquad\qquad\quad -8
\end{array}
$$

$$\therefore \frac{2x - 5x^2 + 6x^3 - 5}{2x + 1} = 3x^2 - 4x + 3 + \frac{-8}{2x + 1}. \quad \text{Answer.}$$

The following example illustrates the division process for polynomials involving two variables. In this case you first arrange the terms in order of decreasing degree in *one* of the variables.

EXAMPLE 2 Divide $2s^3 + 5s^2t - 4t^3$ by $2s^2 + st - 2t^2$.

SOLUTION As given, the terms are in order of decreasing degree in the variable s. Note that the dividend has no first-degree term in s. When dividing, insert any such "missing" term with a 0 as its coefficient.

$$
\begin{array}{rl}
 & \quad s + 2t \\
2s^2 + st - 2t^2 & \overline{)\,2s^3 + 5s^2t + 0st^2 - 4t^3} \\
 & \underline{2s^3 + s^2t - 2st^2} \\
 & \qquad\quad 4s^2t + 2st^2 - 4t^3 \\
 & \qquad\quad \underline{4s^2t + 2st^2 - 4t^3} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$\therefore \frac{2s^3 + 5s^2t - 4t^3}{2s^2 + st - 2t^2} = s + 2t. \quad \text{Answer.}$$

The quotient of a polynomial and a monomial can be expressed as the sum of the quotients obtained by dividing each term of the polynomial by the monomial.

EXAMPLE 3 Express $\dfrac{9y^4 + 27y^3 - y^2 + 12}{3y^3}$ as a sum by division.

SOLUTION Divide each term in the dividend by the monomial divisor:

$$\frac{9y^4 + 27y^3 - y^2 + 12}{3y^3} = 3y + 9 - \frac{1}{3y} + \frac{4}{y^3}. \quad \text{Answer}$$

Oral Exercises

Transform the given quotient into a sum by dividing.

1. $\dfrac{6x^3 - 10x}{2x^2}$

2. $\dfrac{3a^2b + ab}{ab}$

3. $\dfrac{9x^6 - 6x^2 + 3}{3x^3}$

State the first term of the quotient.

4. $x - 5\overline{)3x^2 - 4x + 1}$

5. $2x + 1\overline{)6x^3 - x^2 + 7x - 4}$

6. $x^2 - 2x + 1\overline{)5x^3 - x + 8}$

7. $4x^2 - 3\overline{)-12x^3 + 5x^2 - 2}$

Written Exercises

Transform the given rational expression into a sum by dividing.

A

1. $\dfrac{15y^4 - 25y^3 + 10y^2}{5y^2}$

2. $\dfrac{12a^2b^2 - 4ab - 20}{4ab}$

3. $\dfrac{c^2d^2 - cd^2 + cd}{-cd}$

4. $\dfrac{27r^2s + 15rs^2 + r}{3rs}$

5. $\dfrac{x^2 + 5x - 66}{x - 5}$

6. $\dfrac{18t^2 + 3t - 10}{3t - 2}$

7. $\dfrac{-8x^2 - 14x + 11}{2x + 5}$

8. $\dfrac{u^2 - 8}{u - 3}$

9. $\dfrac{3z^3 - 7z^2 + 6z - 8}{z - 2}$

10. $\dfrac{a^3 + a^2 - 22a + 8}{a - 4}$

11. $\dfrac{3r^3 + 7r^2 - 7r - 2}{3r + 1}$

12. $\dfrac{8v^3 - 10v^2 - 17v + 15}{4v - 3}$

13. $\dfrac{2c^3 - 5c^2 + 18}{2c + 3}$

14. $\dfrac{27t^3 - 6}{3t - 2}$

15. $\dfrac{12x^3 - 3x^2 - 20x + 7}{4x - 1}$

16. $\dfrac{10x^3 - 4x^2 + 15x - 6}{5x - 2}$

17. $\dfrac{8d^3 + 125}{2d + 5}$

18. $\dfrac{x^4 + 2x^3 - 11x^2 - 7x + 20}{x + 4}$

B

19. $\dfrac{2y^4 - 7y^3 - 8y^2 + 17y - 4}{y^2 - 5y + 4}$

20. $\dfrac{t^4 - 2t^3 - 5t + 5}{t^2 + t + 2}$

21. $\dfrac{5t^4 - 3t^3 - 8t^2 - 7}{t^2 - 2}$

22. $\dfrac{b^3 + 27}{b^2 - 3b + 9}$

23. $\dfrac{2x^3 + 7x^2y - xy^2 + 12y^3}{x + 4y}$

24. $\dfrac{32r^5 + 4}{2r + 1}$

C **25.** By trying examples such as $\frac{x^5 - 32}{x - 2}$, make a conjecture about the pattern of the quotient of any division of the form $\frac{a^n - b^n}{a - b}$, in which n is a positive integer.

26. Repeat the process mentioned in Exercise 25 to make a conjecture about the quotient of any division of the form $\frac{a^n + b^n}{a + b}$, in which n is a positive *odd* integer.

6-8 Multiplying and Dividing Rational Expressions

You can multiply two rational expressions by using the same rule as that for multiplying rational numbers.

For all r, s, t, and $u \in \mathbb{R}$, t and $u \neq 0$,

$$\frac{r}{t} \times \frac{s}{u} = \frac{rs}{tu}.$$

EXAMPLE 1 Simplify $\frac{p^2 - 3p - 4}{p^3} \cdot \frac{p^2 + 2p}{2p - 8}$.

SOLUTION

$$\frac{p^2 - 3p - 4}{p^3} \cdot \frac{p^2 + 2p}{2p - 8} = \frac{(p - 4)(p + 1)}{p^3} \cdot \frac{p(p + 2)}{2(p - 4)}$$

$$= \frac{p(p - 4)(p + 1)(p + 2)}{p(p - 4)2p^2}$$

$$= \frac{p^2 + 3p + 2}{2p^2}. \quad \text{Answer.}$$

The relationship between multiplication and division (page 23) and the fact that the reciprocal of $\frac{s}{u}$ is $\frac{u}{s}$ if $s \neq 0$ and $u \neq 0$ lead to the following result.

Theorem. For all r, s, t, and $u \in \mathbb{R}$, s, t, and $u \neq 0$,

$$\frac{r}{t} \div \frac{s}{u} = \frac{r}{t} \cdot \frac{u}{s} = \frac{ru}{ts}.$$

EXAMPLE 2 Simplify $\dfrac{v^2 + 4v + 4}{v^3 - 9v} \div \dfrac{v^2 - 4}{v^2 + 2v - 15}$.

SOLUTION

$$\begin{aligned}
\frac{v^2 + 4v + 4}{v^3 - 9v} \div \frac{v^2 - 4}{v^2 + 2v - 15} &= \frac{v^2 + 4v + 4}{v^3 - 9v} \cdot \frac{v^2 + 2v - 15}{v^2 - 4} \\
&= \frac{(v + 2)(v + 2)}{v(v + 3)(v - 3)} \cdot \frac{(v + 5)(v - 3)}{(v - 2)(v + 2)} \\
&= \frac{(v + 2)(v + 5)}{v(v + 3)(v - 2)} \\
&= \frac{v^2 + 7v + 10}{v^3 + v^2 - 6v}. \quad \text{Answer.}
\end{aligned}$$

EXAMPLE 3 Simplify $(3x + x^{-1}) \div \left(2 - \dfrac{x}{2}\right)$.

SOLUTION

$$\begin{aligned}
\frac{3x + x^{-1}}{2 - \frac{x}{2}} = \frac{3x + \frac{1}{x}}{2 - \frac{x}{2}} = \frac{\frac{3x^2 + 1}{x}}{\frac{4 - x}{2}} = \frac{3x^2 + 1}{x} \cdot \frac{2}{4 - x} &= \frac{2(3x^2 + 1)}{x(4 - x)} \\
&= \frac{2 + 6x^2}{4x - x^2}. \quad \text{Answer.}
\end{aligned}$$

Oral Exercises

Express each product or quotient in lowest terms.

1. $\dfrac{4y^2}{3}\left(\dfrac{3}{y}\right)^3$

2. $\dfrac{7x - 14}{3} \cdot \dfrac{9}{x - 2}$

3. $\dfrac{5}{x + 3} \cdot \dfrac{x^2 - 9}{25}$

4. $\dfrac{x^2 + 5x + 6}{y^2} \cdot \dfrac{y}{x + 2}$

5. $\dfrac{(x + 2)^2}{4} \cdot \dfrac{16}{x^2 - 4}$

6. $\dfrac{(a + 2)^2}{3^4} \div \dfrac{a + 2}{9}$

7. $\dfrac{4xy^2}{7^2} \div \dfrac{xy}{21}$

8. $\dfrac{(x - 3)^2}{y^2} \div \dfrac{6x - 18}{5y}$

9. $\dfrac{(a + 4)^2}{a^3} \div \dfrac{5a + 20}{a^2b}$

Written Exercises

Express each product or quotient in lowest terms.

A

1. $\dfrac{3p - 6}{p} \cdot \dfrac{p^2 + 2p}{p^2 - 4}$

2. $\dfrac{5a^2 + 15a}{25a^3} \cdot \dfrac{a}{(a + 3)^2}$

3. $\dfrac{9x^2 - 1}{(3x + 1)^2} \cdot \dfrac{2x^2}{12x^2 - 4x}$

4. $\dfrac{3}{b^2 - 3b} \cdot \dfrac{b^2 - b - 6}{3b + 6}$

5. $\dfrac{7r - 35}{8} \div \dfrac{r^2 - 25}{4}$

6. $\dfrac{d^2 + 4d}{d^2 + 3d - 4} \div \dfrac{d^3}{2d - 2}$

B 7. $\dfrac{y^2 - 7y + 12}{y^2 + y - 12} \div \dfrac{(y-4)^2}{3y + 12}$

8. $\dfrac{t^3 - 49t}{(t-2)^2} \div \dfrac{t^2 - 5t - 14}{t^2 - 4}$

9. $\dfrac{4x^2 + 4x + 1}{x^3 - x^2} \cdot \dfrac{x^3 - 2x^2 + x}{2x^2 - x - 1}$

10. $\dfrac{a^2 - b^2}{(a-b)^2} \cdot \dfrac{a^2 + b^2}{(a+b)^2}$

11. $\dfrac{z^2 - 4z}{z^2 + 2z - 8} \div \dfrac{z^3 - 4z^2}{z^2 + 4z}$

12. $\dfrac{4k^2 - 25}{4k^2 - 16k + 15} \div \dfrac{4k^2 - 4k - 15}{4k^2 - 9}$

13. $\dfrac{x^3 - 8}{x^3 - 4x} \cdot \dfrac{x^3 + 2x^2}{x^2 + 2x + 4}$

14. $\dfrac{n^4 - 1}{n^2 + 1} \div \dfrac{(n-1)^2}{n^2 - 1}$

15. $a\left(a - \dfrac{b^2}{a}\right)(a - b)^{-2}$

16. $(4x - x^{-1}) \div (2x^{-1} + x^{-2})$

17. $\left(\dfrac{r}{s} - \dfrac{s}{r}\right) \div (r^{-1} - s^{-1})$

18. $\left(4k + \dfrac{1}{k+1}\right) \cdot \left(2k + 1 - \dfrac{1}{2k+1}\right)$

19. $\left(n + \dfrac{1}{n-1}\right)\left(1 + \dfrac{2-n}{n^2 - 1}\right)^{-1}$

20. $\left(\dfrac{x^3}{y^3} - 1\right)\left(\dfrac{x}{y} - 1\right)^{-2}\left(\dfrac{x}{y} - 1\right)$

C 21. $\dfrac{kx^2 - 1}{k} \div \dfrac{k^2}{k^2x^4 + kx^2 + 1}$

22. $\left(\dfrac{x}{y} - y\right)\left(\dfrac{x^4}{y^4} + x^2 + y^4\right)\left(\dfrac{x}{y} + y\right)$

6-9 Adding and Subtracting Rational Expressions

Two rational numbers having the same denominator can be added or subtracted in accordance with the following theorem (see Exercise 19 on page 25).

> For all a, b, and $c \in \mathcal{R}$, $c \neq 0$,
>
> $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

The same rule applies in the case of rational expressions. For example,

$$\frac{3x^2}{x^2 - 1} - \frac{x+2}{x^2 - 1} = \frac{3x^2 - x - 2}{x^2 - 1} = \frac{(3x+2)(x-1)}{(x+1)(x-1)} = \frac{3x+2}{x+1}.$$

If the denominators differ, then you must find a common denominator before adding or subtracting. Just as with fractions, it is simplest to use the **least common denominator** (LCD), that is, the polynomial of least degree and least positive constant factor that has each denominator as a factor.

EXAMPLE 1 Simplify $\frac{a^2}{a-1} - \frac{3}{2} + \frac{2a-4}{2a-2}$.

SOLUTION To find the LCD, first factor the denominators completely.

$$\frac{a^2}{a-1} - \frac{3}{2} + \frac{2a-4}{2(a-1)}.$$

$\therefore$ the LCD is $2(a-1)$.

Next replace each rational expression with an equivalent one having the LCD as denominator, and then simplify:

$$\frac{2a^2}{2(a-1)} - \frac{3(a-1)}{2(a-1)} + \frac{2a-4}{2(a-1)} = \frac{2a^2 - 3a + 3 + 2a - 4}{2(a-1)}$$

$$= \frac{2a^2 - a - 1}{2(a-1)} = \frac{(2a+1)(a-1)}{2(a-1)}$$

$$= \frac{2a+1}{2} \quad (a \neq 1). \quad \text{Answer.}$$

EXAMPLE 2 Simplify $\frac{x^{-2} - y^{-2}}{x^{-1}y^{-1}}$.

SOLUTION

$$\frac{x^{-2} - y^{-2}}{x^{-1}y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} \cdot \frac{1}{y}} = \frac{\frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2}}{\frac{1}{xy}} = \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{1}{xy}} = \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{1}$$

$$= \frac{(y^2 - x^2)xy}{(xy)(xy)} = \frac{y^2 - x^2}{xy}. \quad \text{Answer.}$$

Oral Exercises

State the least common denominator of the terms of the expression.

1. $\frac{5}{a} - \frac{7}{2b}$
2. $\frac{1}{x^2y} + \frac{7x}{y^2}$
3. $\frac{4}{x-2} + \frac{3}{x+2}$
4. $\frac{1}{a+5} - \frac{a}{5}$
5. $\frac{1}{2x^2y} + \frac{2y}{x^3} + \frac{5x}{y^3}$
6. $\frac{4x}{3x+2} + \frac{3x}{3x-2}$

Written Exercises

Simplify the given rational expression.

A

1. $\frac{3x}{x-2} - \frac{6}{x-2}$
2. $\frac{10}{5b^2} + \frac{15}{b}$
3. $\frac{2}{a^2b} - \frac{2}{ab^2} + \frac{4}{a^2b^2}$
4. $\frac{2r}{r+3} + \frac{12}{2r+6}$

5. $\dfrac{2y + 2}{(y + 1)^2} + \dfrac{y - 1}{y + 1}$

6. $\dfrac{t - 15}{3t^2} - \dfrac{t + 2}{6t} + \dfrac{60t}{2t^3}$

7. $\dfrac{c}{c - 3} - \dfrac{27 - 9c}{c^2 - 9}$

8. $\dfrac{x}{x + y} - \dfrac{2y^2}{x^2 - y^2} + \dfrac{y}{x - y}$

9. $3 + \dfrac{k^2}{k - 5} + \dfrac{4k}{2k - 10}$

10. $\dfrac{a + b}{a^2 - 2ab + b^2} - \dfrac{2a}{a^2 - b^2}$

11. $\dfrac{x}{x^2 - 4} - \dfrac{x + 2}{x^2 - x - 2}$

12. $\dfrac{d^2 + 2}{d^2 - 3d} + \dfrac{2d}{2d - 6}$

13. $\dfrac{x - 5}{x^2 + 2x - 3} - \dfrac{x - 1}{x^2 + 6x + 9}$

14. $\dfrac{2x - 1}{x^2 - x - 12} + \dfrac{x + 1}{x^2 + x - 6}$

15. $\dfrac{c^2d^{-1} - d}{cd^{-1} - 1}$

16. $\dfrac{1 - 4z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$

17. $\dfrac{4a^2}{a^2 - b^2} + \dfrac{a + b}{a - b} - \dfrac{a - b}{a + b}$

18. $\dfrac{4p^2 + q^2}{(2p - q)^3} - \dfrac{2p + q}{(2p - q)^2}$

B 19. $\dfrac{x}{x^2 - y^2} - \dfrac{x - y}{x^2 - xy - 2y^2} - \dfrac{x}{x^2 - 3xy + 2y^2}$

20. $\dfrac{2t^3 + 4t}{t^3 - 8} - \dfrac{t}{t - 2}$

21. $\left(1 - \dfrac{2(c - 3)}{c^2 - 4c + 3}\right)\left(1 + \dfrac{2(c + 1)}{c^2 - 2c - 3}\right)$

22. $\left(r - s + \dfrac{2s^2}{r + s}\right)^{-1}\left(r^2 - \dfrac{s^4}{r^2}\right)$

23. $\dfrac{(x - y)(x^{-1} + y^{-1})}{(x + y)(x^{-1} - y^{-1})}$

C 24. Theorem: Any rational expression of the form $\dfrac{Ax + B}{(x - 1)(x - 2)}$, where A and B are integers, can be written equivalently in the form $\dfrac{p}{x - 1} + \dfrac{q}{x - 2}$ where p and q are integers. Prove the theorem by solving the equation $\dfrac{Ax + B}{(x - 1)(x - 2)} = \dfrac{p}{x - 1} + \dfrac{q}{x - 2}$ for p and q in terms of A and B.

6-10 Using Polynomials with Rational Coefficients

The mathematical description of practical problem situations often involves equations whose members are polynomials with rational coefficients.

EXAMPLE 1 One high-speed computer system can prepare the weekly sales summary of a large company in 10 h. A faster system can do the job in 6 h. If both systems were in operation, how rapidly could the sales summary be prepared?

SOLUTION

1. The problem asks for the number of hours required for the systems to prepare the sales summary together.
2. Let x represent the number of hours for the systems to prepare the sales summary together.

   $\frac{1}{10}$ = rate of the first system (one-tenth of the job in 1 h).
   $\frac{1}{6}$ = rate of the second system (one-sixth of the job in 1 h).
   1 = total work done together (one whole job) in x h.

3. $$\underbrace{\textbf{Total}\text{ work}}_{1}\ \underset{=}{\text{is}}\ \underbrace{\textbf{part}\text{ done by the first}}_{\frac{1}{10}x}\ \underset{+}{\text{plus}}\ \underbrace{\textbf{part}\text{ done by the second.}}_{\frac{1}{6}x}$$

4. $$\mathbf{30} \cdot 1 = \mathbf{30} \cdot \tfrac{1}{10}x + \mathbf{30} \cdot \tfrac{1}{6}x$$
   $$30 = 3x + 5x$$

   Completing Step 4 is left to you to find that working together the systems would require 3 h 45 min. Checking the work (Step 5) is also left to you.

A percent is equivalent to a fraction whose denominator is 100. For example, 41% = $\frac{41}{100}$, or 0.41, and 165% = $\frac{165}{100}$, or 1.65. When you multiply a number called the **base (b)**, by a **percent (r)**, the product is called the **percentage (p)**. The formula $p = br$ is a basic tool in solving many problems in science and business.

EXAMPLE 2 Twenty grams of a 60% solution of alcohol in water is to be diluted by a 42% solution. At most, how many grams of the weaker solution can be added if the resultant solution is to be at least 50% alcohol?

SOLUTION

1. The problem asks for the maximum number of grams of the 42% solution to be added.
2. Let y denote the number of grams of the 42% solution to be added. Then:

   $20 + y$ = the number of grams in the resultant solution
   $0.50(20 + y)$ = 50% of the resultant solution
   $0.42y$ = the amount of alcohol in the added solution
   $0.60(20)$ = the amount of alcohol in the original solution

3. Alcohol in the resultant solution

   $$\overbrace{\underbrace{\text{Alcohol in original solution}}_{0.60(20)}\ \underset{+}{\text{plus}}\ \underbrace{\text{alcohol in added solution}}_{0.42y}}\ \underbrace{\text{is at least}}_{\geq}\ \underbrace{\text{50\% of the resultant solution.}}_{0.50(20 + y)}$$

4. Multiplying each member of the inequality by 100, you find:

$$60(20) + 42y \geq 50(20 + y)$$

By solving this inequality, show that *at most* 25 g of the 42% solution can be added. Check your work.

Problems

A

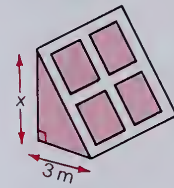
1. At a water pollution control facility one pipe can fill a purification tank in 4 h. Another pipe can fill the tank in 6 h. How many hours will it take to fill the tank if both pipes are working together?
2. In the tank described in Exercise 1, there is a drain which can empty the tank in 60 h. How long will it take to fill the tank if both pipes are working together and the drain is open?
3. Each of two canceling machines can process a bag of mail in 10 min. A third machine can process a bag of mail in 15 min. How long will it take to process 8 bags of mail if all three machines work together?
4. An office has three copying machines, two of which can make a copy in 4 s and one of which can make a copy in 6 s. How long will it take to make 500 copies if all three machines work together?
5. Harriet Smith bicycles at 12 km/h and walks at 5 km/h. What is the minimum distance she must bicycle in order to cover a 22 km course in 3 h or less?
6. How much glacial (pure) acetic acid must be added to 180 g of a solution that is 35% acetic acid to produce a solution that is 50% acetic acid?
7. How much of a 30% salt solution should be added to 200 g of a 12% salt solution to produce a 20% salt solution?
8. Kevin Crawford took a business trip by car and averaged 50 km/h going, but only 40 km/h coming back because he took a different route that was 10 km shorter. If he stopped for 2 h at his destination and took 8.5 h for the trip, what was the distance by his original route?

B

9. How many full minutes elapse after 4:00 before the hour and minute hands of a clock coincide?
10. Each of two road graders can pave a certain section of road in 15 h working alone. After the two machines have been working for 2 h they are joined by an older model grader that would take 20 h to pave the section of road by itself. How long after the first two machines began working will it take the three machines to pave the section of road?

11. At 10:00 A.M. a dentist's car leaves the Euclidville National Bank heading north on Route 5, at 60 km/h. At 10:10 A.M. a police car starting from a point 5 km south of the bank on Route 5 gives chase at 100 km/h to deliver an emergency message. How far from the bank will the police car overtake the dentist's car?

C 12. A lean-to greenhouse extending 3 m from a wall, as shown, is to have one end and four-fifths of its square slanting side made from transparent plastic. What is the tallest it can be if enough plastic is available to cover 9.9 m²?



6-11 Fractional Equations

An equation involving one or more rational expressions in which a variable appears in the denominator is called a **fractional equation.**

To solve the fractional equation

$$1 + \frac{30}{a^2 - 9} - \frac{5}{a - 3} = 0, \quad (1)$$

you can begin by multiplying both members of the equation by the LCD, $(a + 3)(a - 3)$, or $a^2 - 9$:

$$(a^2 - 9)\left(1 + \frac{30}{a^2 - 9} - \frac{5}{a - 3}\right) = (a^2 - 9) \cdot 0$$

Then you have:

$$\begin{aligned} a^2 - 9 + 30 - 5(a + 3) &= 0 \\ a^2 + 21 - 5a - 15 &= 0 \\ a^2 - 5a + 6 &= 0 \quad (2) \\ (a - 3)(a - 2) &= 0 \\ a = 3,\ a &= 2 \end{aligned}$$

Checking the solution set $\{3, 2\}$ of Equation (2) in the original Equation (1), we have:

$$1 + \frac{30}{3^2 - 9} - \frac{5}{3 - 3} \stackrel{?}{=} 0 \qquad\qquad 1 + \frac{30}{2^2 - 9} - \frac{5}{2 - 3} \stackrel{?}{=} 0$$

$$1 + \frac{30}{0} - \frac{5}{0} \stackrel{?}{=} 0 \qquad\qquad 1 + (-6) - (-5) \stackrel{?}{=} 0$$

$$0 \stackrel{?}{=} 0$$

Since $a = 3$ produces zero divisors, 3 is not an admissible root of Equation (1). Hence, although the solution set of Equation (*2*) is $\{3, 2\}$, the solution set of Equation (*1*) is simply $\{2\}$.

Thus you can see that when you transform a fractional equation by multiplying both members by their LCD, the resulting equation is not

necessarily equivalent to the original one. The solution set of the transformed equation will, however, include all the roots of the original equation. *Always check the roots back in the original equation* to see which ones are admissible.

EXAMPLE A plane is to seed a field measuring 1 km wide and 1.2 km long. If the ratio of the time it takes to fly the length of the field with a 10 km/h tail wind to the time required to fly back against that same wind is 8:9, what is the speed of the plane in still air?

SOLUTION

1. The problem asks for the speed of the plane in still air.
2. Let r = speed of the plane in still air.
   $r + 10$ = speed of the plane with tailwind.
   $r - 10$ = speed of the plane against tailwind.
   t = time to fly the length with the wind.
   t' = time to fly the length against the wind.

   Then $\dfrac{t'}{t} = \dfrac{9}{8}$.
3. $$\underbrace{\text{Time} \times \text{rate with wind}}_{t(r+10)} \overset{}{\underset{\downarrow}{=}} \underbrace{\text{distance}}_{1.2} \underset{\downarrow}{=} \underbrace{\text{time} \times \text{rate against wind}}_{t'(r-10)}$$
   $$t(r + 10) = 1.2 = t'(r - 10)$$
4. Dividing both members of $t(r + 10) = t'(r - 10)$ by $t(r - 10)$, you obtain:
   $$\frac{r+10}{r-10} = \frac{t'}{t} = \frac{9}{8}$$
   Multiply both members of $\dfrac{r+10}{r-10} = \dfrac{9}{8}$ by $8(r - 10)$:
   $$8(r + 10) = 9(r - 10)$$
   $$r = 170$$
5. Check the solution.

$\therefore$ the speed of the plane in still air is 170 km/h. Answer.

Oral Exercises

State the least common denominator of the terms of the equations and any restrictions on the variable.

1. $\dfrac{3}{4x} - \dfrac{1}{3x} = 5$

2. $\dfrac{1}{2} + \dfrac{7}{4a} = \dfrac{15}{4a^2}$

3. $\dfrac{4}{y} = \dfrac{6}{5y} + 2$

4. $\dfrac{2}{5} = \dfrac{3d - 2}{4d + 1}$

5. $\dfrac{3z}{z + 1} - \dfrac{2}{z - 1} = 0$

6. $\dfrac{4x}{x - 2} - \dfrac{12}{x^2 - 4} - \dfrac{3}{x + 2} = 0$

Written Exercises

Determine the solution set over $\mathcal{R}$. Check all apparent solutions.

A **1–6.** Solve Oral Exercises 1–6.

7. $\dfrac{4t^2}{t^2-9} - \dfrac{2t}{t+3} = \dfrac{3}{t-3}$

8. $\dfrac{b+4}{b} + \dfrac{3}{b-4} = \dfrac{-16}{b^2-4b}$

9. $\dfrac{12}{a^2-1} - 2 = \dfrac{3a}{a-1}$

10. $\dfrac{r-2}{2} = \dfrac{r^2+8}{2r-10} - \dfrac{r^2-3}{r-5}$

11. $p - \dfrac{1}{2} = \dfrac{5+p^2}{3p-6}$

12. $1 - \dfrac{n}{2n+1} - \dfrac{20}{4n^2-1} = 0$

B **13.** $\dfrac{n}{n-5} + \dfrac{17}{25-n^2} = \dfrac{1}{n+5}$

14. $\dfrac{1-2k}{1-k} - \dfrac{3k}{2k+1} = \dfrac{k^2+11}{2k^2-k-1}$

15. $\dfrac{y+3}{y+2} = 2 - \dfrac{3}{y^2+5y+6}$

16. $\dfrac{1}{x-3} - \dfrac{2x+3}{(x-3)^2} = \dfrac{-10}{(x-3)^3}$

17. $\dfrac{c-2}{c^2-3c} + \dfrac{8-c}{c^2-5c+6} = \dfrac{2c+2}{c^2-2c}$

18. $\dfrac{k-3}{k^2-k-2} + \dfrac{2k-7}{k^2-3k+2} = \dfrac{k}{k^2-1}$

19. $\dfrac{2z}{z-1} - \dfrac{6z}{z^3-1} - \dfrac{3}{z^2+z+1} = 2$

20. $\dfrac{-42m}{m^3+8} + \dfrac{3m}{m+2} = 3$

C **21.** $\dfrac{2x+y}{x-y} = \dfrac{3}{4}$

$\dfrac{x}{y} = -2$

22. $\dfrac{-1}{x} + \dfrac{3}{y} = \dfrac{4}{xy}$

$\dfrac{3}{x} - \dfrac{2}{y} = \dfrac{9}{xy}$

23. $\dfrac{2}{x-y} + \dfrac{3}{x+y} = \dfrac{3-2y}{x^2-y^2}$

$\dfrac{3}{x+y} - \dfrac{1}{x-y} = \dfrac{11-3y}{x^2-y^2}$

24. $\dfrac{3}{2x+5} = \dfrac{4}{y-1}$

$\dfrac{2}{3x-1} = \dfrac{-4}{y+3}$

25. $\dfrac{x+2}{x+1} - \dfrac{x+3}{x+2} = \dfrac{x+4}{x+3} - \dfrac{x+5}{x+4}$

26. $\dfrac{-3}{2x+y} = \dfrac{6}{x+3y}$; $\dfrac{5}{x+2y} = \dfrac{10}{2xy}$

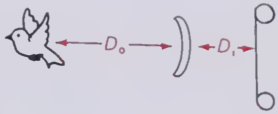
Problems

A **1.** If a flight that is being chartered for \$19,200 could acquire 4 more passengers, each one would pay \$20 less for the flight. How many passengers are there?

2. One worker can assemble a computer module in 30 min working alone. With a second, faster worker assisting, the job can be done in 12 min. How long would it take the second worker to assemble the module working alone?

3. If two resistors of resistances R_1 and R_2 are connected in parallel, the total resistance R of the circuit is given by the equation $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}$. If R_2 is 40 Ω greater than R_1 and the total resistance of the circuit is 15 Ω, what is R_1?

4. Every camera lens has a characteristic measurement f, called the focal length, such that when an object is in focus, its distance D_0 from the lens and the distance D_i from the lens to the film satisfy the equation $\frac{1}{D_0} + \frac{1}{D_i} = \frac{1}{f}$. If $D_0 = 90$ cm and D_i is 3 cm greater than the focal length, what is the focal length of the lens?



5. Dave increased his running speed by 40 m/min and completed a 3600 m course in 1 min less than his previous time. What was his original speed?

6. Sue rowed the first 1200 m of a race at a constant speed. For the next 2600 m of the race she increased her speed by 5 m/min, and for the last 1200 m of the race she returned to her original speed. If she completes the race in 1 h 20 min, what was her original speed?

7. Marvin Fine received a 25¢ per hour raise for his part-time job. He can now earn his former weekly salary of \$33 in 1 h less of working time. How many hours does he work each week?

8. Ten minutes after Jane Bennett left for school on her bicycle, her brother noticed that she had left her baseball glove at home and started after her on his moped. If the moped averages 4 km/h faster than the bicycle, and Jane's brother overtook her after 8 km, how fast does Jane bicycle?

B

9. The college crew team can maintain a speed equivalent to 20 km/h in still water. If it takes the team 32 min to row 5 km upstream and back, what is the rate of the current in the river? (*Hint:* Express the time in hours.)

10. A car averages 2 km/L less when hauling a trailer than when traveling by itself. On a trip requiring 25 L of gas, it hauled a trailer for 90 km and traveled 80 km by itself. What is the car's average fuel consumption when traveling by itself?

11. A canoeist can paddle 12 km upstream and 12 km back downstream in the same amount of time as she can paddle 25 km in still water. If the rate of the current is 2 km/h, what is her rate in still water?

12. A day laborer takes three more days to paint a house than an apprentice painter. A master painter takes three days fewer than the apprentice. The master painter can do as much work in seven days as the day laborer and the apprentice working together can accomplish in six days. How long would it take the apprentice alone to paint the house?

C 13. In a certain city a birth occurs on the average every 24 min and a death every half hour. A resident moves out of the city every 1.5 h, and a new person moves into the city every 4.5 h. How long does it take on the average for the population to increase by 1 person?

Self-Test 3

VOCABULARY

rational algebraic expression (p. 192)
fraction in lowest terms (p. 192)
least common denominator (p. 199)
base (p. 202)
percent (p. 202)
percentage (p. 202)
fractional equation (p. 204)

1. Simplify $\frac{4y^3 - 36y}{2y^2 + 6y}$. *Obj. 1, p. 191*

2. Transform $\frac{6x^3 - 13x^2 + 3x - 20}{2x - 5}$ into a sum by dividing. *Obj. 2, p. 191*

3. Express $\frac{a^2 - a - 2}{(a - 2)^2} \div \frac{a^3 + a^2}{2a^2 - 8}$ in lowest terms. *Obj. 3, p. 191*

4. Transform $\frac{x}{x - 4} - \frac{16}{x^2 - 16} - \frac{2}{x + 4}$ into an equivalent rational expression in lowest terms. *Obj. 4, p. 191*

5. Solve $\frac{8}{9 - x^2} - \frac{x}{3 - x} - 2 = 0$. *Obj. 5, p. 191*

6. Each of three steam rollers can smooth a section of road in 21 h working alone. When these three are joined by a faster steam roller, the section of road can be smoothed in 5 h. How long would it take the faster steam roller to smooth the section of road working alone?

Check your answers with those printed at the back of the book.

Chapter Summary

1. Laws for working with positive integral exponents are extended to any integral exponent by defining b^0 and $b^{-n}(b \neq 0, n > 0)$:

$$b^0 = 1, \text{ and } b^{-n} = \frac{1}{b^n}.$$

See page 172 for a list of the laws of exponents.

2. Two polynomials may be multiplied by multiplying each term of one of the polynomials by each term of the other and then adding all the products.

3. A polynomial is *reducible* if it can be expressed as a product of two or more polynomials of lower positive degree; otherwise, it is *irreducible*. The *greatest monomial factor* of a polynomial is the monomial with greatest numerical coefficient and greatest degree that is a factor of each term of the polynomial. A polynomial is said to be *factored completely* when it is expressed as a product of factors each of which is a constant and one or more irreducible polynomials whose greatest monomial factor is 1.

4. You may use the fact that a product of real numbers is zero if and only if at least one of the factors is zero to solve polynomial equations. For example, the quadratic polynomial in the equation

$$3x^2 + x - 4 = 0$$

may be factored to give the equivalent equation

$$(3x + 4)(x - 1) = 0.$$

Then it is a simple matter to set the factors equal to zero to determine that the solution set is $\{-\frac{4}{3}, 1\}$.

5. You may use the following fact to help solve inequalities by factoring. For a and $b \in \mathcal{R}$, $ab > 0$ if and only if a and b are of the same sign, and $ab < 0$ if and only if a and b are of opposite signs.

6. To reduce a rational expression to *lowest terms,* you can factor the numerator and denominator completely and then divide both by all their common factors.

7. By using the division algorithm, you can transform a nonzero rational expression into the sum of a polynomial and a rational expression in which the degree of the numerator is less than that of the denominator.

8. Operations can be performed with rational expressions by using the corresponding rules for operations with rational numbers.

Chapter Review

1. Give an expression equivalent to $(5x^2y^{-3})(7xy^5)$. 6-1

   a. $35xy$ b. $-35x^3y$ c. $35x^3y^2$

2. Express the product $(2x + 5y)^2$ in simple form. 6-2

   a. $4x^2 + 25y^2$ b. $4x^2 + 20xy + 25y^2$ c. $4x^2 + 10xy + 25y^2$

3. Factor the polynomial $2x^2 - 10x + 12$ completely. 6-3

   a. $(2x - 6)(x - 2)$ b. $2(x - 3)(x - 2)$ c. $2(x - 3)(x + 2)$

4. Solve $x^2 + 5x - 24 = 0$. 6-4

   a. $\{3, 8\}$ b. $\{-8, -3\}$ c. $\{3, -8\}$

5. Find the solution set of the inequality $x^2 + 3x - 28 < 0$ over $\mathcal{R}$. 6-5

   a. $\{x\colon x > 4 \text{ or } x < -7\}$ b. $\{x\colon -7 < x < 4\}$ c. $\{x\colon x > 7\}$

6. Simplify $(30a^5b - a^2)2a^{-2}$. 6-6

   a. $60a^3b - 1$ b. $60a^3 - 2$ c. $60a^3b - 2$

7. Transform $\dfrac{2x^2 + x - 12}{x - 2}$ into a sum by dividing. 6-7

   a. $2x + 3$ b. $2x + 5 - \dfrac{2}{x - 2}$ c. $2x + 5 + \dfrac{2}{x - 2}$

8. Express $\dfrac{y^2 - y - 6}{2y + 10} \cdot \dfrac{y^2 - 25}{y - 3}$ in lowest terms. 6-8

   a. $\dfrac{y^2 - 3y - 10}{2}$ b. $\dfrac{y^2 - 3y + 10}{2}$ c. $\dfrac{y^2 - 25}{y + 2}$

9. Simplify $\dfrac{5}{k^2 - 4} + \dfrac{2}{k + 2}$. 6-9

   a. $\dfrac{7k - 6}{k^2 - 4}$ b. $\dfrac{k^2 - 2}{k^2 - 4}$ c. $\dfrac{2k + 1}{k^2 - 4}$

10. The average of two numbers is 20, and one of the numbers is 2 less than $\frac{3}{4}$ of the other. Letting x represent the greater number, write an equation to solve the problem. 6-10

    a. $x + \dfrac{3}{4}x = 20$ b. $\dfrac{x + \left(\frac{3}{4}x - 2\right)}{2} = 20$ c. $\dfrac{1}{2}\left(\dfrac{3}{4}x\right) = 20$

11. Solve $\dfrac{8}{x^2 - 9} - \dfrac{1}{x - 3} = 0$ over $\mathcal{R}$. 6-11

    a. $\{1\}$ b. $\{3\}$ c. $\{2\}$ d. $\{5\}$

Chapter Test

Write as an equivalent expression containing only positive exponents. Assume no variable equals zero.

1. $(4x^4y^{-1}z)(-3x^{-2}y^{-2}z^5)$ 2. $\dfrac{x^5y^{-2}}{x^3y^{-2}z^{-3}}$ 6-1

3. Write the product $(2x^2 + 3x + 9)(x - 3)$ in simple form. 6-2

Factor completely.

4. $x^2 + 14xy - 51y^2$ 5. $24x^3 - 3$ 6-3

Solve over $\mathcal{R}$ by factoring.

6. $y^2 - 4y - 45 = 0$ 7. $x^3 - 25x = 0$ 6-4

Find the solution set of the given inequality over $\mathcal{R}$ and graph its solution set.

8. $x^2 - 8 > 2x$ 9. $x^2 - 7x \leq -10$ 6-5

Simplify.

10. $\dfrac{k^2 + 10k + 25}{k^3 + 125}$ 11. $\dfrac{x^3 + 7x^2 + 12x}{2x^2 + 16x + 32}$ 6-6

12. Express $\dfrac{x^3 - 3x^2 + 6}{x + 2}$ as a sum by dividing. 6-7

Express in lowest terms.

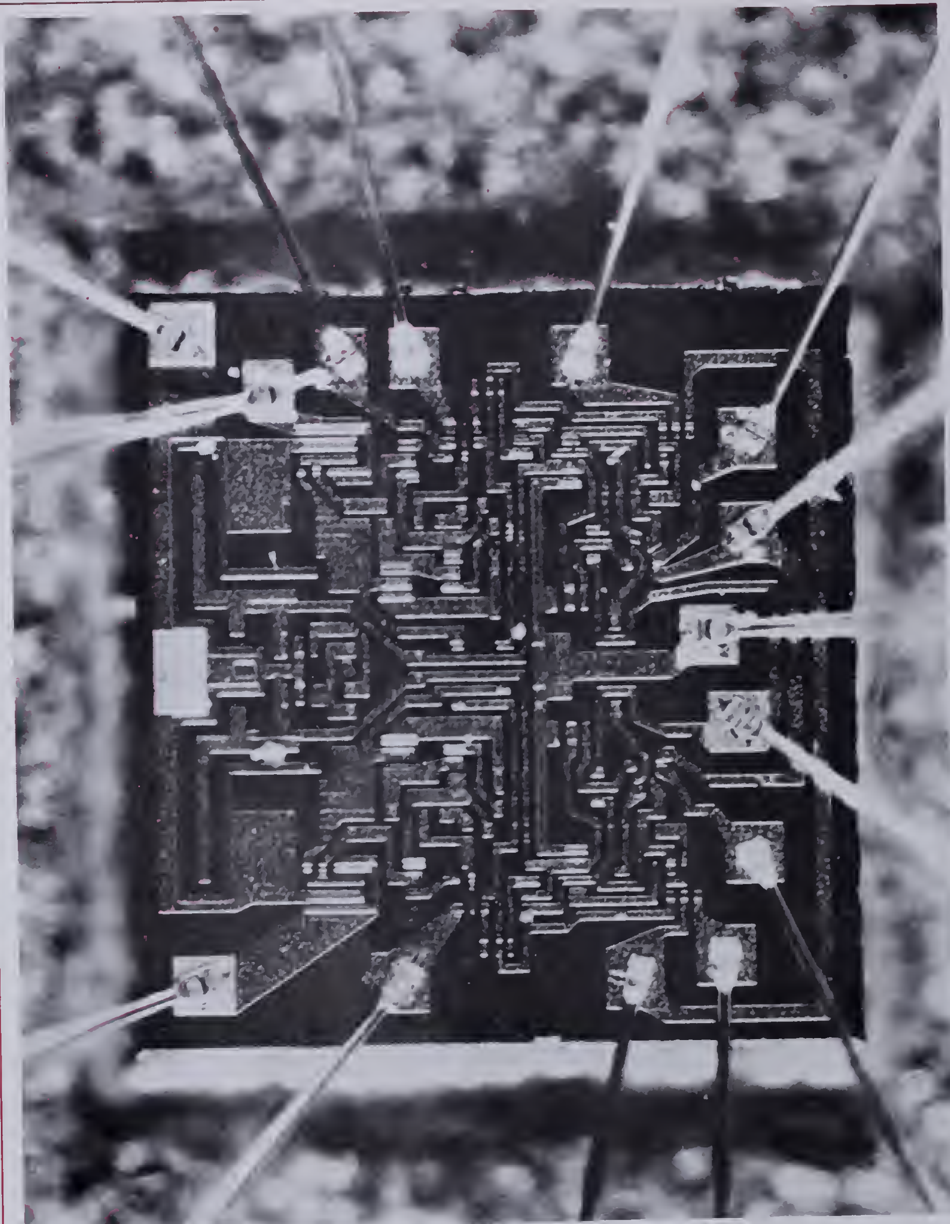
13. $\dfrac{x^2 - y^2}{x^2 - xy} \cdot \dfrac{2xy}{x + y}$ 14. $\dfrac{x^4 - 16}{x + 2} \div \dfrac{x^2 + 4}{7}$ 6-8

Simplify.

15. $1 + \dfrac{1}{x - 5} - \dfrac{2x}{3x - 15}$ 16. $\dfrac{-x}{x^2 - 6x + 9} + \dfrac{2}{x - 3}$ 6-9

17. The sum of $\frac{1}{2}$ a number and $\frac{1}{3}$ the sum of the number and 7 equals 9. Find the number. 6-10

18. Solve $\dfrac{(x - 1)^2}{x} + \dfrac{2x - 1}{x} = 2$ over $\mathcal{R}$. 6-11



This microscopic chip holds between eight and sixteen thousand "bytes" of memory. Each byte consists of eight "bits," which are pieces of information in binary form.

7 Sequences and Series

Arithmetic Sequences and Series

***OBJECTIVES** for Sections 7-1 through 7-3:*

1. ***Determine an arithmetic sequence when the first term and a rule for computing each successive term from the preceding term are given.***
2. ***Find a specified term of an arithmetic sequence when two terms, or one term and the common difference, are given.***
3. ***Find the sum of a given arithmetic series.***
4. ***Solve practical problems involving arithmetic sequences and series.***

7-1 Arithmetic Sequences

Sue had $18 in her savings account when she joined the payroll savings plan. She decided to have $20 of her weekly salary deposited in her savings account. Starting with the initial amount, and continuing through five successive deposits, the number of dollars in the account formed the following *sequence:*

$$18, 38, 58, 78, 98, 118.$$

The numbers in a sequence are called the **terms** of the sequence. In the sequence given above, the first term is 18, the second is 38, and so on. A sequence which has a last term is called a **finite sequence**. A sequence which you think of as continuing forever, such as

$$18, 38, 58, 78, 98, 118, \ldots,$$

has no last term and is called an **infinite sequence**.

The sequences shown on page 213 are called *arithmetic* (pronounced ar-ith-*met*-ic in this usage) *sequences*. An **arithmetic sequence,** or an **arithmetic progression,** is any sequence in which each term after the first is obtained by adding a fixed number, called the **common difference,** to the preceding term. In the sequence 18, 38, 58, 78, 98, 118, the common difference is 20. The terms in an arithmetic sequence are said to be **in arithmetic progression.**

EXAMPLE 1 The first three terms of an arithmetic sequence are $-7, -3, 1$. What are the common difference and the fourth term of this sequence?

SOLUTION 1. To find the common difference, subtract any term from its successor.

$$(-3) - (-7) = 4 \qquad [\text{or } 1 - (-3) = 4]$$

$\therefore$ the common difference is **4**. Answer.

2. To find the fourth term, add the **common difference** to the third term.

$$1 + 4 = 5$$

$\therefore$ the fourth term is 5. Answer.

To refer to the terms of any sequence, you often use subscript notation. For instance, you might call the first term of a sequence a_1, the second term a_2, and so on, with a_n denoting the nth term.

You can use this subscript notation to write a rule for forming the successive terms of an arithmetic sequence. In the sequence 18, 38, 58, 78, 98, 118, . . . ,

$$a_1 = 18, \qquad a_2 = 38 = a_1 + 20, \qquad a_3 = a_2 + 20.$$

In general:

> If the first term of an arithmetic sequence is a_1, and the common difference is d, then the successive terms are obtained from the rule
>
> $$a_{n+1} = a_n + d, \qquad n = 1, 2, 3, \ldots.$$

EXAMPLE 2 Name the first term and give a rule for finding the successive terms in the arithmetic sequence 23, 19, 15, ...

SOLUTION $a_1 = 23$

$d = a_2 - a_1 = 19 - 23 = -4 \qquad [\text{or} \quad d = a_3 - a_2 = 15 - 19 = -4]$

Therefore, successive terms are computed according to the rule

$$a_{n+1} = a_n + (-4) = a_n - 4. \quad \text{Answer.}$$

Oral Exercises

State the second and third terms of the arithmetic sequence whose first term and common difference are given.

1. $a_1 = 3, d = 2$
2. $a_1 = 17, d = 4$
3. $a_1 = 18, d = -3$
4. $a_1 = -1, d = -5$
5. $a_1 = 0, d = 4$
6. $a_1 = -9, d = 10$
7. $a_1 = 7, d = \frac{1}{2}$
8. $a_1 = 5, d = x$
9. $a_1 = 0, d = 0.2$

Tell whether each of the following is an arithmetic sequence. If it is, give the next term.

10. $1, 3, 5, 7, \ldots$
11. $1, 4, 9, 16, \ldots$
12. $1, 2, 4, 8, \ldots$
13. $6, 3, 0, -3, \ldots$
14. $-3, 2, 7, 12, \ldots$
15. $-2, -3, -5, -8, \ldots$

In each of the following, three consecutive terms of an arithmetic sequence are given. State the terms that immediately precede and immediately follow the three given terms.

16. $\ldots, 3, 7, 11, \ldots$
17. $\ldots, -2, -5, -8, \ldots$
18. $\ldots, 10, 60, 110, \ldots$
19. $\ldots, 0, -2, -4, \ldots$
20. $\ldots, 1, -1, -3, \ldots$
21. $\ldots, -11, -6, -1, \ldots$
22. If k is a real number, is the sequence $k, k, k, k, \ldots$ an arithmetic progression? Explain.

Written Exercises

Give the second, third, and fourth terms of an arithmetic sequence with the given first term and common difference.

A

1. $a_1 = -11, d = 10$
2. $a_1 = 5, d = -2$
3. $a_1 = -8, d = -4$
4. $a_1 = 0, d = \frac{3}{2}$
5. $a_1 = -1, d = \frac{1}{2}$
6. $a_1 = \frac{5}{2}, d = -2$

Give the first four terms of a sequence determined by the given first term and a rule for finding successive terms. If the sequence is an arithmetic progression, give the common difference.

7. $a_1 = 3, a_{n+1} = 3a_n$
8. $a_1 = -5, a_{n+1} = a_n + 7$
9. $a_1 = 2, a_{n+1} = (a_n)^2$
10. $a_1 = -2, a_{n+1} = a_n - 10$
11. $a_1 = 4, a_{n+1} = a_n + 2k$
12. $a_1 = 7, a_{n+1} = 7 - a_n$

Give a rule for finding successive terms of the following sequences. Tell whether or not the sequence is an arithmetic progression.

13. $15, 18, 21, 24$
14. $48, 24, 12, 6$
15. $5, -10, 20, -40$
16. $8, -2, -12, -22$
17. $-\frac{1}{2}, 3, \frac{13}{2}, 10$
18. $\frac{3}{16}, \frac{3}{8}, \frac{3}{4}, \frac{3}{2}$
19. a, ab, ab^2, ab^3
20. $a, a + k^2, a + 2k^2, a + 3k^2$

B 21. Show that if a_1, a_2, and a_3 are the first terms of an arithmetic progression, then $a_2 = \dfrac{a_1 + a_3}{2}$.

22. Show that if a_1, a_2, a_3, and a_4 are the first four terms of an arithmetic progression, then $\dfrac{2a_1 + a_4}{3} = a_2$.

In Exercises 23–26 find $a_2 - a_1$ and $a_3 - a_2$, and determine whether a_1, a_2, and a_3 are in arithmetic progression for all values of the variables that are without subscripts.

23. $a_1 = b - 4k$, $a_2 = b - 3k$, $a_3 = b - 10k$

24. $a_1 = 1 + r + 2r^2$, $a_2 = 1 - r + r^2$, $a_3 = 1 - 3r$

25. $a_1 = cd$, $a_2 = c(d - c)$, $a_3 = d(c - d)$

26. $a_1 = (x + y)^2$, $a_2 = x^2 + y^2$, $a_3 = (x - y)^2$

C 27. Give a rule of succession for the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . and give the next three terms.

28. Find x if it is known that the sequence $2x - 1, 5x - 3, 4x + 3$ is an arithmetic progression.

7-2 Arithmetic Means

To determine a particular term of an arithmetic sequence, $a_1, a_2, a_3, \ldots$, such as a_{20}, it is not necessary to compute each preceding term. Notice that you have:

$$\begin{aligned}
&a_1\\
&a_2 = a_1 + d = a_1 + 1d\\
&a_3 = a_2 + d = (a_1 + d) + d = a_1 + (d + d) = a_1 + 2d\\
&a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + (2d + d) = a_1 + 3d
\end{aligned}$$

and so on. This suggests the following fact:

> The nth term of an arithmetic sequence whose first term is a_1 and whose common difference is d is
>
> $$a_n = a_1 + (n - 1)d.$$

EXAMPLE 1 Find the twentieth term in the arithmetic sequence: $-3, 2, 7, \ldots$

SOLUTION $a_1 = -3$, $d = 7 - 2 = 5$. Since $a_n = a_1 + (n - 1)d$,

$$a_{20} = -3 + (20 - 1) \cdot 5 = -3 + 19 \cdot 5 = 92.$$

$\therefore a_{20} = 92.$ Answer.

The terms between two given terms of an arithmetic sequence are called **arithmetic means** between the given terms. For example, the set of three arithmetic means between 18 and 98 is {38, 58, 78}, because

$$18, \mathbf{38}, \mathbf{58}, \mathbf{78}, 98$$

is an arithmetic sequence.

A single arithmetic mean inserted between two numbers is the **average,** or *the* **arithmetic mean,** of the two numbers.

EXAMPLE 2 Find the four arithmetic means between the terms 5 and 25.

SOLUTION You can draw a diagram of the part of the sequence from the term 5 to the term 25 as follows:

$$5, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, 25$$

First, determine d for the sequence whose first term is 5 and whose sixth term is 25. Replacing a_n with **25,** a_1 with **5,** and n with **6** in $a_n = a_1 + (n - 1)d$, you find:

$$\begin{aligned} 25 &= 5 + (6 - 1)d \\ 25 &= 5 + 5d \\ d &= 4 \end{aligned}$$

The required means are found by successive additions of 4:

$$5, \mathbf{9}, \mathbf{13}, \mathbf{17}, \mathbf{21}, 25$$

$\therefore$ the four arithmetic means are 9, 13, 17, 21. Answer.

EXAMPLE 3 What is the first term of an arithmetic sequence whose fifth term is 2 and whose ninth term is 8?

SOLUTION You can draw a diagram of the sequence:

$$\underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, 2, \underline{\quad}, \underline{\quad}, \underline{\quad}, 8$$

1. To find the common difference, consider the part of the sequence beginning with the term 2; in this sequence, the *fifth* term is then 8, and so:

$$\begin{aligned} a_5 &= a_1 + (5 - 1)d \\ 8 &= 2 + 4d \\ d &= 1\tfrac{1}{2} \end{aligned}$$

2. To find the first term of the original sequence, you can use:

$$\begin{aligned} a_5 &= a_1 + (5 - 1)d \\ 2 &= a_1 + (4)(1\tfrac{1}{2}) \\ 2 &= a_1 + 6 \\ a_1 &= -4 \end{aligned}$$

$\therefore$ the first term of the given sequence is -4. Answer.

Oral Exercises

For each of the following state the values that you would use for a_1, $n - 1$, and d in order to find the specified term of the given arithmetic sequence by means of the formula on page 216. Do not compute the value of a_n.

1. $7, 5, 3, \ldots; a_{15}$

2. $4, 104, 204, \ldots; a_{12}$

3. $-7, -2, 3, \ldots; a_{21}$

4. $-2, -17, -32, \ldots; a_{13}$

5. $-3, -\frac{1}{2}, 2, \ldots; a_{29}$

6. $0.2, 0.15, 0.1, \ldots; a_{18}$

7. $\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}, \ldots; a_{17}$

8. $1.08, 1.14, 1.2, \ldots; a_{26}$

State the arithmetic mean of the given numbers.

9. 5 and 7

10. 0 and 22

11. -3 and 7

12. -1 and -13

13. 7 and -17

14. 3.5 and 6.5

15. -2.5 and 8.5

16. $\frac{1}{7}$ and $\frac{5}{7}$

Written Exercises

A **1–8.** Find the specified term of the arithmetic sequences given in Oral Exercises 1–8.

In Exercises 9–20, find the requested value, using the given values for an arithmetic sequence.

9. $a_1 = 3, d = 7, a_{101} = ?$

10. $a_1 = 9, a_{15} = 135, d = ?$

11. $a_{21} = 336, d = 17, a_1 = ?$

12. $a_1 = 110, a_{26} = -65, d = ?$

13. $a_1 = -3, a_{18} = -88, d = ?$

14. $a_1 = \frac{1}{2}, d = -\frac{1}{2}, a_{31} = ?$

15. $a_{30} = 62, d = 3, a_1 = ?$

16. $a_{49} = -12, d = -3.5, a_1 = ?$

17. $a_1 = 21, d = -8, a_n = -99, n = ?$

18. $a_1 = 12, d = -\frac{1}{3}, a_n = 5, n = ?$

19. $a_1 = -46, d = \frac{3}{2}, a_n = -4, n = ?$

20. $a_1 = -0.8, d = 2.6, a_n = 98, n = ?$

Insert the stated number of arithmetic means between the given numbers.

21. Three, between 8 and 60

22. Five, between -2 and 16

23. Three, between 9 and 23

24. Seven, between 3 and -3

25. Nine, between -1 and -5

26. Four, between $\frac{3}{5}$ and $\frac{7}{5}$

Given the specified values for the terms of an arithmetic sequence, find d and a_1.

27. $a_{10} = 7, a_{14} = 15$

28. $a_6 = 5, a_{12} = -13$

29. $a_{13} = 8, a_{17} = 48$

30. $a_5 = 1, a_9 = 7$

31. $a_7 = -\frac{1}{2}, a_{10} = -1$

32. $a_4 = -5, a_{10} = 22$

In Exercises 33–36, find x under the given conditions.

B 33. The arithmetic mean of $x + 1$ and $4x + 3$ is 7.

34. The arithmetic mean of $2x - 1$ and $x + 9$ is -2.

35. The arithmetic mean of $x - 8$ and $2x + 1$ is x.

36. The arithmetic mean of $4x + 1$ and $x - 9$ is $x + 2$.

C 37. Prove that if a, c, b is an arithmetic sequence, then $c = \frac{1}{2}(a + b)$. Thus, the arithmetic mean of a and b is $\frac{1}{2}(a + b)$.

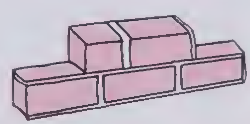
38. Prove that in an arithmetic sequence, a_n is the arithmetic mean of a_1 and a_{2n-1}, for any integer n greater than 1.

Problems

A 1. The Joneses have a variable-rate mortgage loan from a savings bank, which has a monthly payment of \$460 during the first year. If the monthly payment were to increase by \$25 per year for each succeeding year of the life of the loan, what would their monthly payment be during the twelfth year?

2. The Walkers' tomato patch produces 14 tomatoes on the first day of the season and 9 more on each succeeding day. How many tomatoes will the plants yield on the fifteenth day of the season?

3. Each row of bricks on the gable end of a house is made up of $1\frac{1}{2}$ fewer bricks than the preceding row. If the bottom row of bricks is made up of 67 bricks, how many bricks will be needed in the sixteenth row? If the topmost row consists of a single brick, how many rows will there be?



4. Between March 1 and March 31, the sunrise at 40° North Latitude occurs about 1.6 min earlier each day than the preceding day. If the sun rose at 6:33 A.M. on March 1, at what time did it rise on March 21? On what day did the sun rise at 5:53 A.M.?

5. Susan owns 100 shares of the HYM Company. Each share is worth \$35. Susan expects the value of a share to increase by \$1.80 each year. Estimating the annual dividend to be 4% of the value of the share, estimate the total annual dividend Susan will receive twenty years from now.

6. An object given an initial upward velocity *loses* 9.8 m/s of that velocity every second, or *gains* 9.8 m/s every second in a downward direction. If an object has an upward velocity of 49 m/s at the end of 1 s, what will be its velocity at the end of the 7 s? After how many seconds will it have a *downward* velocity of 49 m/s?

B 7. Each year, John Harris spends \$3400 in mortgage loan payments for his garage. At the end of the first year of operation, the garage produced a gross income of \$12,000. In the following years, this income increased by \$800 per year. After how many years will the difference between the gross income and the loan payment reach \$19,000?

8. During a laboratory observation period it is found that the diameter of a tree increases the same amount each year. If the diameter was 61 mm at the end of the sixth year and 76 mm at the end of the tenth year, what was it at the end of the first year?

9. In its original form, the width of the Great Pyramid at Giza decreased by 1.57 m for each successive meter of height. If the width was 229.22 m measured at a height of 1 m, at what height was the width 103.62 m? How high was the Great Pyramid?

10. In a laboratory experiment the temperature of a test sample rises a constant amount each minute. If the temperature was 7° after 13 min and 15° after 19 min, what was it after 1 min?

11. A building company finds that its profit from the sale of the first condominium apartment in a new building is −\$1000 (a loss of \$1000). The company's profit on the sale of the second unit is −\$300. On selling more units within the building, the company finds that the profit increases by \$700 on each additional unit sold. How many units did the company sell if the profit on the last unit was \$3200?

12. Jane Evans drove her car 8400 km during the first year she owned it. In each successive year, she drove the car 750 km more than she did the previous year. During the first year, the car cost 12¢ per kilometer to run; the cost per kilometer increased by 2¢ each year. How much did it cost Jane to run her car in its seventh year?

7-3 Arithmetic Series

From the terms of any given sequence, such as

$$4,\ 7,\ 10,\ 13,\ 16,$$

you can construct an associated sequence S_1, S_2, S_3, S_4, S_5 of *sums:*

$$\begin{aligned}
S_1 &= 4\\
S_2 &= 4 + 7 = 11\\
S_3 &= 4 + 7 + 10 = 21\\
S_4 &= 4 + 7 + 10 + 13 = 34\\
S_5 &= 4 + 7 + 10 + 13 + 16 = 50
\end{aligned}$$

Each of the indicated sums S_1, S_2, S_3, S_4, and S_5 is called a *series* associated with the sequence 4, 7, 10, 13, 16.

In general, given any sequence

$$a_1, a_2, a_3, \ldots$$

with n or more terms, the associated **series** of n terms, S_n, is

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

The Greek letter **Σ (sigma),** called the **summation sign,** is used to abbreviate the writing of a series. For example, to abbreviate the series

$$S_5 = 4 + 7 + 10 + 13 + 16,$$

first observe that 4, 7, 10, 13, 16 is an arithmetic sequence with nth term, or general term, $4 + (n - 1)3$. Therefore, the series can be denoted by the symbol:

$$\sum_{n=1}^{5} [4 + (n - 1)3]$$

(read "the summation of $4 + (n - 1)3$ from $n = 1$ to $n = 5$"). This means that you successively replace n with 1, 2, 3, 4, and 5, and then write an expression denoting the sum of the resulting values. The letter n is called the **index** (plural, **indexes** or **indices**) and the replacement set of n is the **range of summation.** Note that any letter, such as i, j, k, m, or n, may be chosen as the index.

EXAMPLE 1 Write $\sum_{k=0}^{5} 5k$ in expanded form.

SOLUTION Replace k with the numerals 0, 1, 2, 3, 4, and 5, in turn and write the sum.

$$\sum_{k=0}^{5} 5k = 5(0) + 5(1) + 5(2) + 5(3) + 5(4) + 5(5). \quad \text{Answer.}$$

EXAMPLE 2 Use summation notation to write the series

$$1 + 5 + 9 + 13 + 17 + 21 + 25.$$

SOLUTION Each term is of the form $1 + 4(n - 1) = 1 + 4n - 4 = 4n - 3$. Since there are seven terms,

$$1 + 5 + 9 + 13 + 17 + 21 + 25 = \sum_{n=1}^{7} (4n - 3). \quad \text{Answer.}$$

A series, such as $1 + 5 + 9 + 13 + 17 + 21 + 25$, whose terms are in arithmetic progression is called an **arithmetic series.** Because

$$1 + 5 + 9 + 13 + 17 + 21 + 25 = 91,$$

we say that the **value,** or **sum,** of this series is 91.

You can find a formula for the sum of any arithmetic series by noticing that there are two ways to obtain the terms of the series:

(1) Start with the first term a_1 and successively add the common difference d.

(2) Start with the nth term a_n and successively subtract d.

Thus, (1) $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d]$;

(2) $S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d]$.

Adding the corresponding members of these equations, you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n),$$

where $a_1 + a_n$ occurs n times. Hence,

$$2S_n = n(a_1 + a_n),$$

or

$$S_n = \frac{n}{2}(a_1 + a_n).$$

EXAMPLE 3 Find the sum of the first one hundred positive odd integers.

SOLUTION $a_1 = 1$, $d = 2$, $n = 100$,

$$a_n = a_1 + (n - 1)d = a_{100}$$
$$= 1 + (100 - 1)2 = 199$$

and

$$S_{100} = \frac{100}{2}(1 + 199) = 10{,}000. \quad \text{Answer.}$$

Note: Example 3 illustrates the following interesting fact: *The sum of the first n positive odd integers is n^2* (see Exercise 33, page 224).

In applying the formula for S_n in Example 3, we used the fact that $a_n = a_1 + (n - 1)d$. Thus $S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[a_1 + a_1 + (n - 1)d]$, or

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

In finding the two formulas for S_n for an arithmetic series, you have proved the following theorem.

Theorem. If S_n is the sum of the first n terms of an arithmetic sequence whose first term is a_1, whose common difference is d, and whose nth term is a_n, then

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{and} \quad S_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

Oral Exercises

In Exercises 1–8 state the given series in expanded form.

1. $\sum_{i=1}^{4} 2i$
2. $\sum_{i=3}^{7} i$
3. $\sum_{k=1}^{5} (k + 2)$
4. $\sum_{n=1}^{4} (n - 3)$
5. $\sum_{m=3}^{6} (2m + 1)$
6. $\sum_{j=2}^{5} (3 - j)$
7. $\sum_{i=7}^{10} (10i - 1)$
8. $\sum_{i=6}^{10} (-5i)$

For each of the following arithmetic series, tell which of the two formulas in the theorem on page 222 you would use to find the sum of the series. State the values of those of the constants a_1, d, n, and a_n needed to apply the formula.

9. A series of eleven terms beginning $5 + 7 + 9 + \cdots$
10. The series whose first term is 7 and whose twelfth term is -26.
11. The sum of the multiples of 3 between 3 and 48 inclusive.
12. A series of seventeen terms beginning with -9 and ending with 15.
13. A series of seven terms beginning $\frac{3}{4} + \frac{5}{4} + \frac{7}{4} + \cdots$
14. The sum of the multiples of 5 between 45 and 115 inclusive.
15. The sum of the odd numbers between 11 and 37 inclusive.
16. A series of fifteen terms beginning $-9 - 4 + 1 + \cdots$
17. A series of twelve terms beginning $-1 - 8 - 15 \cdots$
18. $\sum_{n=1}^{25} (n - 6)$
19. $\sum_{k=1}^{17} (4k - 1)$
20. $\sum_{i=3}^{9} (7 - 2i)$

Written Exercises

In Exercises 1–20, find the sum of the series.

A 1–12. Find the sums of the series given in Oral Exercises 9–20.

13. $a_1 = 2,\ a_6 = 17,\ n = 6$
14. $a_1 = 5,\ a_{13} = -19,\ n = 13$
15. $a_1 = -310,\ a_{27} = -50,\ n = 27$
16. $a_1 = 29,\ d = -3,\ n = 15$
17. $a_1 = -14,\ d = 5,\ n = 27$
18. $a_1 = 6,\ a_{21} = 16,\ n = 21$
19. $a_1 = 11,\ a_2 = 5,\ n = 16$
20. $a_1 = 56,\ a_{11} = -14,\ n = 15$

Write each of the following using summation notation.

21. $6 + 8 + 10 + 12$
22. $4 + 7 + 10 + 13$
23. $-3 + 2 + 7 + 12 + 17$
24. $8 + 1 - 6 - 13 - 20$

Use the given data about an arithmetic series to find the requested value.

B 25. $a_1 = -7$, $S_{15} = -77$, $d = \underline{?}$

26. $a_1 = 48$, $S_{14} = -329$, $d = \underline{?}$

27. $a_1 = -5$, $a_n = 49$, $S_n = 616$, $n = \underline{?}$

28. $a_{10} = 79$, $S_{10} = 430$, $a_1 = \underline{?}$

29. $d = 4$, $S_9 = -189$, $a_1 = \underline{?}$

30. $a_1 = -12$, $d = 3$, $S_n = 33$, $n = \underline{?}$

C 31. The sum of the first n positive integers is 4950. What is the value of n?

32. Find the sum of all the multiples of 3 between 100 and 200.

33. Show that the sum of the first n positive odd integers is n^2.

34. Show that if m is a positive integer, then the sum of all the integers between m and m^2, inclusive, is $\dfrac{m(m^3 + 1)}{2}$.

Problems

A 1. Greta Kimmel had \$25 of her first monthly paycheck deposited in the credit union. During each of the next 15 months, she increased the monthly deposit by \$5. How much money did she deposit in the credit union during the 16-month period?

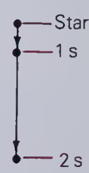
2. A city's population grew by 4200 persons in 1980. During each year of the next decade its *rate* of population growth is expected to decrease by 20 persons per year. What is the city's expected total population growth from 1980 to 1990 inclusive?

3. Marvin Beauchamps earned an annual salary of \$13,400 during his first year as a computer programmer, and he received raises of \$1400 during each of the next 9 years. If he invested 10% of his salary in shares of the company for which he worked, find the amount of money he invested over the 10-year period.

4. An antique clock chimes as many times as the hour. How many times does it chime between 8:00 A.M. and 7:00 P.M. inclusive?

5. A motorist drives 1000 km in January, 1240 km in February, and an additional 240 km in each succeeding month of the year. If gas costs 20¢/L and the car averages 8 km/L of gas, how much can the motorist expect to spend on gas for the year?

6. A free-falling object drops 9.8 m farther during each second than it did during the preceding second. If an object falls 4.9 m during the first second of its descent, how far will it fall in 5 s?



B 7. A cucumber farmer picks 120 cucumbers on the first day of the harvest and 40 more on each succeeding day. How many days will it take for the farmer to pick a total of 3000 cucumbers?

8. A city tax assessor wishes to offer a fixed-payment tax plan to be in effect for a period of twenty years to the developer of a new shopping mall. The first year's taxes are to be a fixed amount. In each of the succeeding years of the plan, the taxes will increase by an amount equal to the first year's taxes. What should the first year's taxes be for the city to collect $945,000 over a twenty-year period?

9. Eileen Yazzi earns $16,200 during her first year of employment as an environmental engineer. Sarah Smith, a co-worker who was employed at the same time, earns $14,400 during her first year. If Eileen gets annual raises of $600 and Sarah gets annual raises of $700, how long will it take for them to have earned the same total amount?

C 10. For a period of 42 d, a mailbox received 4 more letters each day than the previous day. The total number of letters received during the first 24 d equals the total number received during the last 18 d. How many letters were received during the entire period?

11. A number of the form $T_n = 1 + 2 + 3 + 4 + \cdots + n$ is called a triangular number. Show that for any positive integer n: $T_n + T_{n-1} = n^2$.

12. Using the definition of T_n given in Problem 11, show that for any positive integer n: $T_{2n+1} - 2T_n = (n + 1)^2$. Explain this result.

Self-Test 1

VOCABULARY

terms (p. 213)	common difference (p. 214)
finite sequence (p. 213)	arithmetic means (p. 217)
infinite sequence (p. 213)	average (p. 217)
arithmetic sequence (p. 214)	arithmetic series (p. 221)

1. An arithmetic sequence has first term -7, and its terms are related by the equation $a_{n+1} = a_n + 6$. Find a_4, a_5, and a_6. *Obj. 1, p. 213*

2. Find the first term of the arithmetic sequence whose fifth term is 7 and whose eleventh term is -5. *Obj. 2, p. 213*

3. Find the sum of the arithmetic series $\sum_{j=1}^{8} (9j - 2)$. *Obj. 3, p. 213*

4. Find the sum of the positive integers that are less than 100 and are divisible by 3. *Obj. 4, p 213*

5. Attendance at the Archimedes High School football games increased by 150 people from game to game during the school's ten-game football season. If 1250 people attended the first game, what was the total attendance for the season?

Check your answers with those at the back of the book.

programming in BASIC

Study the following program. It will print out the first n terms of an arithmetic sequence when you input values for a_1, d, and n.

```
10  PRINT "INPUT A1,D,N";
20  INPUT A1,D,N
30  FOR I=1 TO N-1
40  PRINT A1;",";
50  LET A1=A1+D
60  NEXT I
70  PRINT A1
80  END
```

Exercises

1. Write a program that will print out the nth term of an arithmetic sequence when you input values for a_1, d, and n.
2. Write a program that will print out the first n terms of an arithmetic series and the sum of those terms when you input values for a_1, d, and n.
3. The Fibonacci numbers are the numbers in the sequence

   $$1, 1, 2, 3, 5, 8, \ldots$$

   where each number after the second is the sum of the two preceding numbers. Write a program that will print out the first n Fibonacci numbers.

Geometric Sequences and Series

***OBJECTIVES** for Sections 7-4 through 7-6:*
1. *Find a specified term of a geometric sequence when two terms, or one term and the common ratio, are given.*
2. *Insert any number of geometric means between two given numbers.*
3. *Find the sum of a given geometric series.*
4. *Solve practical problems involving geometric sequences and series.*

7-4 Geometric Sequences

In Section 7-1 you learned that in an arithmetic sequence each term after the first is obtained by adding a fixed number, the common difference, to the preceding term. Notice that in the sequence

$$2, 10, 50, 250$$

each term after the first is obtained by *multiplying* the preceding term by a fixed number, namely 5.

$$a_2 = 10 = 2 \cdot 5 = a_1 \cdot 5$$
$$a_3 = 50 = 10 \cdot 5 = a_2 \cdot 5$$
$$a_4 = 250 = 50 \cdot 5 = a_3 \cdot 5$$

Thus, $$a_{n+1} = a_n \cdot 5 \quad \text{for } n = 1, 2, 3.$$

Any sequence in which each term after the first is the product of the preceding term and a fixed number is called a **geometric sequence** or a **geometric progression.** The fixed number is called the **common ratio.** In general:

> In a geometric sequence whose first term is a_1 and whose common ratio is r, successive terms are obtained from the rule
>
> $$a_{n+1} = a_n \cdot r, \qquad n = 1, 2, 3, \ldots.$$

The terms of a geometric sequence are said to be **in geometric progression.**

EXAMPLE 1 The first three terms of a geometric sequence are 54, 18, 6. Find the common ratio and the fourth term of this sequence.

SOLUTION 1. To find the **common ratio,** divide any one term into its successor.

$$18 \div 54 = \tfrac{1}{3} \qquad [\text{or } 6 \div 18 = \tfrac{1}{3}]$$

$\therefore$ the common ratio is $\frac{1}{3}$. Answer.

2. To find the fourth term, multiply the third term by the common ratio.

$$6 \times \tfrac{1}{3} = 2$$

$\therefore$ the fourth term is 2. Answer.

To discover a formula for the general term in any geometric sequence, let us continue the equations above for the terms of the geometric sequence 2, 10, 50, 250, . . . in the following way.

$$a_2 = a_1 \cdot 5 = a_1 \cdot 5^1$$
$$a_3 = a_2 \cdot 5 = a_1 \cdot 5^2$$
$$a_4 = a_3 \cdot 5 = a_1 \cdot 5^3$$

Do you see that the nth term ($n > 1$) of the sequence is given by the formula

$$a_n = a_1 \cdot 5^{n-1}?$$

Now consider a geometric sequence whose first term is a_1 and whose common ratio is r. The first few terms of the sequence are

$$a_1, a_1r^1, a_1r^2, a_1r^3, a_1r^4,$$

so that the nth term is given by

$$a_n = a_1r^{n-1}, \qquad n > 1.$$

Notice that this formula applies, whatever value r may have. Of course, if the value of r is 0, all terms after the first are also 0.

Does the preceding formula apply in case n is 1? In that case, you find

$$a_1 = a_1r^{1-1}, \qquad \text{or} \qquad a_1 = a_1r^0.$$

For this formula to be valid, r^0 must be 1. In fact, as you saw on page 173, for every nonzero real number r, r^0 is defined to be 1. With this definition you can state the following fact:

> The nth term of a geometric sequence whose first term is a_1 and whose common ratio is a nonzero number r is
>
> $$a_n = a_1r^{n-1}.$$

In finding the terms of a geometric progression, you may use the following table.

Short Table of Powers

N	N^2	N^3	N^4	N^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1,024
5	25	125	625	3,125
6	36	216	1,296	7,776
7	49	343	2,401	16,807
8	64	512	4,096	32,768
9	81	729	6,561	59,049
10	100	1,000	10,000	100,000
11	121	1,331	14,641	161,051
12	144	1,728	20,736	248,832
13	169	2,197	28,561	371,293
14	196	2,744	38,416	537,824
15	225	3,375	50,625	759,375
16	256	4,096	65,536	1,048,576
17	289	4,913	83,521	1,419,857
18	324	5,832	104,976	1,889,568
19	361	6,859	130,321	2,476,099
20	400	8,000	160,000	3,200,000

EXAMPLE 2 Find the sixth term of the geometric progression

$$-2, \ -22, \ -242, \ \ldots .$$

SOLUTION $a_1 = -2$, $r = -22 \div (-2) = 11$.

Since $a_n = a_1 r^{n-1}$,

$$a_6 = -2(11)^{6-1}, \quad \text{or} \quad a_6 = -2(11)^5.$$

From the preceding table, $11^5 = 161{,}051$ and

$$a_6 = -2(161{,}051) = -322{,}102. \quad \text{Answer.}$$

Oral Exercises

In Exercises 1–12, state whether the given sequence is arithmetic, geometric, or neither.

1. $\frac{1}{2}, 1, 2, 4$
2. $4, 9, 16, 25$
3. $11, 8, 5, 2$
4. $8, 4, 0, -4$
5. $12, 6, 3, \frac{3}{2}$
6. $0, 2, 0, 2$
7. $5, -10, 20, -40$
8. $7, 7, 7, 7$
9. $1, 0.1, 0.01, 0.001$
10. $4, -4, 4, -4$
11. $\frac{a}{b}, \frac{a^2}{b}, \frac{a^3}{b}, \frac{a^4}{b}$
12. $x + 2y, x + y, x, x - y$
13. Is one of the following statements always true for any geometric sequence: $|a_1| \le |a_2| \le |a_3| \le \ldots$ or $|a_1| \ge |a_2| \ge |a_3| \ge \ldots$?

Written Exercises

Find the first four terms of the geometric sequence with the given first term and common ratio. Use the table on page 228 if necessary.

A

1. $a_1 = 3, r = 3$
2. $a_1 = 6, r = -6$
3. $a_1 = 6, r = \frac{1}{2}$
4. $a_1 = 12, r = \frac{1}{4}$
5. $a_1 = -2, r = \frac{1}{3}$
6. $a_1 = \frac{4}{3}, r = \frac{1}{2}$
7. $a_1 = \frac{25}{4}, r = -\frac{2}{5}$
8. $a_1 = \frac{c}{d^2}, r = \frac{d}{c}$
9. $a_1 = \frac{4}{9}, r = -3$

Find the specified term of the geometric sequence with the given first term and common ratio.

10. $a_1 = \frac{1}{3}, r = 3, a_4 = \underline{\ ?\ }$
11. $a_1 = 5, r = -2, a_6 = \underline{\ ?\ }$
12. $a_1 = \frac{8}{3}, r = \frac{1}{2}, a_5 = \underline{\ ?\ }$

Find the requested term in the given geometric sequence.

13. $\frac{3}{16}, \frac{3}{8}, \frac{3}{4}, \ldots$; find $a_8 = \underline{\ ?\ }$
14. $54, -18, 6, \ldots; a_6 = \underline{\ ?\ }$
15. $\frac{25}{2}, -\frac{5}{2}, \frac{1}{2}, \ldots; a_7 = \underline{\ ?\ }$
16. $\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \ldots; a_7 = \underline{\ ?\ }$
17. $0.2, 0.02, 0.002, \ldots; a_6 = \underline{\ ?\ }$
18. $0.0007, 0.07, 7, \ldots; a_6 = \underline{\ ?\ }$

Each of the following sequences is either arithmetic or geometric. Determine which it is, then find the requested term.

B 19. $16, 4, -8, \ldots ; a_6 = \underline{?}$

20. $5, 3, \frac{9}{5}, \ldots ; a_6 = \underline{?}$

21. $333\frac{1}{3}, 33\frac{1}{3}, 3\frac{1}{3}, \ldots ; a_6 = \underline{?}$

22. $\frac{4}{5}, -\frac{2}{5}, -\frac{8}{5}, \ldots ; a_7 = \underline{?}$

23. $-0.04, 0.16, 0.36, \ldots ; a_8 = \underline{?}$

24. $a^8b^2, a^6b^3, a^4b^4, \ldots ; a_7 = \underline{?}$

C 25. Show that in any geometric sequence,

$$a_{n-1}a_{n+1} = (a_n)^2 \quad \text{for any integer } n > 1.$$

26. Use the table on page 228 to determine x so that $2^x = 1{,}048{,}576$.

27. Determine x so that $3^x = 59{,}049$. (*Hint:* See Exercise 26.)

28. If $a_1, a_2, \ldots, a_n$ is a geometric sequence, is $a_n, a_{n-1}, \ldots, a_1$ also a geometric sequence? Justify your answer.

Problems

A 1. A manufacturer of artificial greenery is considering a new model tree with 4 limbs. Each limb holds 4 branches, each branch holds 4 twigs, and each twig holds 4 leaves. How many leaves will the new model tree require?

2. At a telephone switching station the first level of a circuit has 3 lines. At the second level each of the lines branches into five lines, at the third level each of the lines branches into five lines, and so on. How many lines emerge from the fifth level?

3. A population of insects being observed in an experiment grows by 10% every two weeks. In other words, at the end of two weeks there will be 1.1 times the original number of insects. If there are 10,000 insects at the beginning of the experiment, how many will there be at the end of eight weeks?

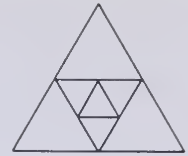
4. Each year the value of a certain car is 70% of what it was the previous year. If its value was \$5000 at the end of the first year, what was it at the end of the fifth year?

5. How many great-great-grandparents do 5 people have, assuming that they have no common ancestors in the previous 4 generations?

6. A legendary coin bank has the power of doubling the amount of money in it in each day. If 1¢ is deposited in it on January 1, how much will be in the bank on January 31? (*Note:* You may use the approximation $2^{10} \approx 1020$.)

B 7. The half-life of nitrogen-13, an isotope of the gas nitrogen, is about 10 min; that is, at the end of any 10 min period half of the amount present at the beginning of the period will remain. If 4 mg of nitrogen-13 is observed to be present at 3:10 P.M. in the course of a laboratory experiment, how much will be present at 4:00 P.M.?

8. A side of an equilateral triangle is 64 cm long. A second equilateral triangle is inscribed in it by joining the midpoints of the sides of the first triangle. The process is continued, as shown in the accompanying diagram. Find the perimeter of the sixth *inscribed* equilateral triangle.



In Exercises 9–11 use the following compound-interest law: P dollars earning interest at $r\%$ $\left(\frac{r}{100}\right)$ per year compounded d times a year amounts to $P\left(1+\frac{r}{100d}\right)^{d}$ at the end of one year and to $P\left(1+\frac{r}{100d}\right)^{nd}$ at the end of n years.

9. Explain why the successive amounts computed at the end of each interest period according to the compound-interest law form a geometric progression. What is the common ratio?
10. The interest rate in a certain bank is 8% compounded semiannually. If you deposit \$5000 in an account and make no other deposits or withdrawals, how much money will be in your account at the end of two years?
11. Find the compound interest due at the end of 5 years if \$2000 is invested at 10% compounded annually.

7-5 Geometric Means

The terms between two given terms of a geometric sequence are called **geometric means** between the given terms. In the sequence

$$-2,\ 6,\ -18,\ 54,\ -162,$$

6, −18, and **54** are the three geometric means between −2 and −162.

EXAMPLE 1 Find the two geometric means between the terms 3 and 375.

SOLUTION Schematically, the sequence looks like this: 3, ____, ____, 375. To determine r, replace a_1 with 3, a_n with 375, and n with 4 in

$$a_n = a_1 r^{n-1}$$
$$375 = 3r^{4-1}$$
$$125 = r^3$$
$$r = 5 \text{ (from the table on page 228 or from memory)}$$

To complete the sequence, multiply 3 by 5 and the result by 5:

$$3,\ 15,\ 75,\ 375.$$

$\therefore$ the required means are 15 and 75. Answer.

EXAMPLE 2 Find the second term of a geometric sequence whose third term is 36 and whose fifth term is 4. Give all possible correct answers.

SOLUTION A diagram of the sequence is ___, ___, 36, ___, 4.

1. To find r, determine the common ratio of the sequence whose first term is 36 and whose third term is 4:

$$a_3 = a_1 r^{3-1}$$
$$4 = 36r^2$$
$$\tfrac{1}{9} = r^2$$
$$r = \tfrac{1}{3} \quad \text{or} \quad r = -\tfrac{1}{3}$$

2. Working from the term 36 with a common ratio of $\frac{1}{3}$, you find that the sequence is

$$\mathbf{324},\ \mathbf{108},\ 36,\ \mathbf{12},\ 4;$$

if the common ratio is $-\frac{1}{3}$, the sequence is

$$\mathbf{324},\ \mathbf{-108},\ 36,\ \mathbf{-12},\ 4.$$

$\therefore$ the second term is either 108 or -108. Answer.

Examples 1 (page 231) and 2 illustrate the two situations that occur when you wish to solve over $\mathcal{R}$ an equation of the form

$$r^n = b,$$

where n is a positive integer and b is a positive real number. In case n is odd, the equation has one real root, which is a positive number. Thus, the equation

$$r^3 = 125 \text{ has the single real root } 5.$$

In case n is even, the equation has two real roots, one a positive number and the other a negative number. Both roots have the same absolute value. For instance,

$$r^2 = \tfrac{1}{9} \text{ has the two real roots } \tfrac{1}{3} \text{ and } -\tfrac{1}{3}.$$

Now suppose that b is a negative number. For example, what are the roots in $\mathcal{R}$ of

$$r^3 = -125?$$

Do you see that the one and only real root is -5? On the other hand,

$$r^2 = -\tfrac{1}{9}$$

has no real root because the square of every real number is either 0 or a positive number.

The preceding discussion suggests the following theorem, which we shall accept without proof.

Theorem. For all real numbers b:

1. If n is a positive odd integer, then the equation $r^n = b$ has one and only one real root.
2. If n is a positive even integer, then $r^n = b$ has:
   i. one real root if b is 0;
   ii. two real roots with the same absolute value if b is a positive number;
   iii. no real root if b is a negative number.

A single geometric mean inserted between two numbers is called the **geometric mean** or **mean proportional** of the numbers. If the real number m is the geometric mean of two nonzero real numbers a and b, then a, m, b is a geometric sequence and

$$\frac{m}{a} = \frac{b}{m}.$$

$$\therefore m^2 = ab.$$

Since m^2 is positive, a and b must be both positive or both negative. If both a and b are positive, then *the* geometric mean of the numbers is usually defined to be the *positive* root of the equation $m^2 = ab$, whereas if both a and b are negative, then their geometric mean is the *negative* root of the equation.

EXAMPLE 3 Find the geometric mean of 18 and 50.

SOLUTION The required mean satisfies the equation $m^2 = 18 \cdot 50$, or

$$m^2 = 900$$
$$m = 30 \qquad \text{or} \qquad m = -30.$$

$\therefore$ the geometric mean of 18 and 50 is 30. Answer.

Oral Exercises

State the geometric mean of the given pair of numbers.

1. 1, 9
2. $-2, -18$
3. 4, 25
4. $\frac{1}{2}$, 32
5. $-4, -9$
6. $-\frac{3}{4}, -12$
7. a, a^3; $a > 0$
8. a^2, b^2; $ab > 0$

State all roots of the given equation. Use the table on page 228.

9. $x^3 = -8$
10. $x^4 = 81$
11. $x^2 = 121$
12. $x^5 = 243$
13. $x^4 = -16$
14. $x^3 = -125$
15. $x^2 = 225$
16. $x^6 = 64$

17. In a geometric sequence, if $a_1 < 0$ and $a_4 < 0$, is r a positive number or a negative number? Is a_3 a positive number or a negative number?

18. In a geometric sequence, if $a_1 < 0$ and $a_4 > 0$, is r positive or negative?

19. Of what number and 2 is 8 the geometric mean?

20. Of what number and -12 is -6 the geometric mean?

21. Of what number and a is a^3 the geometric mean?

Written Exercises

Find the common ratio of a geometric sequence having the given terms.

A

1. $a_1 = 5,\ a_4 = 40$
2. $a_1 = -3,\ a_4 = 192$
3. $a_1 = 135,\ a_4 = -5$
4. $a_1 = 250,\ a_4 = 2$
5. $a_3 = 9,\ a_5 = 16$
6. $a_2 = -80,\ a_6 = -5$

Give the first two terms of a geometric sequence having the given terms.

7. $a_3 = 12,\ a_6 = 96$
8. $a_3 = 5,\ a_6 = -135$
9. $a_4 = -6,\ a_7 = -48$
10. $a_3 = 10,\ a_6 = \frac{2}{25}$
11. $a_5 = \frac{5}{4},\ a_8 = 10$
12. $a_3 = -12,\ a_6 = \frac{32}{9}$

Insert the given number of geometric means between the given numbers and write the resulting geometric sequence. Give all possible correct answers.

13. Two between 3 and 375.
14. Two between -7 and -56.
15. Two between 108 and -4.
16. Two between $-\frac{2}{3}$ and 18.
17. Three between 176 and 11.
18. Three between $-\frac{25}{2}$ and $-\frac{8}{25}$.
19. Three between $\frac{5}{27}$ and 15.
20. Three between -8 and $-\frac{81}{2}$.

In Exercises 21–26 find the specified value using the data given about a geometric sequence.

B

21. $a_1 = 3,\ a_2 = 2,\ a_n = \frac{32}{81},\ n = \underline{\ ?\ }$
22. $a_1 = \frac{125}{16},\ a_2 = \frac{25}{8},\ a_n = \frac{4}{125},\ n = \underline{\ ?\ }$
23. $a_5 = -\frac{1}{2},\ a_8 = \frac{1}{128},\ a_3 = \underline{\ ?\ }$
24. $a_6 = 160,\ a_9 = -1280,\ a_4 = \underline{\ ?\ }$
25. $a_4 = \dfrac{p^2}{q^3},\ a_7 = \dfrac{p^2}{q^9};\ a_1 = \underline{\ ?\ }$
26. $a_5 = \dfrac{d}{c^5},\ a_8 = \dfrac{d^4}{c^{11}},\ a_1 = \underline{\ ?\ }$

27. In a laboratory experiment, there were 6400 microorganisms present in a culture at the end of 1 h. At the end of an additional 3 h, there were 21,600. If the population increases geometrically, by what factor did it increase each hour?

28. When the value of a certain block of stock is calculated using the closing prices on the last business day of each of six successive years, it is observed that the value of the stock decreased by a fixed percent from year to year. The value as of the first year was \$10,240; the value as of the sixth year was \$2430. By what percent did the value of the stock decrease each year?

29. Find all possible values of x that make the sequence $x - 1, x + 2, 3x$ a geometric progression.

30. Find all possible values of x that make $x + 3$ the geometric mean of $x - 2$ and $3x - 1$.

For Exercises 31 and 32 let $a_1, a_1r, a_1r^2, a_1r^3, \ldots$ represent a geometric sequence.

C 31. Show that the sequence formed by squaring each term of the sequence above is also a geometric sequence by finding the common ratio.

32. Assuming that $a_1r \neq 0$, show that the sequence formed by taking the reciprocal of each term is also a geometric sequence by finding the common ratio.

33. Prove that for $t \neq 0, t \neq 1, t \neq -1$, if the sequence $t^a, t^b, t^c, t^d, \ldots$ is geometric, then the sequence $a, b, c, d, \ldots$ is an arithmetic sequence.

34. Show that if $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are geometric sequences, then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots$ is also a geometric sequence.

35. Let $a, b \geq 0$. Show that the arithmetic mean of a^2 and b^2 is greater than or equal to the geometric mean of a^2 and b^2. (Hint: Start with the fact that $(a - b)^2 \geq 0$ for any real numbers a and b.)

36. Show that for any geometric sequence if $r > 0$, then a_n is the geometric mean of a_1 and a_{2n-1}, for any positive integer n.

7–6 Geometric Series

A series, such as $3 + 6 + 12 + 24$, whose terms are in geometric progression is called a **geometric series.** To find an expression for the sum S_n of a geometric series of n terms, first write S_n in expanded form and below it write the product of $-r$ and S_n as follows:

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} \\ -rS_n &= - a_1r - a_1r^2 - a_1r^3 - \cdots - a_1r^{n-1} - a_1r^n \end{aligned}$$

Then add the corresponding members of these equations to obtain

$$S_n - rS_n = a_1 + (a_1r - a_1r) + (a_1r^2 - a_1r^2) + \ldots + (a_1r^{n-1} - a_1r^{n-1}) - a_1r^n.$$

Since $a_1r - a_1r = 0$, $a_1r^2 - a_1r^2 = 0$, and so on, you have

$$S_n - rS_n = a_1 - a_1r^n,$$
$$(1 - r)S_n = a_1 - a_1r^n.$$
$$\therefore S_n = \frac{a_1 - a_1r^n}{1 - r}, \qquad r \neq 1.$$

EXAMPLE 1 Find the sum of the terms of the sequence 3, 6, 12, 24.

SOLUTION Replace a_1 with 3, r with 2, and n with 4 in $S_n = \dfrac{a_1 - a_1r^n}{1 - r}$:

$$S_4 = \frac{3 - 3(2)^4}{1 - 2}$$
$$S_4 = \frac{3 - 48}{-1} = \frac{-45}{-1} = 45$$
$$\therefore S_4 = 45. \quad \text{Answer.}$$

Note that $a_1r^n = r(a_1r^{n-1}) = ra_n$. Hence, from $S_n = \dfrac{a_1 - a_1r^n}{1 - r}$,

$$S_n = \frac{a_1 - ra_n}{1 - r}, \qquad r \neq 1.$$

EXAMPLE 2 Find the sum of a geometric series whose first term is 2, whose last (nth) term is 13122, and whose common ratio is 3.

SOLUTION Replace a_1 with 2, a_n with 13122 and r with 3 in $S_n = \dfrac{a_1 - ra_n}{1 - r}$.

$$S_n = \frac{2 - 3(13122)}{1 - (3)}$$
$$= \frac{2 - 39366}{-2} = 19{,}682$$

The following theorem summarizes the results obtained in this section.

Theorem. If S_n is the sum of the first n terms of a geometric sequence whose first term is a_1, whose common ratio is r, and whose nth term is a_n, then

$$S_n = \frac{a_1 - a_1r^n}{1 - r} \quad \text{and} \quad S_n = \frac{a_1 - ra_n}{1 - r}, \qquad r \neq 1.$$

Oral Exercises

State whether the given series is an arithmetic series or a geometric series.

1. $\sum_{n=1}^{5} 3n$ 2. $\sum_{n=1}^{6} 8 \cdot 5^n$ 3. $\sum_{n=1}^{18} \frac{2n+1}{5}$ 4. $\sum_{n=1}^{15} 5 + \frac{n}{8}$

5. $32 + 48 + 72 + 108 + 162$ 6. $16 + 4 - 8 - 20 - 32$

7. The sum of the multiples of 5 between 5 and 125, inclusive.

8. The sum of the powers of 2 between 4 and 64, inclusive.

For each of the following geometric series, state which of the two formulas in the theorem on page 236 you would use to find the sum of the series. State the values of those of the constants a_1, r, n, and a_n needed to apply the formula. Do not compute S_n.

9. The sum of the integral powers of 3 between 1 and 243, inclusive.

10. The sum of the first seven positive integral powers of 2, beginning with 2^1.

11. $2 + 10 + \cdots + 31{,}250$ 12. The first five terms of $162 - 54 + 18 - \cdots$

13. $\sum_{k=1}^{5} 3(2)^{k-1}$ 14. $\sum_{m=1}^{6} -(-4)^{m-1}$ 15. $\sum_{i=1}^{6} 12(\frac{1}{2})^{i-1}$ 16. $\sum_{n=1}^{8} \frac{1}{2}(-2)^{n-1}$

17. $\sum_{n=1}^{7} \frac{1}{40}(-2)^{n-1}$ 18. $\sum_{k=1}^{6} 18(\frac{2}{3})^{k-1}$ 19. $\sum_{j=0}^{5} -5(-\frac{1}{2})^j$ 20. $\sum_{m=0}^{7} \frac{1}{3}(-4)^m$

21. Explain why, in a geometric series, if $a_1 = 1$ and $0 < r < 1$, S_n is always less than $\frac{1}{1-r}$ for any positive integer n.

Written Exercises

Find the sum S_n of the geometric series described.

A 1. $a_1 = \frac{1}{81}$, $r = 3$, $n = 6$ 2. $a = 1$, $r = -2$, $n = 7$

3. $a_1 = 5$, $r = 4$, $n = 5$ 4. $a_1 = -27$, $r = -\frac{1}{3}$, $n = 6$

5. $a_1 = 1000$, $r = \frac{1}{2}$, $n = 7$ 6. $a_1 = 125$, $r = -\frac{2}{5}$, $n = 5$

7–18. Find the sums of the geometric series in Oral Exercises 9–20.

In Exercises 19–28 use the given data for a geometric series to find the requested value.

B 19. $r = 3$, $S_6 = 3640$, $a_1 = ?$ 20. $r = -\frac{1}{5}$, $S_5 = 1042$, $a_1 = ?$

21. $r = \frac{1}{3}$, $a_n = 5$, $S_n = 1820$, $a_1 = ?$ 22. $a_1 = 16$, $r = \frac{3}{2}$, $S_n = 211$, $n = ?$

23. $a_1 = 7$, $r = 2$, $S_n = 889$, $n = ?$ 24. $r = -\frac{1}{2}$, $a_n = -\frac{3}{2}$, $S_n = 31\frac{1}{2}$, $n = ?$

25. $r = 0.5$, $S_5 = 1.9375$, $a_5 = ?$

26. $r = -0.2$, $S_4 = 0.6656$, $a_1 = ?$

27. $a_1 = 3$, $S_3 = \frac{19}{3}$, $r = ?$

28. $a_1 = 1$, $S_3 = \frac{3}{4}$, $r = ?$

(*Hint:* For Exercises 27 and 28, use the fact that $\dfrac{1 - r^3}{1 - r} = 1 + r + r^2$.)

29. The sum of the first 3 terms of a geometric series in which $a_1 = 2$ and $r = x$ is $2x^2 + 3$. What is the value of x?

C **30.** Prove: $\displaystyle\sum_{k=1}^{n} (2^{-k}) = 1 - \frac{1}{2^n}$

31. Prove the first formula of the theorem on page 236 by writing

$$S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r} = a_1\left(\frac{r^n - 1}{r - 1}\right)$$

and using long division.

32. Show that if $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = S_n$,

then $a_1 + \dfrac{a_1}{r} + \dfrac{a_1}{r^2} + \cdots + \dfrac{a_1}{r^{n-1}} = \dfrac{1}{r^{n-1}} \cdot S_n \quad (r \neq 1, 0)$.

Problems

A **1.** After the accelerator pedal of a car is released, the driver waits 5 s before applying the brakes. During each second after the first, the car rolls 0.9 times the distance it rolled during the preceding second. If the car went 20 m during the first second, how far does it go before the brakes are applied?

2. How many ancestors from parents through great-great-great grandparents do 3 unrelated people have?

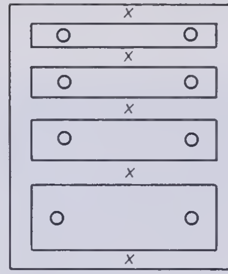
3. A maple tree loses 384 leaves during the first week of fall and $\frac{3}{2}$ as many each successive week. At the end of 7 weeks all the leaves have fallen. How many leaves fell from the tree?

4. Each coil of a tapering spring is to be 0.8 the length of the preceding one. If the wire out of which the spring is to be formed is 18.45 cm long and there are to be 4 coils, how long should the first coil be?

B **5.** A set of five mass pieces to be used on a balance scale has a total mass of 930 g. If they are arranged in order from lightest to heaviest, the second will have twice the mass of the first, the third will have twice the mass of the second, and so on. What is the mass of the heaviest one?

6. A ball is dropped straight down from a height of 81 cm. It rebounds $\frac{2}{3}$ of its previous maximum height with each successive bounce. When it hits the ground for the fifth time, what is the total distance the ball has traveled after being dropped? (Count distances traveled up and distances traveled down.)

7. A bureau with four drawers is to have a height of 80 cm. The height of the first drawer is to be 12.8 cm, and the height of each of the lower drawers is to be 1.25 times the height of the drawer above it. If the five spaces above and below the drawers are to be equal, find their height (x).



C 8. An annuity is a form of savings in which a fixed amount of money is deposited regularly in an account earning compound interest. Consider an annuity in which one dollar is deposited at the beginning of each of n time periods, and for which compound interest at the rate i (expressed as a decimal) is calculated at the end of each time period. Show that the expression below gives the amount of money in the annuity at the beginning of the nth time period:

$$\frac{(1+i)^n - 1}{i}$$

(*Hint:* The expression $(1 + i)^{n-1}$ gives the amount of money that will accumulate from the first dollar deposited. Form a geometric series of such terms.)

Self-Test 2

VOCABULARY geometric sequence (p. 227) common ratio (p. 227) geometric means (p. 231) geometric series (p. 235)

1. Find the first four terms of a geometric sequence if $a_1 = 25$ and $r = \frac{2}{5}$. *Obj. 1, p. 226*
2. Find the sixth term of the geometric sequence $\frac{3}{4}, \frac{3}{2}, 3, \ldots$
3. Insert three geometric means between 27 and $\frac{1}{3}$. Give all possible correct answers. *Obj. 2, p. 226*
4. Find the sum of the geometric series $\sum_{i=1}^{5} \frac{2}{125}(-5)^{i-1}$. *Obj. 3, p. 226*
5. Atmospheric pressure, which is usually expressed in kilopascals (kPa), is halved for each 4.8 km of elevation above sea level. How far above a point at which the atmospheric pressure is 100 kPa is a point at which the atmospheric pressure is 12.5 kPa? *Obj. 4, p 226*

Check your answers with those at the back of the book.

Careers

in Environmental Protection

Concern over the pollution of the environment has increased greatly in recent years, and the number of careers in environmental protection has increased as well. Air-pollution control is one area of environmental protection. This field includes determining the effects of pollutants on human health and the environment, setting standards for acceptable levels of pollutants, enforcing those standards, and developing ways to meet them. Similar tasks occur in the fields of water- and noise-pollution control.

Other functions of environmental workers include the management of solid wastes—recycling what can be used and properly disposing of other materials; the study of pesticides—their efficiency, side effects, and safety level; and the study of the effects of exposure to radiation for long periods of time. Workers in all of these fields are trying to make the earth a better and safer place to live.

A worker cleans up after an oil spill (above). A machine reproduces the contour of the ocean floor (below).

EXAMPLE In order to meet emission standards set by the Environmental Protection Agency, a certain company must remove at least 80% of the particles from the smoke pouring out of its smokestack. If each filter removes 20% of the particles passing through it, how many of these filters will the company have to install in its smokestack?

SOLUTION Let x be the amount of particles originally in the smoke. After passing through the first filter, there are $0.8x$ particles; after passing through the second, $(0.8)(0.8)x$ particles, and so on. As you can see, this is a geometric sequence with $r = 0.8$:

$$\underbrace{\text{(the amount after the first filter)}}_{a_1 = 0.8x} \quad \text{and} \quad \underbrace{\text{(the amount after the } n\text{th filter)}}_{a_n = (0.8)^{n-1}a_1 = (0.8)^n x}$$

Thus, the number of filters needed will be n, where

$$(0.8)^n x \leq 0.20x.$$

Since

$$(0.8)^7 = 0.2097152 \qquad \text{and} \qquad (0.8)^8 = 0.16777216,$$

the company will need 8 filters.

Infinite Sequences and Series

OBJECTIVES for Section 7-7 and Section 7-8:

1. ***Find the absolute value of the difference between the limit of a convergent sequence and a term in the sequence.***
2. ***Find the sum of a convergent geometric series.***

7-7 Limit of a Sequence

Figure 1 pictures the first few terms of the infinite sequence

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ \ldots,\ 2-(\tfrac{1}{2})^{n-1},\ \ldots.$$

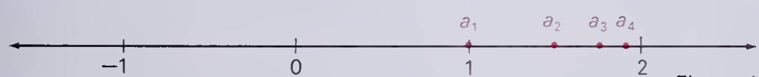


Figure 1

This figure suggests that the graphs of the terms of the given sequence eventually crowd in on the graph of 2. Notice that if you think of each term of the sequence as an approximation of 2, then the error of approximation, that is, the absolute value of the error that you make in considering a_n to be 2, is $(\frac{1}{2})^{n-1}$. As shown in the table at the right, this error is halved each time n is increased by 1. Therefore, by choosing n large enough, you can make the error less than any given positive number, however small. For this reason, we say that the *limit* of the sequence is 2, and we write

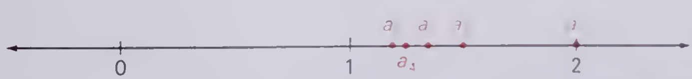
n	a_n	$\lvert 2-a_n \rvert$
1	1	1
2	$1\frac{1}{2}$	$\frac{1}{2}$
3	$1\frac{3}{4}$	$\frac{1}{4}$
4	$1\frac{7}{8}$	$\frac{1}{8}$

$$\lim_{n\to\infty}\left[2-(\tfrac{1}{2})^{n-1}\right]=2,$$

read, "the limit of $2-(\frac{1}{2})^{n-1}$ as n increases without bound is 2." An infinite sequence having a limit is said to **converge** or be **convergent.**

EXAMPLE Find the limit of the sequence $2,\ 1\frac{1}{2},\ 1\frac{1}{3},\ 1\frac{1}{4},\ \ldots,\ 1+\dfrac{1}{n},\ \ldots.$

SOLUTION Show the first few terms of the sequence on a number line.



The diagram suggests that the limit is 1; in fact, the error made in considering the nth term a_n to be 1 is $|a_n-1|$, or $\dfrac{1}{n}$, and $\dfrac{1}{n}$ is as small a positive number as you like if n is great enough.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1. \quad \text{Answer.}$$

In general, an infinite sequence has a **limit** L if you can make the error of approximation, $|L - a_n|$, less than any positive number, however small, by choosing n great enough. Notice how the successive terms of the sequence

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ \ldots,\ 2 - (\tfrac{1}{2})^{n-1},\ \ldots$$

compare with each other. Because

$$1 \leq 1\tfrac{1}{2} \leq 1\tfrac{3}{4} \leq 1\tfrac{7}{8} \ldots,$$

you say that the sequence is *nondecreasing.* Any sequence in which each term is less than or equal to the following term is called a **nondecreasing sequence.**

The sequence $2,\ 1\tfrac{1}{2},\ 1\tfrac{1}{3},\ 1\tfrac{1}{4},\ \ldots,\ 1 + \dfrac{1}{n},\ \ldots$, on the other hand, is *nonincreasing,* because

$$2 \geq 1\tfrac{1}{2} \geq 1\tfrac{1}{3} \geq 1\tfrac{1}{4} \ldots.$$

Any sequence in which each term is greater than or equal to the following term is called a **nonincreasing sequence.**

Notice also that each term of the sequence

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ \ldots,\ 2 - (\tfrac{1}{2})^{n-1},\ \ldots$$

is less than 2 in absolute value, and that each term of the sequence

$$2,\ 1\tfrac{1}{2},\ 1\tfrac{1}{3},\ 1\tfrac{1}{4},\ \ldots,\ 1 + \frac{1}{n},\ \ldots$$

is less than or equal to 2 in absolute value. Whenever there exists a number which equals or exceeds the *absolute value* of *every* term of a sequence, you say that the sequence is **bounded** by the number. Thus, each of the sequences given above is bounded by 2. Of course, each of the sequences is also bounded by 10, for example.

The fact that both of the sequences

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ \ldots,\ 2 - (\tfrac{1}{2})^{n-1},\ \ldots$$

and

$$2,\ 1\tfrac{1}{2},\ 1\tfrac{1}{3},\ 1\tfrac{1}{4},\ \ldots,\ 1 + \frac{1}{n},\ \ldots$$

are convergent illustrates the last axiom needed to characterize the set $\mathcal{R}$ of real numbers. This assumption is stated below.

Axiom of Completeness

Every bounded, nondecreasing (or nonincreasing) sequence of real numbers converges, and its limit is a real number.

Not all infinite sequences converge. An infinite sequence that does not have a limit is said to **diverge** or to be **divergent**. For example, the sequence

$$3,\ 9,\ 27,\ \dots,\ 3^n,\ \dots$$

diverges because it contains terms that are arbitrarily large in absolute value. Do you see that this sequence is nondecreasing but that it is not bounded? The sequence

$$1,\ -1,\ 1,\ -1,\ \dots,\ (-1)^{n-1},\ \dots$$

is also divergent because its terms are alternately 1 and -1. Do you see that this sequence is bounded, but that it is neither nondecreasing nor nonincreasing?

Oral Exercises

For each of the following sequences: (a) state the first four terms; (b) tell whether the sequence is nondecreasing, nonincreasing or neither; (c) tell whether or not the sequence is bounded; and (d) tell whether or not the sequence seems to be convergent, and if so guess the limit.

1. $a_n = (\frac{1}{5})^n$
2. $a_n = 4 - (\frac{1}{2})^n$
3. $a_n = (-\frac{1}{3})^n$
4. $a_n = (-1)^n 2$
5. $a_n = (-1)^{n-1}\dfrac{n}{100}$
6. $a_n = \dfrac{n}{n+1}$
7. $a_n = \dfrac{n^2}{n^2-2}$
8. $a_n = (-1)^n 3n$
9. $a_n = 1 + (\frac{1}{3})^n$
10. $a_n = (-1)^n \dfrac{1}{n^2}$
11. $a_n = 3 - \dfrac{1}{n}$
12. $a_n = 2 + (-1)^n \dfrac{1}{n}$
13. Give an example of a sequence that is neither nondecreasing nor nonincreasing but is convergent.
14. If the sequence $a_1, a_2, a_3, \dots$ is convergent, must the sequence $a_2, a_4, a_6, \dots$ (consisting of just the even terms of the first sequence) also be convergent? Explain why or give a counterexample.

Written Exercises

Write the first four terms of the sequence whose nth term is given and guess the limit or state that the sequence is not convergent.

A

1. $a_n = \dfrac{2n-1}{n}$
2. $a_n = \dfrac{3n^2+1}{n^2}$
3. $a_n = \dfrac{n^2}{100}$
4. $a_n = 3 - \dfrac{1}{2^n}$
5. $a_n = (-1)^n \dfrac{n^2}{n^2+1}$
6. $a_n = 5\left(1 + \dfrac{1}{n}\right)$

In Exercises 7–12, a formula for the nth term of an infinite sequence and its limit L are given. For $n = 1, 2, 3$, and 4, find $|L - a_n|$, and then give the general formula for $|L - a_n|$.

7. $a_n = \dfrac{3n + 1}{n}$; $L = 3$

8. $a_n = -2 + (\frac{2}{3})^n$; $L = -2$

9. $a_n = \dfrac{2 + n}{4n}$; $L = \frac{1}{4}$

10. $a_n = 4 + (-1)^n \dfrac{1}{n}$; $L = 4$

B **11.** $a_n = \dfrac{3n}{4n + 1}$; $L = \frac{3}{4}$

12. $a_n = \dfrac{4n^2 + 1}{2n^2 - 1}$; $L = 2$

13–18. Find a value of n that will make $|L - a_n| < \frac{1}{10}$ for Exercises 7–12 above. Give the numerical value of $|L - a_n|$ for this n.

19. Suppose $a_1, a_2, a_3, \ldots$ is an infinite sequence such that $2 - \dfrac{1}{n} < a_n < 2$. Explain why $\lim\limits_{n \to \infty} a_n = 2$.

C **20.** Suppose the sequence $a_1, a_2, a_3, \ldots$ has limit L and the sequence $b_1, b_2, b_3, \ldots$ is defined by the relation $b_n = ka_n$. Show that the second sequence has limit kL by showing that the expression $|b_n - kL|$ has limit 0. (*Hint:* Recall that $|ab| = |a||b|$.)

7-8 Infinite Geometric Series

Figure 2 pictures the numbers

$$1,\ 1 + \tfrac{1}{2},\ 1 + \tfrac{1}{2} + \tfrac{1}{4},\ 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8},$$

and suggests three facts:

1. The more terms you add in the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots + (\tfrac{1}{2})^{n-1} + \cdots,$$

the greater is the sum obtained.
2. The sum never exceeds 2, no matter how many terms you add.
3. If enough terms are added, the sum will approximate 2 as closely as you may demand.

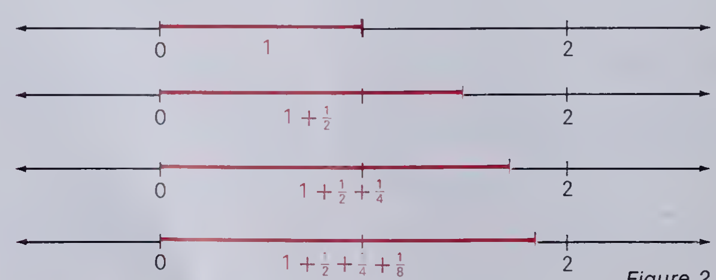


Figure 2

You can never add all the terms, so that you cannot refer to the "sum of the infinite series" without first defining what you mean by such a sum. To define the sum of the *infinite geometric series*

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots + (\tfrac{1}{2})^{n-1} + \cdots,$$

consider the sequence of *partial sums*

$$S_1 = 1,\ S_2 = 1 + \tfrac{1}{2},\ S_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4},$$

and in general

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + (\tfrac{1}{2})^{n-1} = \sum_{i=1}^{n} (\tfrac{1}{2})^{i-1}, \qquad n \geq 1.$$

Since S_n is the sum of the first n terms of a geometric sequence whose first term is 1 and whose common ratio is $\frac{1}{2}$, you have:

$$S_n = \frac{[1 - 1(\frac{1}{2})^n]}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

Hence (see page 241), $\lim_{n \to \infty} S_n = 2$. Accordingly, we *define* the sum of this infinite series to be 2, and we write

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + (\tfrac{1}{2})^{n-1} + \cdots = 2, \qquad \text{or} \qquad \sum_{i=1}^{\infty} (\tfrac{1}{2})^{i-1} = 2.$$

In general, for any infinite series $a_1 + a_2 + \cdots + a_n + \cdots$,

$$S_n = \sum_{i=1}^{n} a_i$$

is called a **partial sum.** If the sequence $S_1, S_2, \ldots, S_n, \ldots$ of partial sums converges and if $\lim_{n \to \infty} S_n = S$, then the **sum of the infinite series**

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is defined to be S. You write

$$\sum_{k=1}^{\infty} a_k = S,$$

and you say that the series **converges** or **is convergent.**

On the other hand, if the sequence of partial sums diverges, then the series **diverges** or **is divergent,** and its sum is *not* defined. For example, the series

$$1 + 2 + 3 + 4 + \cdots + n + \cdots$$

diverges, because the sequence of partial sums, below, diverges.

$$1,\ 3,\ 6,\ 10,\ \ldots,\ \frac{n(n+1)}{2},\ \ldots$$

The series

$$1 - 1 + 1 - 1 + \cdots$$

also is divergent because the sequence of partial sums 1, 0, 1, 0, ... is a divergent sequence.

Consider each of the following cases for any infinite geometric series,

$$a_1 + a_1 r + a_1 r^2 + \cdots.$$

Case 1. $a_1 = 0$. In this case, every term of the series is 0, so that the series is $0 + 0 + 0 + \cdots$, which has the sum 0.

Case 2. $a_1 \neq 0$ and $r = 1$. In this case, every term of the series is a_1, so that the series is $a_1 + a_1 + a_1 + \cdots$. This series diverges because the sequence of partial sums $a_1, 2a_1, \ldots$ diverges.

Case 3. $a_1 \neq 0$ and $r = -1$. In this case, the terms of the series are alternately a_1 and $-a_1$, so that the series is $a_1 - a_1 + a_1 - a_1 + \cdots$. This series diverges because the sequence of partial sums, $a_1, 0, a_1, 0, \ldots$ diverges.

Case 4. $a_1 \neq 0$ and $|r| \neq 1$. In this case, the nth partial sum is

$$S_n = \frac{a_1}{1 - r} - \frac{a_1}{1 - r} r^n.$$

If $|r| < 1$, then r^n $\left(\text{and therefore, } \left|\frac{a_1}{1 - r} r^n\right|\right)$ can be made to approximate 0 as closely as you wish by taking n great enough. Because $\left|S_n - \frac{a_1}{1 - r}\right| = \left|\frac{a_1}{1 - r} r^n\right|$, it follows that

$$S = \lim_{n \to \infty} S_n = \frac{a_1}{1 - r}, \qquad |r| < 1.$$

If $|r| > 1$, $|r^n|$ increases steadily with n; so r^n does not have a limit as n increases without bound and neither does S_n.

The following theorem summarizes these cases.

Theorem. The infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + \cdots$$

converges and has the sum $\frac{a_1}{1 - r}$ if $|r| < 1$. If $a_1 = 0$, the series converges and has the sum 0. If $|r| \geq 1$ and $a_1 \neq 0$, the series diverges.

EXAMPLE Determine the sum of the infinite geometric series

$$\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots.$$

SOLUTION $a_1 = \frac{9}{10}$, $r = \frac{1}{10}$, $|r| < 1$. Since $S = \frac{a_1}{1 - r}$, you have

$$S = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

The series $\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots$ is often written as the infinite decimal $0.999\ldots$, or $0.\overline{9}$, where the bar shows that the indicated digit is repeated without end. Thus, you have

$$0.999\ldots = 0.\overline{9} = 1$$

and similarly, for example,

$$3.\overline{9} = 4, \qquad 5.6\overline{9} = 5.7, \qquad \text{and} \qquad 0.75\overline{9} = 0.76.$$

Oral Exercises

For each of the following infinite geometric series, (a) state the common ratio, (b) tell whether or not the series converges, and (c) state the value of S_1, S_2, and S_3.

1. $24 + 12 + 6 + \cdots$
2. $2 - 2 + 2 - \cdots$
3. $3 + 1 + \frac{1}{3} + \cdots$
4. $\frac{25}{4} + \frac{5}{4} + \frac{1}{4} + \cdots$
5. $\frac{3}{8} - \frac{3}{4} + \frac{3}{2} - \cdots$
6. $7 + 7 + 7 + \cdots$
7. $2 - \frac{1}{2} + \frac{1}{8} - \cdots$
8. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$
9. $0.6 + 0.06 + 0.006 + \cdots$
10. $0.18 + 0.0018 + 0.000018 + \cdots$

State each of the following nonterminating decimals in the form of an infinite geometric series, and state the common ratio.

EXAMPLE $0.333\ldots$

SOLUTION Three tenths plus three hundredths plus three thousandths, and so on; common ratio: one tenth.

11. $0.555\ldots$
12. $0.323232\ldots$
13. $0.040404\ldots$
14. $0.9009009\ldots$
15. $0.767676\ldots$
16. $0.810810810\ldots$

Give an example that shows each of the following statements is false.

17. Every infinite geometric series converges.
18. Any infinite geometric series in which the common ratio is a positive number has a positive sum.

Written Exercises

Find the sum of the given infinite geometric series if it converges. If it is not convergent, so state.

A

1. $54 + 18 + 6 + \cdots$
2. $1000 - 200 + 40 - \cdots$
3. $7 + 3 + \frac{9}{7} + \cdots$
4. $3 - 3 + 3 - \cdots$
5. $\frac{3}{2} - 1 + \frac{2}{3} - \cdots$
6. $\frac{4}{5} + \frac{2}{25} + \frac{1}{125} + \cdots$
7. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \cdots$
8. $0.7 + 0.07 + 0.007 + \cdots$
9. $49 + 14 + 4 + \cdots$
10. $0.3 - 0.003 + 0.00003 - \cdots$
11. $\sum_{n=1}^{\infty} 3(\frac{1}{4})^{n-1}$
12. $\sum_{n=1}^{\infty} 5(-\frac{1}{10})^{n-1}$
13. $\sum_{n=1}^{\infty} \frac{1}{25}(\frac{5}{2})^{n-1}$
14. $\sum_{n=1}^{\infty} \frac{2}{3}(-\frac{3}{4})^{n-1}$

In Exercises 15–20 find the requested value for the infinite geometric series described.

15. $r = \frac{1}{2}$, $S = 12$, $a_1 = \underline{?}$
16. $r = -\frac{2}{3}$, $S = 8$, $a_1 = \underline{?}$
17. $r = \frac{1}{5}$, $S = -25$, $a_1 = \underline{?}$
18. $a_1 = 8$, $S = 10$, $r = \underline{?}$
19. $a_1 = \frac{5}{2}$, $S = \frac{3}{2}$, $r = \underline{?}$
20. $S = 5a_1$, $r = \underline{?}$

In Exercises 21–28 convert the given nonterminating decimal to a fraction by rewriting the decimal as an infinite geometric series (see Oral Exercises 11–16) and finding the sum of the series.

21. 0.555 ...
22. 0.090909 ...
23. 0.373737 ...
24. 0.0666 ...
25. 0.181818 ...
26. 0.135135135 ...
27. 0.06006006 ...
28. 0.270270270 ...

B

29. The sum of an infinite geometric series whose first term is 2 and whose common ratio is x is $\dfrac{4}{x}$. Find x.
30. The sum of an infinite geometric series whose first term is 36 and whose common ratio is x is $-25x$. Find x.
31. For what value of x will the sum of the infinite geometric series $x + x^2 + x^3 + \cdots$ be 3?
32. Find x so that the sum of the infinite geometric series $x + x^3 + x^5 + \cdots$ will be $\frac{3}{8}$.
33. Find all possible values of x for which the sum of the series $12x + 36x^3 + 108x^5 + \cdots$ is 24.

C

34. Suppose that the two infinite geometric series $a_1 + a_1r + a_1r^2 + \cdots$ and $2a_1 + 2a_1s + 2a_1s^2 + \cdots$ have the same sum for some $r \neq \frac{1}{2}$, $0 < r < 1$. Find s in terms of r and give an example of two series with this relationship.

Problems

A 1. A ball is dropped straight down from a height of 50 cm. It rebounds $\frac{1}{3}$ of its previous maximum height with each successive bounce. How far will the ball travel before coming to rest? (Count distances traveled up and distances traveled down.)

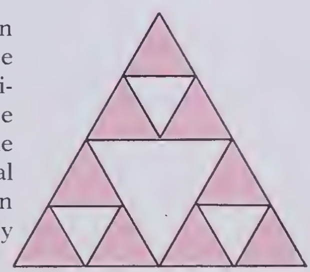
2. A square piece of paper with sides of 40 cm is cut into four smaller squares, each with sides of 20 cm. Three of these squares are placed side by side. The remaining square is cut into four smaller squares, each with sides of 10 cm. Three of these squares are placed beside the bigger squares. The fourth square is cut into four equal smaller squares, and the process is continued indefinitely. What will be the length of the string of squares?

3. A tortoise moving along a straight line traveled 2 m in 1 min. In the next minute, it moved 1 m. In each succeeding minute, it traveled half as far as it did in the previous minute. If the tortoise went on traveling this way forever, how far would it go?

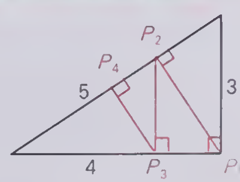
4. Friction and air resistance cause each swing (after the first) of a pendulum bob to be 0.8 as long as that of the preceding swing. If the path of the first swing is 20 cm long, find the total distance traveled by the bob before it comes to rest.

B 5. A rotating flywheel rotates 400 revolutions in the first minute. In the next minute it rotates 240 revolutions. In each succeeding minute, it rotates $\frac{3}{5}$ as many times as it did in the previous minute. How many revolutions will the wheel make before coming to rest?

C 6. The segments joining the midpoints of the sides of an equilateral triangle are drawn, and the area of the triangle they form is removed from the original triangle. The segments connecting the midpoints of the remaining triangles are joined, and the areas of the triangles they form are removed from the original triangle. The remaining area is the shaded region in the diagram. If this process is repeated infinitely many times, how much of the original area will be left?

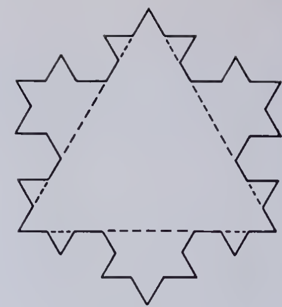


7. In a right triangle with dimensions of 3 cm, 4 cm, and 5 cm, perpendicular segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, and so on are constructed as indicated at the right. Determine the sum of the lengths of all the constructed perpendicular segments. (*Hint:* The new triangles formed are similar to the original triangle.)



8. If $|x| < 1$, find the sum S of the series $1 + 2x + 3x^2 + 4x^3 + \cdots$ (*Hint:* Consider $S - xS$.)

9. A "snowflake" curve is constructed as follows: The sides of an equilateral triangle are trisected, and the middle third of the trisection serves as a base for a new equilateral triangle. This segment is then removed from the figure. The process is continued. If the side of the initial equilateral triangle is of length 1, what is the area enclosed by the snowflake curve if the process is continued without end?



10. Show that the figure described in Problem 9 has no perimeter, that is, that the curve is of unbounded length.

Self-Test 3

VOCABULARY

convergent infinite sequence (p. 241)
limit of an infinite sequence (p. 242)
divergent infinite sequence (p. 243)
nondecreasing sequence (p. 242)
nonincreasing sequence (p. 242)
bounded sequence (p. 242)
partial sum of an infinite series (p. 245)
sum of an infinite series (p. 245)
convergent infinite series (p. 245)
divergent infinite series (p. 245)

In Exercises 1 and 2, the nth term of an infinite sequence is given. Give a_1, a_2, a_3, and a_4 for this sequence, guess the limit L, and give an expression for $|L - a_n|$ in terms of n.

1. $a_n = 2 + \frac{1}{n^2}$ 2. $a_n = \frac{n-1}{n}$ *Obj. 1, p. 241*

Find the sum of the infinite geometric series.

3. $12 - 3 + \frac{3}{4} - \cdots$ 4. $\sum_{n=1}^{\infty} \frac{2}{3}(-\frac{2}{3})^{n-1}$ *Obj. 2, p. 241*

Convert to a fraction by writing as an infinite geometric series and finding the sum.

5. $0.444\ldots$ 6. $0.545454\ldots$

Check your answers with those at the back of the book.

Chapter Summary

1. An *arithmetic sequence,* or *arithmetic progression,* is any sequence in which each term after the first is obtained by adding a fixed number, d, called the *common difference,* to the preceding term. For an arithmetic sequence, you have

$$a_{n+1} = a_n + d \quad \text{and} \quad a_n = a_1 + (n - 1)d \text{ for } n = 1, 2, 3, \ldots.$$

2. The terms between two given terms of an arithmetic sequence are called *arithmetic means* between the given terms; a single arithmetic mean between two numbers is the *average,* or *the arithmetic mean,* of the two numbers.
3. The sum of the first n terms of a given sequence is the associated series, S_n. For an arithmetic series, you have

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n - 1)d].$$

4. You can use the *summation sign* to abbreviate the writing of a series, using an *index* to indicate the *range of summation.* Thus,

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

5. A *geometric sequence,* or *geometric progression,* is any sequence in which each term after the first is the product of the preceding term and a fixed number, r, called the *common ratio.* For a geometric sequence, you have $a_{n+1} = a_n \cdot r$ and $a_n = a_1 r^{n-1}$, $n = 1, 2, 3, \ldots.$
6. The terms between two given terms of a geometric sequence are called *geometric means* between the given terms. A single geometric mean between two numbers is called a *geometric mean* or *mean proportional* of the two numbers; *the* geometric mean of two positive numbers is usually defined to be positive, and of two negative numbers to be negative.
7. A series whose terms are in geometric progression is called a geometric series. For a geometric series, you have

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1 - ra_n}{1 - r}, \qquad r \neq 1.$$

8. An infinite sequence has a *limit* L if you can make the error of approximation, $|L - a_n|$, less than any positive number, however small, by choosing n great enough. Any infinite sequence which has a limit is said to *converge,* or to be *convergent;* if the sequence does not converge, it is said to *diverge,* or to be *divergent.*
9. *Axiom of Completeness:* Every bounded, nondecreasing (or nonincreasing) sequence of real numbers converges, and its limit is a real number.

10. For any infinite series $a_1 + a_2 + \cdots + a_n + \ldots$, $S_n = \sum_{i=1}^{n} a_i$ is called a *partial sum*. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ of partial sums converges to S, then the infinite series is said to *converge*, or to be *convergent*, and its *sum* is defined to be S.
11. For an infinite geometric series with $|r| < 1$, you have the sum

$$S = \frac{a_1}{1 - r}.$$

Chapter Review

1. Give the fourth term of an arithmetic sequence with first term 16 and common difference -2. 7-1

   a. 24 b. 22 c. 10 d. 8

2. Find the value of a_1 for an arithmetic sequence in which $d = 7$ and $a_{11} = 82$. 7-2

   a. 12 b. 5 c. 159 d. 157

3. Insert two arithmetic means between 6 and 18.

   a. 9, 13 b. 10, 14 c. 10, 12 d. 10, 16

4. In an arithmetic series of 7 terms, $a_1 = -4$ and $a_7 = 14$. Find its sum. 7-3

   a. 68 b. 70 c. 10 d. 35

5. Find the value of $\sum_{k=1}^{15} (3k - 1)$.

   a. 345 b. 240 c. 47 d. 360

6. Give the fourth term of a geometric sequence with first term 32 and common ratio $\frac{1}{2}$. 7-4

   a. 256 b. 2 c. 4 d. 128

7. Find the common ratio for a geometric sequence in which $a_1 = 54$ and $a_6 = -\frac{2}{9}$. 7-5

   a. $-\frac{2}{3}$ b. $\frac{1}{3}$ c. 3 d. $-\frac{1}{3}$

8. Insert two geometric means between 4 and -32.

   a. -8, 16 b. 8, -16 c. 6, -6 d. 8, 16

9. Find the sum S_5 for a geometric series in which $a_1 = 8$ and $r = \frac{1}{2}$. 7-6

   a. 15 b. 16 c. $15\frac{1}{2}$ d. 31

10. Find the value of $\sum_{i=1}^{6} 12(-\frac{1}{2})^{i-1}$

a. $5\frac{5}{8}$ b. $-6\frac{1}{2}$ c. $7\frac{7}{8}$ d. $\frac{63}{64}$

11. Guess the limit of the sequence $a_n = 2 - (\frac{1}{2})^n$. 7-7

a. $1\frac{1}{2}$ b. 2 c. $2\frac{1}{2}$ d. 0

12. Find the value of $\sum_{h=1}^{\infty} 8(\frac{2}{3})^{h-1}$. 7-8

a. 24 b. 8 c. 64 d. 12

13. Convert the nonterminating decimal 0.181818 . . . to a fraction.

a. $\frac{9}{50}$ b. $\frac{1}{18}$ c. $\frac{2}{11}$ d. $\frac{7}{8}$

Chapter Test

1. Determine the first four terms of an arithmetic sequence in which a_2 is 9 and the common difference d is -4. 7-1
2. Give a rule for finding successive terms of the arithmetic progression $-11, -2, 7, 16$.
3. Find a_{24} for an arithmetic sequence in which $a_1 = 5$ and $d = 6$. 7-2
4. Insert three arithmetic means between -6 and 6.
5. Find the value of the sum S_8 for the arithmetic sequence 8, 12, 16, ... 7-3
6. Find the value of $\sum_{k=1}^{7} (2k - 1)$.
7. Find a_5 for a geometric sequence in which $a_1 = 8$ and the common ratio r is 2. 7-4
8. The population of a city is observed to increase by 10% a year. What will be the population of the city at the end of the third year after a census year in which its population was found to be 20,000?
9. Insert two geometric means between -2 and 128. 7-5
10. Find the sum of the geometric series for which $a_1 = 8$, $r = -2$, and $n = 6$. 7-6
11. If the nth term a_n of an infinite sequence is $\dfrac{6n - 1}{8n + 1}$ and its limit L is $\frac{3}{4}$, find a_5, $|L - a_5|$, and the general formula for $|L - a_n|$. 7-7
12. Write the decimal 0.272727 . . . as a fraction. 7-8

This loggerhead turtle, a member of an endangered species, was photographed at the Steinhart Aquarium in San Francisco.

8

Radicals and Irrational Numbers

Power Functions, Roots, and Radicals

***OBJECTIVES** for Sections 8-1 through 8-3:*
1. *Use power functions in solving variation problems.*
2. *Determine all real* n*th roots of a given real number.*
3. *Determine all rational roots of an equation with integral coefficients.*

8-1 Power Functions and Variation

A function f defined by an equation of the form

$$f(x) = x^n$$

is called a **power function**. Figure 1 shows the graphs of power functions for the values $n = 1$, 2, 3, and 4.

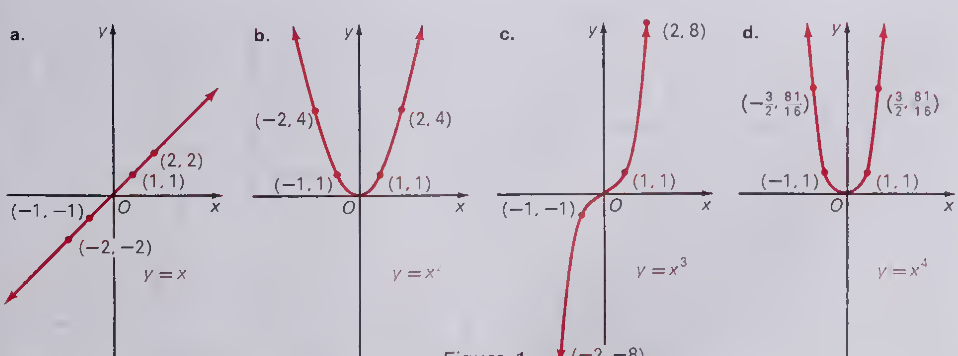


Figure 1

A comparison of the graphs in (a) and (c) of Figure 1 indicates the fact that when n is *odd,* the graph of $y = x$ is symmetric with respect to the *origin.* Thus when the function contains the ordered pair (a, b) it also contains the pair $(-a, -b)$; a function with this property is said to be an **odd function.**

When n is *even,* as in (b) and (d) of Figure 1, the graph is symmetric with respect to the *vertical axis.* Thus when the function contains (a, b), it also contains $(-a, b)$; such a function is called an **even function.**

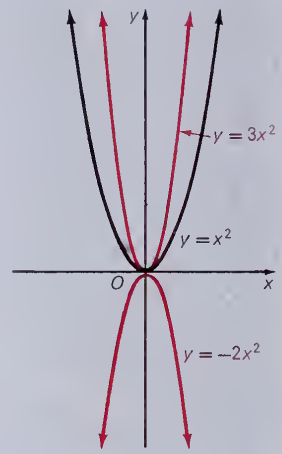


Figure 2

Closely related to the power function is the function defined by

$$y = ax^n, n > 0, a \neq 0.$$

This function is termed a **variation.** We say that *y varies directly as,* or *is directly proportional to, the nth power of x,* and that a is the **constant of variation,** or **proportionality.** When $n = 1$, the function becomes the direct variation $y = ax$ (see page 89).

Figure 2 shows a comparison of the graphs of $y = 3x^2$ and $y = -2x^2$ with that of the power function $y = x^2$.

The concept of variation arises frequently in problems related to the physical world.

EXAMPLE 1 The distance that a body near the earth's surface will fall from rest varies directly as the square of the number of seconds it has been falling. If a boulder falls from a cliff a distance of 176.4 m in 6 s, approximately how far will it fall in 4 s?

SOLUTION Let d = distance (in meters), t = time (in seconds), and a = constant of variation. Then the problem can be solved by either of the following methods.

Method I

$$d = at^2$$

$$176.4 = a(6)^2;\ a = \frac{176.4}{36} = \frac{44.1}{9}$$

$$d = 4.9t^2$$

When $t = 4$,

$$d = 4.9(4)^2 = 78.4$$

Method II

$$d_1 = a(t_1)^2;\ d_2 = a(t_2)^2$$

$$\frac{d_1}{(t_1)^2} = \frac{d_2}{(t_2)^2}$$

$$\frac{176.4}{6^2} = \frac{d_2}{4^2}$$

$$d_2 = \frac{176.4 \times 16}{36} = 78.4$$

$\therefore$ the boulder falls approximately 78.4 m in 4 s. Answer.

Oral Exercises

State (a) whether the function is even, odd, or neither and (b) whether its graph is symmetric to the y-axis, the origin, or neither.

1. $f(x) = -x^4$ **2.** $f(x) = 4x^3 + 1$ **3.** $y = \frac{1}{x}$

4. $y = x^2 + 1$ **5.** $f(x) = |x|$ **6.** $f(x) = x|x|$

Written Exercises

In Exercises 1–6, graph each pair of equations on the same set of axes. Be sure to label each graph clearly.

A **1.** $y = x^2$, $y = -x^2$ **2.** $y = \frac{1}{2}x^2$, $y = 2x^2$ **3.** $y = x^3$, $y = -x^3$

4. $y = x^4$, $y = \frac{x^4}{4}$ **5.** $y = \frac{x^3}{4}$, $y = \frac{x^2}{4}$ **6.** $y = \frac{-x^3}{6}$, $y = \frac{x^2}{6}$

Find the value of a for which the point with the coordinate pairs given lies on the graph of the equation.

7. $(3, -8)$; $y = ax^2$ **8.** $(2, 12)$; $y = ax^3$ **9.** $(-2, -6)$; $y = ax^3$

10. $(\frac{1}{4}, \frac{3}{2})$; $y = ax^2$ **11.** $(\frac{1}{2}, -\frac{1}{24})$; $y = ax^3$ **12.** $(-\frac{1}{2}, \frac{1}{6})$; $y = ax^4$

In Exercises 13–16, assume y varies directly as x.

13. If y is 6 when x is 2, find y when x is 4.

14. If y is 8 when x is -4, find y when x is 6.

B **15.** If y is $\frac{3}{4}$ when x is $\frac{1}{2}$, find y when x is -2.

16. If y is -6 when x is 3, find y when x is $\frac{1}{2}$.

In Exercises 17–20, assume that $y = ax^2z^3$, $a \neq 0$. (We say that y varies as the square of x and the cube of z.) State what happens to the value of y under the given conditions.

17. The value of x is doubled; the value of z remains the same.

18. The value of z is halved; the value of x remains the same.

19. The values of both x and z are doubled.

20. The value of x is doubled; the value of z is halved.

C **21.** If $f(x)$ is an even function and $g(x)$ is an odd function, is $f(g(x))$ an even or an odd function? Prove your answer.

22. If $f(x)$ and $g(x)$ are both odd functions, is $f(g(x))$ an even or an odd function? Prove your answer.

Problems

A 1. Over a fixed period of time, the energy developed by an electric circuit of a fixed resistance varies directly as the square of the current. If the circuit develops 30 kW·h in the given time period when the current is 10 A, how much energy will it develop when the current is 15 A?

2. The power required to propel a boat varies as the cube of its speed. If a certain motorboat requires 400 kW to run at a steady speed of 10 km/h, how many kilowatts are needed to run the boat at 20 km/h?

3. In structural engineering the maximum deflection of a flat rectangular plate varies as the fourth power of the width of the plate. If the maximum deflection is 0.008 cm for a plate 2 m wide, what is the maximum deflection for a plate 5 m wide?

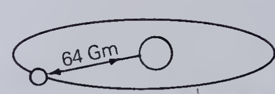
4. The power of a capacitor in an electric circuit varies directly as the square of the voltage. If the power is 0.0968 W when the voltage is 110 V, what is the power when the voltage drops to 100 V?

5. The mass of a spherical drop of mercury of diameter 0.4 cm is 0.456 g. If the mass of a drop of mercury varies directly with the cube of its diameter, what is the mass of a drop of diameter 0.2 cm?

6. The kinetic energy of an 1100 kg car moving at 90 km/h is equivalent to about 0.1 kW·h of electricity. If the kinetic energy varies as the square of the velocity, what is the kinetic energy, to the nearest 0.1 kW·h, of a car moving at 100 km/h?

B 7. The radiancy (energy radiated per unit time per unit area) of an object that absorbs all the radiation that falls on it varies as the fourth power of the Kelvin temperature of the object. If such an object at 300° K has a radiancy of $405 (J/s)/m^2$, what is the radiancy of the object at 100° K?

8. One of Kepler's laws of planetary motion states that the *square* of period of revolution of a planet varies directly as the *cube* of its distance from the sun. Use the information given about the planet Mercury to find the value of the constant k in the formula $T^2 = kr^3$ expressing this relationship. The period T of a planet is expressed in days and its distance r from the sun in gigameters. (A gigameter is equal to 1,000,000,000 m; its symbol is Gm.) Mercury is about 64 Gm from the sun and has a period of revolution of about 102.4 d.



9. The square of the period of revolution of an artificial earth satellite varies directly as the cube of its distance from the center of the earth. A satellite 20,000 km from the center of the earth has a period of revolution of 8 h. What would the period of a satellite 80,000 km from the center of the earth be?

8-2 The Real *n*th Roots of a Number

By observing the graphs in Figures 3, 4, and 5 of the power functions specified by (1) $f(x) = x^2$, (2) $g(x) = x^3$, and (3) $h(x) = x^4$, you can answer questions such as the following about the members of f, g, and h.

(1) How many values in the domain of f are paired with the value $f(x) = 9$ in the range? That is, how many values of x satisfy the equation $x^2 = 9$?

(2) How many values in the domain of g are paired with the value $g(x) = -8$ in the range? That is, how many solutions are there of the equation $x^3 = -8$?

(3) Are there any real values of x such that the pair $(x, -2) \in h$? That is, are there any real values of x that satisfy $x^4 = -2$?

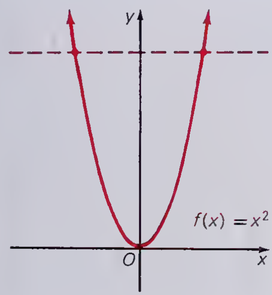


Figure 3

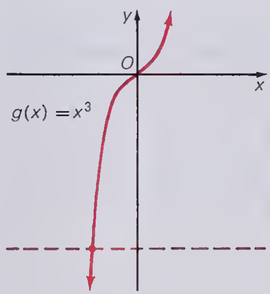


Figure 4

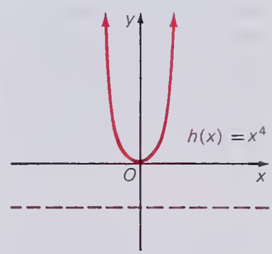


Figure 5

The abscissas of the points shown by red dots on the graphs in Figures 3 and 4 indicate the values of x that satisfy the equation given in each case. In Figure 5 you can see that, since -2 is not an element of the range, there is no pair $(x, -2)$ in h, and hence there is no real solution to $x^4 = -2$.

Each solution of the equation $x^n = b$, for n a positive integer, is called an ***n*th root of *b*.** Thus, 3 and -3 are the 2nd roots (or *square roots*) of 9, and -2 is the 3rd (or *cube*) root of -8.

The facts concerning the *real* nth roots of b, as suggested by Figures 3, 4, and 5, are summarized in the following table.

Number and Nature of Real *n*th Roots of *b*

	$b > 0$	$b < 0$	$b = 0$
n even	one positive root one negative root	no real roots	one root, namely, 0
n odd	one positive root	one negative root	one root, namely, 0

The symbol $\sqrt[n]{b}$ (read "the nth root of b") denotes the principal nth root of b, that is:

1. The *nonnegative* nth root of b if n is even and $b \geq 0$. For example, we write

$$\sqrt[2]{36} = 6, \ -\sqrt[2]{36} = -6, \ \sqrt[4]{0} = 0.$$

2. The single real nth root of b if n is odd. For example,

$$\sqrt[3]{8} = 2, \ -\sqrt[3]{8} = -2.$$

Notice that we are talking about only the *real* nth roots of b; that is, in the set of real numbers, $\sqrt[4]{-16}$ is not defined.

The symbol $\sqrt[n]{b}$ is called a **radical**; b is the **radicand** and n the **index**. (The index 2 for the square root is usually omitted.) Always be careful to include all of the desired radicand under the radical sign. Thus,

$$\sqrt{36 - 9} = \sqrt{27} = 3\sqrt{3}, \quad \text{whereas} \quad \sqrt{36} - 9 = 6 - 9 = -3.$$

Notice that $\sqrt[3]{(-2)^3} = -2$, but $\sqrt[4]{(-2)^4} = 2$, because a radical of even index denotes a *nonnegative* root. In general,

$$\sqrt[n]{b^n} = b \text{ if } n \text{ is odd},$$

and

$$\sqrt[n]{b^n} = |b| \text{ if } n \text{ is even}.$$

Oral Exercises

State the value of the indicated principal root, or state that the expression is not defined as a real number.

1. $\sqrt[3]{27}$
2. $\sqrt[3]{-8}$
3. $\sqrt{-25}$
4. $-\sqrt[4]{16}$
5. $\sqrt[4]{-1}$
6. $\sqrt[5]{0}$
7. $-\sqrt{-\frac{1}{16}}$
8. $-\sqrt[3]{-\frac{1}{8}}$
9. $\sqrt{(-3)^2}$
10. $-\sqrt[4]{(-7)^4}$
11. $\sqrt{-(-6)^2}$
12. $\sqrt[4]{(-\frac{1}{2})^4}$

Is the statement true for all real numbers? If it is false give a counterexample.

13. $\sqrt{a^2} = a$
14. $\sqrt{a^2} = \sqrt{(-a)^2}$
15. $\sqrt[3]{a^3} = a$

Written Exercises

Express each radical as a decimal to the nearest tenth by using Tables 3 and 4 in the Appendix.

A

1. $\sqrt{7}$
2. $\sqrt{1.4}$
3. $\sqrt[3]{25}$
4. $\sqrt{\frac{11}{5}}$
5. $-\sqrt{11}$
6. $\sqrt{0.81}$
7. $\sqrt[3]{-47}$
8. $\sqrt[3]{(-3)^4}$
9. $\sqrt{-670}$
10. $\sqrt{91}$

Solve over the set of rational numbers.

EXAMPLE 1 $1.44x^2 - 1 = 0$

SOLUTION $1.44x^2 - 1 = 0$

$$x^2 = \frac{1}{1.44} = \frac{100}{144}$$

$$x = \pm\frac{10}{12} = \pm\frac{5}{6}$$

$\therefore$ the solution set is $\{\frac{5}{6}, -\frac{5}{6}\}$. Answer.

11. $4x^2 = 36$ **12.** $1.96x^2 - 1 = 0$ **13.** $x^4 - 16 = 0$ **14.** $25x^2 = 9$

15. $x^2 + 4 = 0$ **16.** $3x^3 = 0.024$ **17.** $125x^3 - 8 = 0$ **18.** $32x^5 + 1 = 0$

Find all real values of the variable for which the statement is true.

EXAMPLE 2 $\sqrt{(x-4)^2} = x - 4$

SOLUTION Since $\sqrt{(x-4)^2}$ denotes a nonnegative number, the equation is true if and only if $x - 4$ is nonnegative; that is, $x - 4 \geq 0$, or $x \geq 4$. Answer.

19. $\sqrt{x-7} = 0$ **20.** $\sqrt{(x+2)^2} = x + 2$ **21.** $\sqrt[3]{(y-1)^3} = y - 1$

22. $\sqrt{(x-15)^2} = |x - 15|$ **23.** $\sqrt[3]{x-5} = 0$ **24.** $\sqrt{9 - x^2} \in \mathcal{R}$

Determine the solution set over $\mathcal{R}$.

B **25.** $\sqrt[3]{(x-1)^3} = 4$ **26.** $\sqrt{(x+2)^2} = 3$ **27.** $\sqrt{x^2} = x + 6$

C **28.** Prove: If $x \geq 0$, $y \geq 0$, then $\sqrt{x^2 + y^2} \leq x + y$. (*Hint:* Assume $\sqrt{x^2 + y^2} > x + y$ and show that this leads to a contradiction; see Exercise 34 on p. 49.)

29. Show that if $\sqrt{x^2 + y^2} = x + y$, then $x = 0$ or $y = 0$.

8-3 The Roots of a Polynomial Equation

A *rational number* has been defined (page 192) as any number that can be represented in the form $\frac{a}{b}$, where a and b are integers, $b \neq 0$.

You know that $\sqrt{4}$ is a rational number because $\sqrt{4} = \frac{2}{1}$, but what about $\sqrt{5}$? Since $\sqrt{5}$ is a root of the equation $x^2 - 5 = 0$, we can answer the question by using the following theorem concerning the rational roots, if any, of such a polynomial equation. (Note first that the **leading coefficient** of a polynomial is the coefficient of the term of highest degree.)

Rational Root

Theorem. Let $f(x)$ be a simplified polynomial with integral coefficients. If the equation $f(x) = 0$ has a rational root $\frac{p}{q}$ that is in lowest terms, then p must be an integral factor of the constant term, and q an integral factor of the leading coefficient, of $f(x)$.

EXAMPLE 1 If $f(x) = 4x^3 + 4x^2 - x - 1$, list the possible rational roots of $f(x) = 0$; then find which, if any, actually satisfy that equation.

SOLUTION The numerator of any rational root $\frac{p}{q}$ must be an integral factor of -1, and the denominator an integral factor of 4. That is, $p \in \{1, -1\}$ and $q \in \{1, -1, 2, -2, 4, -4\}$. Hence the possible rational roots $\frac{p}{q}$ are

$$\pm\tfrac{1}{4},\ \pm\tfrac{1}{2},\ \pm 1.$$

Next, check each possible value for $\frac{p}{q}$ in $f(x)$ to see which, if any, satisfy $f(x) = 0$.

$$f(\tfrac{1}{4}) = -\tfrac{15}{16}; \quad f(\tfrac{1}{2}) = 0; \quad f(1) = 6;$$
$$f(-\tfrac{1}{4}) = -\tfrac{9}{16}; \quad f(-\tfrac{1}{2}) = 0; \quad f(-1) = 0.$$

$\therefore$ the rational roots of $4x^3 + 4x^2 - x - 1 = 0$ are $\frac{1}{2}$, $-\frac{1}{2}$ and -1. Answer.

EXAMPLE 2 If $f(x) = x^2 - 5$, determine the rational roots, if any, of $f(x) = 0$.

SOLUTION A rational root $\frac{p}{q}$ must satisfy $p \in \{5, -5, 1, -1\}$ and $q \in \{1, -1\}$. Hence $\frac{p}{q} \in \{5, -5, 1, -1\}$. Checking in $f(x)$:

$$f(5) = f(-5) = 20 \neq 0;$$
$$f(1) = f(-1) = -4 \neq 0.$$

$\therefore x^2 - 5 = 0$ has no rational roots. Answer.

Real numbers that are not rational are called **irrational numbers.** Example 2 establishes the fact that $\sqrt{5}$ and $-\sqrt{5}$, which are the roots of $x^2 - 5 = 0$, are irrational numbers. In general, for any positive integer b and any integer $n > 1$, you can show, by considering the equation $x^n = b$, that $\sqrt[n]{b}$ is an irrational number unless b is the nth power of an integer.

Example 3 below illustrates the following fact: *The sum, difference, product, or quotient of a rational number and an irrational number is an irrational number.* (Exceptions: $0 \cdot x$ and $\frac{0}{x}$, where x is irrational.)

EXAMPLE 3 Determine whether $2 + 3\sqrt{5}$ is a rational or an irrational number.

SOLUTION If $2 + 3\sqrt{5}$ is a rational number, there must be integers a and b, $b \neq 0$, such that

$$2 + 3\sqrt{5} = \frac{a}{b}$$

or

$$\sqrt{5} = \frac{a - 2b}{3b}.$$

Since $a - 2b$ and $3b$ are both integers and $b \neq 0$, $\frac{a-2b}{3b}$ is a rational number. But we know that $\sqrt{5}$ is an irrational number. Thus we have been led to a contradiction, and accordingly, our hypothesis that $2 + 3\sqrt{5}$ is a rational number must be false. Hence $2 + 3\sqrt{5}$ is irrational. Answer.

Oral Exercises

State the possible rational roots of the given equation.

1. $x^3 - 2 = 0$
2. $x^3 + 9 = 0$
3. $x^2 - 5x - 7 = 0$
4. $2x^3 - x + 1 = 0$
5. $4x^3 - 3x^2 - 1 = 0$
6. $x^4 - 5x^3 + 6 = 0$
7. $10x^4 + 3x^3 - x + 1 = 0$
8. $2x^3 - 7x^2 + x - 3 = 0$
9. $3x^3 - x + 4 = 0$

Written Exercises

In Exercises 1–10, show that the number is a root of the given equation. Then use the method of Example 2 to show that the equation has no rational roots, and therefore that the given number must be irrational.

A

1. $\sqrt{2}$; $x^2 - 2 = 0$
2. $\sqrt[3]{5}$; $x^3 - 5 = 0$
3. $\sqrt{6}$; $x^2 - 6 = 0$
4. $-\sqrt[3]{4}$; $x^3 + 4 = 0$
5. $\sqrt[3]{-\frac{1}{2}}$; $2x^3 + 1 = 0$
6. $\sqrt{\frac{9}{2}}$; $2x^2 - 9 = 0$

Use the method of Example 3 to show that the given number is irrational. Use the results of Exercises 1–3 above.

7. $\sqrt{2} + 5$
8. $\frac{\sqrt{2}}{3}$
9. $2\sqrt[3]{5}$
10. $7 - \sqrt[3]{5}$
11. $\frac{7 + \sqrt{6}}{2}$
12. $\frac{2\sqrt{6} - 1}{3}$

Find the rational roots of the given equation by trying each of the possible roots as a value for x.

13. $x^3 + x + 2 = 0$
14. $x^3 - 3x^2 + x - 3 = 0$
15. $x^3 - 3x + 2 = 0$
16. $x^3 - 3x^2 + 4 = 0$
17. $x^4 - 8x^2 - 9 = 0$
18. $x^4 - x^3 - 5x^2 - x - 6 = 0$
19. $2x^3 + x^2 - 2x - 1 = 0$
20. $3x^3 - x^2 + 3x - 1 = 0$

B 21. $2x^3 + 7x^2 + 2x - 3 = 0$
22. $3x^4 - x^3 + x^2 - x - 2 = 0$
23. $4x^3 - 8x^2 - 11x - 3 = 0$
24. $6x^4 - 5x^3 - 15x^2 + 4 = 0$

C 25. This exercise is a partial proof of the theorem on page 262 so the theorem itself cannot be used in the proof.
  a. Prove that if n is an integral root of $ax^3 + bx^2 + cx + d = 0$ (a, b, c, and d are integers) then n must be a factor of d.
  b. Prove that if n is an integer and $\frac{1}{n}$ is a root of $ax^3 + bx^2 + cx + d = 0$ (a, b, c, and d are integers) then n must be a factor of a.

26. Prove that if t is an irrational number, then for any rational numbers a and b, $a \neq 0$, the number $at + b$ is irrational.

Self-Test 1

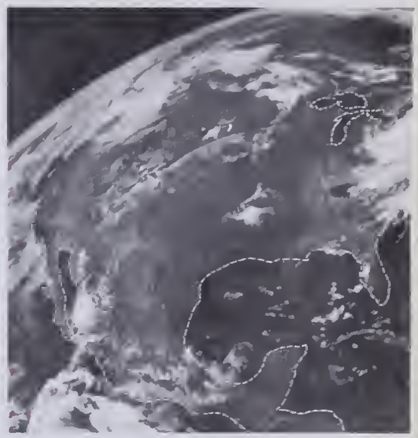
VOCABULARY
power function (p. 255)
odd function (p. 256)
even function (p. 256)
variation (p. 256)
constant of variation (p. 256)
nth root of b (p. 259)
radical (p. 260)
radicand (p. 260)
index of a radical (p. 260)
leading coefficient (p. 261)
irrational number (p. 262)

1. If y varies directly as x^3 and y is 3 when x is 2, find y when $x = 4$. *Obj. 1, p. 255*
2. Determine the value of **(a)** $\sqrt{(-14)^2}$; and **(b)** $\sqrt[3]{-8}$. *Obj. 2, p. 255*
3. Solve $5x^3 = 40$ over the rational numbers.
4. Determine all rational roots of the equation $x^3 - 7x - 6 = 0$. *Obj. 3, p. 255*

Check your answers with those at the back of the book.

Careers

in Meteorology



Meteorology is the study of atmospheric phenomena. In addition to weather forecasting, there are many other specializations in the field, such as climatology (the study of average weather patterns), the design of meteorological instruments, and the study of the chemical composition and physical properties of the atmosphere.

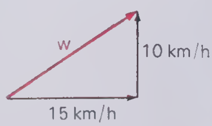
Meteorologists work in a variety of situations. In many countries government and military weather stations employ a large number of meteorologists. The operations of commercial airlines, aerospace industries, and insurance companies rely upon weather information and other atmospheric data. A number of meteorologists teach and do research at universities. The knowledge and skills of meteorologists are also needed in the field of air pollution control.

Satellite pictures (above) of the atmosphere help to predict the weather. Teletype machines (below) provide the latest weather reports.

EXAMPLE Thermal wind in the atmosphere results from temperature differences within a layer of air. To find the direction and speed of a wind at a particular altitude above the ground, meteorologists find the sum of the *vectors* (see Section 16-5) representing the thermal wind and the wind at the surface.

If the wind at the surface is from the south at 10 km/h and the thermal wind between the surface and an altitude of 5450 m is from the west at 15 km/h, what is the velocity of the wind at an altitude of 5450 m?

SOLUTION The diagram at the right is a vector representation of the speed and direction of the winds. Use the Pythagorean Theorem (page 40) to find the speed of the resultant wind, w:



$$\begin{aligned} w^2 &= 100 + 225 \\ w^2 &= 325 \\ w &= \sqrt{325} \\ &= 5\sqrt{13} \\ w &\approx 18.0 \text{ km/h} \end{aligned}$$

Numerals for Rational and Irrational Numbers

***OBJECTIVES** for Sections 8-4 through 8-6:*

1. Express a rational number by a terminating or repeating decimal numeral, and express such a numeral as a fraction.
2. Represent a given number in standard notation.
3. Estimate products and quotients.
4. Find rational and irrational numbers between any two given real numbers.

8-4 Decimal Numerals for Rational Numbers

To find a decimal numeral for a rational number, first express the number as the quotient of two integers and then perform the indicated division.

EXAMPLE 1 Express as a decimal numeral (a) $1\frac{3}{8}$; and (b) $\frac{1}{22}$. Check your answer in part (a).

SOLUTION a. $1\frac{3}{8} = \frac{11}{8} = 11 \div 8$

$$\begin{array}{r} 1.375 \\ 8\overline{)11.000} \\ \underline{8} \\ 30 \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

b. $\frac{1}{22} = 1 \div 22$

$$\begin{array}{r} 0.0454545 \\ 22\overline{)1.0000000} \\ \underline{88} \\ 120 \\ \underline{110} \\ 100 \\ \underline{88} \\ 120 \\ \underline{110} \\ 100 \\ \underline{88} \\ 120 \\ \underline{110} \\ 10 \end{array}$$

Check:

$$\begin{aligned} 1.375 &= 1 + \tfrac{3}{10} + \tfrac{7}{100} + \tfrac{5}{1000} \\ &= \frac{1000 + 300 + 70 + 5}{1000} \\ &= \tfrac{1375}{1000} = \tfrac{11}{8} \end{aligned}$$

$\therefore\ 1\frac{3}{8} = 1.375.$ Answer.

You can see that in Example 1(a), the division process effectively terminates when the remainder 0 occurs, since thereafter only 0's appear in the quotient. Accordingly, the decimal numeral for $\frac{11}{8}$ is called a **terminating decimal.**

In Example 1(b), however, the remainder 0 never occurs. As a result, the quotient consists of an endlessly *repeating block of digits,* or **repetend:** 45. The decimal numeral for $\frac{1}{22}$ is an example of a **repeating (or periodic) decimal,** which we usually denote as follows:

$$\tfrac{1}{22} = 0.0\overline{45}$$

The bar indicates the block of digits that repeats without end.

Example 1 illustrates the following facts concerning the result when an integer p is divided by a positive integer q:

1. The remainder at each step in the process may be any element of $\{0, 1, 2, \ldots, q - 1\}$.
2. After only 0's are left in the dividend, within at most $q - 1$ steps either 0 occurs as a remainder and the division process stops, or one of the possible nonzero remainders recurs, initiating a repeating sequence of dividends in the algorithm process, and hence a repeating block of digits in the quotient.
3. If the repetend is $\overline{0}$, the quotient is a terminating decimal; otherwise, the quotient is a repeating decimal.

> The decimal representation of any rational number $\dfrac{p}{q}$ either terminates or has a repetend of fewer than q digits. Conversely, every terminating or repeating decimal represents a rational number.

You can use either of the following two methods to convert a repeating decimal to a common fraction. Method I, first presented in Section 7-8, uses the formula for the sum of an infinite series. Method II involves multiplying the repeating decimal by 10^n where n is the number of digits that repeat.

EXAMPLE 2 Express $0.\overline{378}$ as a ratio of two integers.

Method I

SOLUTION Use the formula on page 246 for the sum of an infinite geometric series.

$$0.\overline{378} = 0.378 + 0.378(.001) + 0.378(.001)^2 + \cdots,$$

so that

$$a_1 = 0.378 \text{ and } r = .001.$$

$$S = \frac{a_1}{1 - r} = \frac{0.378}{1 - .001} = \frac{0.378}{.999} = \frac{42}{111}. \quad \text{Answer.}$$

Method II

Let $N = 0.\overline{378}$.

$$\begin{aligned} 1000N &= 378.\overline{378} \\ N &= 0.\overline{378} \\ \hline 1000N - N &= 378 \\ 999N &= 378 \\ N &= \frac{378}{999} = \frac{42}{111}. \quad \text{Answer.} \end{aligned}$$

Oral Exercises

State (a) the first three terms of the representation of the repeating decimal as an infinite series; and (b) the common ratio of the series.

EXAMPLE $0.\overline{34}$

SOLUTION $0.\overline{34} = 0.34 + 0.0034 + 0.000034 + \ldots$; common ratio: 0.01.

1. $0.\overline{23}$
2. $0.\overline{7}$
3. $0.0\overline{8}$
4. $0.0\overline{18}$
5. $1.\overline{717}$
6. $0.\overline{358}$
7. $0.\overline{1725}$
8. $0.0\overline{4563}$

Written Exercises

Express as a decimal.

A

1. $\frac{5}{16}$
2. $\frac{9}{40}$
3. $\frac{5}{6}$
4. $\frac{-8}{11}$
5. $\frac{49}{75}$
6. $\frac{-3}{7}$
7. $\frac{9}{37}$
8. $\frac{38}{41}$

Express the given decimal as a fraction in lowest terms.

9. 0.325
10. 0.0046
11. 3.028
12. -5.0036
13. $0.\overline{18}$
14. $0.\overline{57}$
15. $0.\overline{296}$
16. $3.\overline{09}$

B

17. $1.2\overline{7}$
18. $-2.\overline{540}$
19. $0.58\overline{3}$
20. $-1.6\overline{81}$
21. $2.\overline{037}$
22. $1.2\overline{027}$
23. $-1.7\overline{9}$
24. $0.14\overline{864}$

C

25. Show that any repeating decimal of the form $0.\overline{abc}$ (3-digit repetend) can be represented as a fraction $\frac{p}{q}$ in lowest terms where q is an integer of at most 3 digits.

26. Show that if a fraction in lowest terms can be represented by a terminating decimal, its denominator can have only powers of 2 and 5 as factors.

8-5 Standard Notation

We use integer powers of 10 in representing any nonzero real number in *standard notation.* Such a number is given in **standard notation** (also called **scientific notation** when used in the expression of measurements) provided it is named as a product,

$$a \times 10^n,$$

where $1 \leq |a| < 10$ and n is an integer. As illustrated by the last three examples in the table at the right, if $n = 0$ (so that $10^n = 1$), then the second factor ordinarily is not written, and if $n \neq 0$ but $a = 1$, then the first factor ordinarily is not written.

Decimal notation	*Standard notation*
$-4_{\wedge}1.5$	-4.15×10^1
$0.005_{\wedge}8$	5.8×10^{-3}
$3_{\wedge}.207$	3.207
$-1_{\wedge}$	-1
$0.0001_{\wedge}$	10^{-4}

In a numeral, each digit reporting the *number of units* of measure contained in a measurement is called a **significant digit.** Thus, 27.2 and 0.0103 have three significant digits. In each decimal numeral in the table, notice the red caret that is placed after the first significant digit in the numeral. By counting the number of places *from the caret to the decimal point,* you obtain n. Do you see that n is positive or negative according as you count to the right or to the left from the caret?

You can easily compare numbers when they are represented in standard notation.

EXAMPLE 1 $6.3 \times 10^5 > 9.1 \times 10^4$, because $5 > 4$.

EXAMPLE 2 $5.1 \times 10^3 < 5.2 \times 10^3$, because $5.1 < 5.2$.

It is often convenient to break off, or **round** a lengthy decimal, leaving a numeral that represents an approximation of the number named by the given decimal. Using $\approx$ to mean "equals approximately," you may write

$$248.13 \approx 248.1, \qquad 248.13 \approx 248, \qquad 248.13 \approx 250,$$

as approximations of 248.13 to the nearest tenth, the nearest unit, and the nearest multiple of 10, respectively. In rounding, use this rule:

> To round a decimal, add 1 to the last digit retained if the value of the first digit dropped is 5 or more; otherwise leave the retained digits unchanged.

Under this rule, the difference between a number and its approximation (the **rounding error**) is *at most* half the unit of the last digit retained.

For example, the statement $1.31\overline{7} \approx 1.32$ is equivalent to

$$1.32 - 0.005 \leq 1.31\overline{7} < 1.32 + 0.005, \qquad \text{or} \qquad 1.315 \leq 1.31\overline{7} < 1.325.$$

You can round decimals and use standard notation to help you *estimate* products and quotients rapidly.

EXAMPLE 3 Find a one-significant-digit estimate of A, if

$$A = \frac{2120 \times 36.94 \times 194}{365.3}.$$

SOLUTION

1. Round each number to its one-significant-digit approximation. $\quad A \approx \dfrac{2000 \times 40 \times 200}{400}$
2. Express the approximation in standard notation. $\quad A \approx \dfrac{2 \times 10^3 \times 4 \times 10^1 \times 2 \times 10^2}{4 \times 10^2}$
3. Compute and round to one significant digit. $\quad A \approx \dfrac{2 \times 4 \times 2}{4} \times 10^{3+1+2-2}$

$\therefore A \approx 4 \times 10^4$ or 40,000 **Answer.**

(To four decimal places, A is actually equal to 41,589.6064.)

Oral Exercises

State the given number using standard notation.

1. 0.135 2. 3420 3. 0.025 4. 786

State the given number using decimal notation.

5. 9.4×10^3 6. 8.85×10^{-1} 7. 6.15×10^{-2} 8. 4.7×10^5

Round the given number to the nearest tenth.

9. 6.77 10. 4.348 11. 0.65 12. 0.039

State which of the numbers given is greater.

13. 4.63×10^5 or 8.49×10^4
14. 3.7×10^5 or 3.7×10^8
15. 5.1×10^2 or 5.1×10^{-5}
16. 2×10^{-3} or 2.01×10^{-3}

odd 1-25

Written Exercises

Express each of the following in standard notation.

A

1. 45,000 2. 0.0647 3. 0.00015 4. 382
5. 7258 6. 0.0105 7. 10.32 8. 0.006004

Express each of the following in decimal notation.

9. 4.3×10^4
10. 1.4×10^{-3}
11. 5.69×10^{-2}
12. 3.11×10^5
13. 8.08×10^{-1}
14. 7.003×10^{-4}
15. 4.625×10^6
16. 2.172×10^{-5}

Convert each number of the given expression to standard notation; then express it as a single numeral in standard notation.

17. $\dfrac{(60{,}000)(0.04)}{0.008}$
18. $\dfrac{(0.2)(9000)}{3{,}000{,}000}$
19. $\dfrac{(450{,}000)(2800)}{(0.02)(0.00021)}$
20. $\dfrac{(0.0063)(240)}{(4{,}200{,}000)(0.09)}$
21. $\dfrac{(0.33)(0.000028)(2000)}{(0.088)}$
22. $\dfrac{(0.0018)(1600)}{(24{,}000)(12)}$

Find a one-significant-digit estimate of the given number.

B 23. $\dfrac{(397)(0.00584)}{(8250)(0.973)}$
24. $\dfrac{(0.0647)(80{,}990)}{(0.000448)(0.279)}$
25. $\dfrac{(0.00319)(0.0585)(702{,}000)}{0.000092}$
26. $\dfrac{5.98}{(1979)(0.206)(0.00258)}$
27. Let $a = 3.415$ and $b = 2.707$. Write each as a one-significant-digit approximation to estimate $a + b$, $a - b$, $a \times b$, and $a \div b$. To check the reliability of the estimate, calculate the values of these expressions to four significant digits. For which operation was the estimate the least reliable?

C 28. Repeat the process of Exercise 27 using other numbers. Make a generalization about using estimates.

ON THE CALCULATOR

For very large or very small numbers, it is necessary for calculators to use standard notation. The readout usually leaves two spaces before the appropriate exponent of 10; the 10 itself is not shown, however.

EXAMPLE $0.0000369 \div 410{,}000$

SOLUTION 3.69 $\boxed{\text{EE}}$ 5 $\boxed{+/-}$ $\boxed{\div}$ 4.1 $\boxed{\text{EE}}$ 5 $\boxed{=}$ 9. – 11

9. – 11 $= 9 \times 10^{-11}$. Answer.

Exercises

Evaluate. Express your answer in standard notation.

1. $1{,}234{,}000 \times 62{,}000{,}000$
2. $7{,}340 \div 0.0015$
3. 9072×0.00098
4. $0.000048 \div 10{,}368$
5. $821{,}000{,}000 \div 6{,}420{,}000{,}000$
6. $0.00751 \times 0.00000049$

programming in BASIC

In BASIC, answers that are not exact are rounded to six significant digits. Also, numbers greater than 999999 are rounded to six digits and expressed in a manner that corresponds exactly to the standard notation described in Section 8-5.

Multiplying 54321 by 12345 exactly gives the product 670,592,745. The program

```
10  PRINT 54321*12345
20  END
```

gives the output 6.70593E+08. The "E+08" means "times 10 with the exponent positive 8." Thus,

$$6.70593E+08 = 6.70593 \times 10^8.$$

To discover how the computer expresses small numbers, try the following programs.

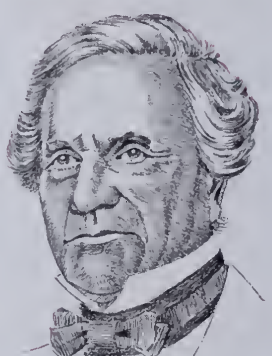
Exercises

1.
```
10  FOR N=2 TO 20
20  PRINT "1 /";N;" =";1/N
30  NEXT N
40  END
```

2.
```
10  LET D=10
20  FOR I=1 TO 10
30  PRINT 5/D
40  LET D=D*10
50  NEXT I
60  END
```

3. In the preceding program change line 30 to: 30 PRINT 3.14159/D

Charles Babbage
1792–1871



Charles Babbage shortly after 1810 sought to revive mathematical research in England. As a result, a group of young Cambridge mathematicians, of which Babbage was a leader, founded the Analytical Society. After more than half a century of waning interest in mathematics in England, the stage was set for renewed interest in mathematics for the latter part of the nineteenth century.

Babbage did not produce any outstanding new results in mathematics. However, he may be called a pioneer in the principles on which our modern computers are based. In 1833 Babbage conceived a "difference engine." In fact, it was a digital computer which would have been able to perform arithmetic operations and even store data using gears, wheels and levers. His "engine" unfortunately was never completed.

8-6 Decimals for Irrational Numbers

The axiom of completeness (page 242) for the set of real numbers can be used to show that every real number has a decimal representation, and conversely. Since the rational numbers are the real numbers named by terminating or repeating decimals, the irrational numbers must be the real numbers represented by the nonterminating, nonrepeating decimals.

It is possible to find successive digits in the infinite decimal representing an irrational number such as $\sqrt{2}$ by various methods. In the geometric method in Figure 6, where the interval from 1 to 2 is subdivided into tenths, you can see that

$$1.4 < \sqrt{2} < 1.5.$$

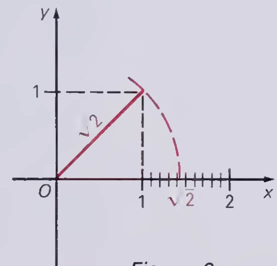


Figure 6

You can verify this algebraically as follows:

$$(1.4)^2 \overset{?}{<} (\sqrt{2})^2 \overset{?}{<} (1.5)^2$$
$$1.96 < 2 < 2.25$$

Next, if the same interval were divided into hundredths and you could observe the location of $\sqrt{2}$, you would find, and again verify by comparing squared members of the following inequality, that

$$1.41 < \sqrt{2} < 1.42.$$

By further subdividing the unit interval into thousandths, ten-thousandths, and so on, you could obtain as many additional digits in the decimal for $\sqrt{2}$ as you desire. In accordance with the usual rules for rounding, when you write $\sqrt{2} \approx 1.414$, you mean

$$1.414 - 0.0005 \le \sqrt{2} < 1.414 + 0.0005.$$

Notice that, for example, the decimal numeral

$$0.2121121112\ldots,$$

which consists, from the decimal point on, of a succession of 2's, separated first by one 1, then by two 1's, then three 1's, and so on, is neither terminating nor repeating. The numeral, therefore, represents an irrational number. No such pattern is known for the decimal expansion of the irrational number $\sqrt{2}$.

Using decimal representations, you can illustrate the following property of **density** of the set of real numbers.

Property of Density

Between any two real numbers, there is another real number.

In fact, between any two real numbers there are both rational and irrational numbers.

EXAMPLE a. Find a rational number between $4.\overline{5211}$ and $4.\overline{521}$.

b. Find an irrational number between $\sqrt{2}$ and $1.41515515551\ldots$.

SOLUTION a. $4.\overline{5211} < 4.5212 < 4.\overline{521}$; $\therefore$ a rational number between $4.\overline{5211}$ and $4.\overline{521}$ is 4.5212.

b. Since by Table 3 in the Appendix you have $\sqrt{2} < 1.415$, you can see that

$$\sqrt{2} < 1.4150505505550\ldots < 1.41515515551\ldots,$$

and $1.4150505505550\ldots$ is one such irrational number.

Oral Exercises

State whether the given number is rational or irrational.

1. $3\sqrt{2}$
2. $4.5\overline{3}$
3. $\sqrt{3} - 0.\overline{14}$
4. $\dfrac{\sqrt{2}}{2.09}$
5. $\dfrac{5}{17}$
6. $\sqrt[3]{64}$
7. $0.0145145145\ldots$
8. $2.3131131113\ldots$
9. $6.5757757775\ldots$
10. The sum of the numbers in Exercises 8 and 9.
11. The product of the numbers in Exercises 1 and 4.
12. The decimal consisting of the positive integers written in order: $0.12345678910111213\ldots$
13. The decimal consisting of 1's separated by successive odd numbers of zeros: $0.1010001000001\ldots$

Written Exercises

State which number is greater.

A

1. $\sqrt{3}$ or 1.732
2. $\sqrt[3]{2}$ or 1.26
3. $\sqrt{\frac{3}{2}}$ or 1.225
4. $\sqrt[3]{9}$ or 2.08
5. $\sqrt{\frac{5}{3}}$ or 1.291
6. $\sqrt[3]{\frac{6}{7}}$ or 0.95

Between the two given numbers find (a) a rational number and (b) an irrational number.

7. 1.4 and 1.5
8. $-\frac{7}{10}$ and $-\frac{3}{4}$
9. $\frac{2}{3}$ and 0.68

B

10. $0.\overline{23}$ and $0.\overline{24}$
11. $\sqrt{2}$ and 1.41
12. $\sqrt[3]{5}$ and 1.712

13. Suppose a, b, c, and d are positive integers such that $ad = bc + 2$. Find a rational number between $\frac{a}{b}$ and $\frac{c}{d}$. (*Hint:* Find a common denominator.)
14. Use the method of Exercise 13 to find a rational number between $\frac{4}{5}$ and $\frac{6}{7}$.
15. Find an irrational number between $\sqrt{\frac{7}{8}}$ and $\sqrt{\frac{5}{6}}$. (*Hint:* See Exercise 14 above.)

C 16. Develop a formula for an irrational number between any two positive rational numbers. (*Hint:* Show that $a + \frac{\sqrt{2}}{2}(b - a)$ is between a and b whenever $a < b$.)
17. Explain why the set of rational numbers has the property of density; that is, between any two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ there is another rational number.

Self-Test 2

VOCABULARY

terminating decimal (p. 266)
repetend (p. 267)
repeating decimal (p. 267)
rounding error (p. 269)
standard notation (p. 269)
significant digit (p. 269)
property of density (p. 273)

1. Express $\frac{6}{11}$ as a terminating or repeating decimal. *Obj. 1, p. 266*
2. Express $0.\overline{297}$ as a fraction.

Express in standard notation. *Obj. 2, p. 266*

3. 4782
4. 0.00783

5. Find a one-significant-digit estimate of $\frac{(38.778)(0.0345)}{(997)(0.711)}$. *Obj. 3, p. 266*
6. Find a rational number and an irrational number between $\frac{4}{9}$ and 0.45. *Obj. 4, p. 266*

Check your answers with those at the back of the book.

Operating with Radicals

OBJECTIVES for Sections 8-7 through 8-10:
1. Use properties of radicals to simplify algebraic expressions.
2. Solve equations involving radicals.
3. Solve quadratic equations by completing the square and by using the quadratic formula.

8-7 Properties of Radicals

Notice that

$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{216} = 6, \qquad \text{and} \qquad \sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3 = 6,$$

and therefore

$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{8} \cdot \sqrt[3]{27}.$$

Likewise,

$$\sqrt{\frac{64}{16}} = \sqrt{4} = 2 \qquad \text{and} \qquad \frac{\sqrt{64}}{\sqrt{16}} = \frac{8}{4} = 2,$$

and therefore

$$\sqrt{\frac{64}{16}} = \frac{\sqrt{64}}{\sqrt{16}}.$$

These examples illustrate the following theorem concerning certain properties of radicals.

Theorem. For all a, b, $\sqrt[n]{a}$, and $\sqrt[n]{b} \in \mathcal{R}$:

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ 2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$

PROOF OF 1

First, note that $\sqrt[n]{ab}$ denotes the *principal* nth root of ab.

1. $(\sqrt[n]{a} \cdot \sqrt[n]{b})^n = (\sqrt[n]{a})^n \cdot (\sqrt[n]{b})^n = a \cdot b$ (Why?)
2. $\therefore$ $\sqrt[n]{a} \cdot \sqrt[n]{b}$ is an nth root of $a \cdot b$. (Why?)
3. If n is even, then a and b must both be nonnegative and accordingly the principal nth root of each of these numbers is nonnegative. Hence their product, $\sqrt[n]{a} \cdot \sqrt[n]{b}$, is nonnegative and must therefore represent the *principal* nth root of $a \cdot b$, $\sqrt[n]{ab}$.
4. If n is odd, and $a \cdot b \geq 0$, then $\sqrt[n]{a} \cdot \sqrt[n]{b} \geq 0$, and also $\sqrt[n]{ab} \geq 0$. Hence $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

The proof of the case in which n is odd and exactly one of a and b is negative, as well as Part 2 of the theorem, will be left as Exercises 34–36, page 279.

If we let $b = a$ in Part 1 we have $\sqrt[n]{a^2} = \sqrt[n]{a} \cdot \sqrt[n]{a} = (\sqrt[n]{a})^2$. This illustrates a special case ($m = 2$) of the following useful theorem concerning another property of radicals.

dosen't work

> **Theorem.** For all b and $\sqrt[n]{b} \in \mathcal{R}$, and m and n positive integers,
> $$\sqrt[n]{b^m} = (\sqrt[n]{b})^m.$$

why? b is a positive number

EXAMPLE 1 Evaluate (a) $\sqrt[3]{(-64)^2}$; and (b) $\sqrt[4]{(81)^3}$.

SOLUTION
a. $\sqrt[3]{(-64)^2} = (\sqrt[3]{-64})^2 = (-4)^2 = 16$
b. $\sqrt[4]{(81)^3} = (\sqrt[4]{81})^3 = 3^3 = 27$

The following theorem enables you to replace the index of a radical with a lesser index.

> **Theorem.** For k and m integers and all b and $\sqrt[km]{b} \in \mathcal{R}$,
> $$\sqrt[km]{b} = \sqrt[k]{\sqrt[m]{b}} = \sqrt[m]{\sqrt[k]{b}}.$$

For example,

$$\sqrt[8]{36} = \sqrt[4 \cdot 2]{36} = \sqrt[4]{\sqrt[2]{36}} = \sqrt[4]{6}.$$

The theorems of this section, together with other number properties, enable you to *simplify* a radical of index n according to the following criteria:

1. The index is as small as possible.
2. There are no radicands containing a fraction or a negative exponent, or radicals appearing in a denominator.
3. No radicand contains the nth power of an integer or polynomial other than 1.

The term **rationalizing the denominator** is often used to describe the process of transforming a term involving radicals and fractions into an equivalent term with the expression for the denominator free of radicals. (See Example 2(b).)

EXAMPLE 2 Simplify each expression:

a. $\sqrt[6]{64x^7}$ b. $\dfrac{2}{\sqrt{2a}}$ c. $\sqrt[5]{\dfrac{128}{b}}$ d. $\sqrt[3]{x^{-3}+(2y)^{-3}}$

SOLUTION

a. $\sqrt[6]{64x^7} = \sqrt[6]{64x^6 \cdot x} = 2x\sqrt[6]{x}$

b. $\dfrac{2}{\sqrt{2a}} = \dfrac{2\sqrt{2a}}{\sqrt{2a}\sqrt{2a}} = \dfrac{2\sqrt{2a}}{2a} = \dfrac{\sqrt{2a}}{a}$

c. $\sqrt[5]{\dfrac{128}{b}} = \dfrac{\sqrt[5]{32}\sqrt[5]{4}\sqrt[5]{b^4}}{\sqrt[5]{b}\sqrt[5]{b^4}} = \dfrac{2\sqrt[5]{4b^4}}{b}$

d. $\sqrt[3]{x^{-3}+(2y)^{-3}} = \sqrt[3]{\dfrac{1}{x^3}+\dfrac{1}{(2y)^3}} = \sqrt[3]{\dfrac{8y^3+x^3}{8x^3y^3}} = \dfrac{\sqrt[3]{8y^3+x^3}}{2xy}$

1-16 1-12 odd

Oral Exercises

Express each of the following in simplified form.

1. $\sqrt{18}$
2. $\sqrt{75}$
3. $\sqrt{27}$
4. $\sqrt[3]{24}$
5. $\sqrt[3]{-54}$
6. $\sqrt{\dfrac{3x^2}{16}}$
7. $\sqrt{\dfrac{12}{x^3}}$
8. $\sqrt[6]{4}$
9. $\sqrt[3]{27^2}$
10. $\sqrt{25^{-3}}$

Written Exercises

Simplify each of the following.

odd **A**

1. $\sqrt{108}$
2. $\sqrt{384}$
3. $\sqrt{\frac{98}{9}}$
4. $\sqrt{\frac{25}{2}}$
5. $\sqrt[3]{-27^2}$
6. $\sqrt[5]{(-32)^4}$
7. $-\sqrt[3]{8^{-5}}$
8. $\sqrt{(-25)^{-4}}$
9. $\sqrt[6]{\frac{8}{27}}$
10. $\sqrt[4]{\frac{25}{9}}$
11. $\sqrt[3]{\sqrt{\frac{125}{8}}}$
12. $\sqrt[8]{16} \cdot \sqrt[6]{125}$

Simplify each of the following; then use Tables 3 and 4 in the Appendix to give approximations correct to the nearest hundredth.

13. $5\sqrt{112}$
14. $\dfrac{5\sqrt{180}}{3}$
15. $\dfrac{3}{2} \cdot \dfrac{\sqrt[3]{750}}{\sqrt[3]{3}}$
16. $\sqrt{1.62}$
17. $\sqrt{\frac{125}{8}} \cdot \sqrt{50}$
18. $\sqrt[3]{\frac{320}{9}}$
19. $\sqrt{\frac{3}{20}} \cdot \sqrt{\frac{35}{3}}$
20. $\sqrt[3]{-12} \cdot \sqrt[3]{54}$

Express each of the following in simple radical form.

21. $\sqrt{\dfrac{a^4}{b^3}}$
22. $\sqrt[3]{\dfrac{16x^6y^4}{z^2}}$
23. $\dfrac{\sqrt[5]{64c^{11}}}{\sqrt[5]{2c^4}}$
24. $\dfrac{x+2}{\sqrt{x^2+4}}$
25. $\sqrt{\dfrac{y^2}{y-3}}$
26. $\sqrt[6]{81p^4q^{-8}}$

B 27. $\sqrt{a^{-2} - (3b)^{-2}}$

28. $\sqrt{4x^6 + 4x^4y^{-4}}$

29. $\sqrt[3]{x^2(x-y)^{-3} - x(x-y)^{-2}}$

30. $\sqrt{4x^{-1} + 4y^{-1}}\sqrt{(x+y)^{-1}}$

31. $\sqrt[4]{a^2(a+b)^{-2}}\sqrt[6]{a^3(a+b)^{-3}}$

32. $\sqrt[5]{(2c)^{-3} + 2c^{-2}}$

Prove each of the following statements.

C 33. If $\sqrt[3]{a^3 + b^3} = a + b$, $a \geq 0$, $b \geq 0$, then $a = 0$ or $b = 0$. (*Hint:* Cube both sides.)

34. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$; n odd, and exactly one of the numbers a, b negative. (*Hint:* First show that $\sqrt[n]{ab}$ and $\sqrt[n]{a}\sqrt[n]{b}$ both denote nonpositive numbers.)

35. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$; $a \geq 0$, $b > 0$.

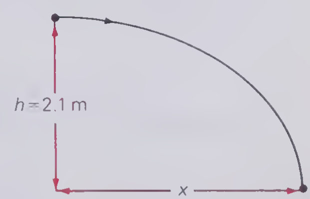
36. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$; n odd; $\dfrac{a}{b} < 0$.

37. $\sqrt[n]{b^m} = (\sqrt[n]{b})^m$; $\sqrt[n]{b}$ denotes a real number, m a positive integer.

Problems

A 1. The power P in watts of a circuit with a total resistance R is related to the current I by the equation $P = I^2R$. The power is expressed in watts (W), the resistance in ohms (Ω), and the current in amperes (A). What current will be produced in a circuit with a total resistance of 10Ω if the power output is 1200 W?

2. An object propelled *horizontally* at a velocity v_0 from a height h will hit the ground $v_0\sqrt{\dfrac{2h}{4.9}}$ meters from its starting point if air resistance is neglected. How far will a baseball that is thrown horizontally at a speed of 40 m/s from a height of 2.1 m travel before hitting the ground?



3. The frequency f of a string on a musical instrument is given by the equation $f = \dfrac{1}{2L}\sqrt{\dfrac{10^5FL}{m}}$, where L is the length of the string, F is the tension on the string, and m is the mass of the string. F is expressed in newtons (N), and f is expressed in hertz (Hz), or vibrations per second. What is the frequency of the D-string on a violin if it has a length of 45 cm and a mass of 0.9 g, and is under a tension of 140 N?

4. The resonant frequency f of a circuit with inductance L and capacitance C is given by the equation $f = \dfrac{1}{2\pi\sqrt{LC}}$. The resonant frequency is expressed in hertz (Hz), the capacitance in farads (F), and the inductance in henrys (H). Find the resonant frequency of a circuit containing an inductance of 1.25×10^{-2} H and a capacitance of 4×10^{-6} F.

5. What is the radius of a sphere whose volume is 264 cm^3? ($V = \frac{4}{3}\pi r^3$; use $\pi \approx \frac{22}{7}$.)

B 6. The speed of sound in air varies directly as the square root of the Kelvin temperature. If the speed of sound is about 340 m/s when the temperature is 289° K, what is its speed at 300° K?

7. Find the side of a cube whose volume is the same as the volume of a sphere of radius 1. (See Exercise 5; use $\pi \approx \frac{22}{7}$.)

8. Find the side of a cube whose surface area is the same as the lateral area of a cone of height 1 and radius 1. (Lateral area $= \pi r\sqrt{r^2 + h^2}$; use $\pi \approx 3.14$.)

9. According to Kepler's third law of planetary motion, the cubes of the average distances of the planets from the sun are proportional to the squares of their times of one revolution around the sun. If Mars is one-sixth as far from the sun as Saturn, find the ratio of their times of revolution.

C 10. According to the theory of relativity, at high speeds the mass m of an object is given by the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the rest mass of the object, v is the velocity of the object, and c is the speed of light. By what factor is the mass of an object increased at $\frac{2}{3}$ the speed of light?

8-8 Operations with Radicals

A sum of radicals can be simplified in accordance with the following rules:

1. First simplify each radical in the sum.
2. Then combine radical terms containing the *same index and radicand,* using the distributive law.

EXAMPLE 1 Simplify $a\sqrt[4]{16a} + 3\sqrt[4]{a^5} - a\sqrt[4]{2401}$.

SOLUTION

$$\begin{aligned} a\sqrt[4]{16a} + 3\sqrt[4]{a^5} - a\sqrt[4]{2401} &= a\sqrt[4]{2^4 \cdot a} + 3\sqrt[4]{a^4 \cdot a} - a\sqrt[4]{7^4} \\ &= 2a\sqrt[4]{a} + 3a\sqrt[4]{a} - 7a \\ &= 5a\sqrt[4]{a} - 7a \\ &= a(5\sqrt[4]{a} - 7). \quad \text{Answer.} \end{aligned}$$

To simplify a product or quotient involving radicals, you can use the theorems in the preceding section.

EXAMPLE 2 Simplify $(\sqrt[3]{a^2} - 1)(\sqrt[3]{a} + a)$.

SOLUTION
$$\begin{aligned}(\sqrt[3]{a^2} - 1)(\sqrt[3]{a} + a) &= \sqrt[3]{a^3} + a\sqrt[3]{a^2} - \sqrt[3]{a} - a\\ &= a + a(\sqrt[3]{a})^2 - \sqrt[3]{a} - a\\ &= a(\sqrt[3]{a})^2 - \sqrt[3]{a}\\ &= \sqrt[3]{a}(a\sqrt[3]{a} - 1). \quad \text{Answer.}\end{aligned}$$

EXAMPLE 3 Rationalize the denominator of $\dfrac{x}{\sqrt{x} - 2}$.

SOLUTION *Plan:* Use the formula for the difference of squares, $(a - b)(a + b) = a^2 - b^2$.

$$\frac{x}{\sqrt{x} - 2} = \frac{x(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{x(\sqrt{x} + 2)}{x - 4}$$

Using radicals, you can factor certain quadratic polynomials which are irreducible over the set of polynomials with *integral* coefficients.

EXAMPLE 4 Factor $x^2 - 18$ completely over the set of polynomials with *real* coefficients.

SOLUTION $x^2 - 18 = x^2 - (\sqrt{18})^2 = (x - \sqrt{18})(x + \sqrt{18})$.

$\therefore x^2 - 18 = (x - 3\sqrt{2})(x + 3\sqrt{2})$.

Oral Exercises

State the factor by which you would multiply the numerator and denominator of the rational expressions in order to rationalize them.

1. $\dfrac{1}{5 + \sqrt{2}}$
2. $\dfrac{a^2}{a - \sqrt{3}}$
3. $\dfrac{17}{\sqrt[3]{x^2} - \sqrt[3]{x} + 1}$
4. $\dfrac{y}{\sqrt[3]{y^2} + 3\sqrt[3]{y} + 9}$

Written Exercises

Simplify each expression.

A

1. $5\sqrt{3} - \sqrt{27}$
2. $2\sqrt{18} + 3\sqrt{50}$
3. $\frac{1}{2}\sqrt{20} - 2\sqrt{45} + 3\sqrt{80}$
4. $2\sqrt{\frac{25}{3}} - \frac{2}{3}\sqrt{48} - \sqrt{\frac{49}{3}}$
5. $3\sqrt[3]{16} + 2\sqrt[3]{54} - \sqrt[3]{2000}$
6. $\sqrt[3]{0.024} + \sqrt[3]{0.081} + 10\sqrt[3]{0.003}$
7. $3\sqrt{16a^3} - a\sqrt{25a} + \sqrt{a^3}$
8. $\sqrt{48x^6} - x\sqrt{27x^4} - 3x^2\sqrt{192x^2}$
9. $2x\sqrt{98x^3} - \frac{1}{2}\sqrt{200x^5} + \sqrt{121x}$
10. $2\sqrt[3]{81x^2} + \sqrt[3]{375x^8} - 2\sqrt[3]{3x^5}$
11. $3\sqrt{10}(2\sqrt{5} - 3\sqrt{20})$
12. $\sqrt{6}\left(\dfrac{4}{\sqrt{3}} + \dfrac{\sqrt{3}}{4}\right)$

13. $\sqrt{\frac{2}{5}}\left(3\sqrt{5} - \frac{7}{\sqrt{5}}\right)$

14. $(\sqrt{7} - \sqrt{11})(\sqrt{7} + \sqrt{11})$

15. $(\sqrt{3} - \sqrt{5})^2$

16. $(2\sqrt{6} - 3\sqrt{2})(2\sqrt{6} + 3\sqrt{2})$

17. $(2\sqrt{7} + \sqrt{3})^2$

18. $(3\sqrt{5} - 2)(\sqrt{5} + 3)$

19. $(\sqrt[3]{4} + 2\sqrt[3]{2})(\sqrt[3]{2} - \sqrt[3]{8})$

20. $(\sqrt[3]{2} - 2\sqrt[3]{3})(\sqrt[3]{9} + 3\sqrt[3]{4})$

21. $\dfrac{3}{\sqrt{5} - 2}$

22. $\dfrac{\sqrt{6}}{5 + \sqrt{3}}$

23. $\dfrac{\sqrt{2} - 3}{\sqrt{2} + 7}$

24. $\dfrac{\sqrt{5} + 2}{2\sqrt{5} - 4}$

B 25. $(\sqrt[3]{2} - 1)(\sqrt[3]{4} + \sqrt[3]{2} + 1)$

26. $(\sqrt[3]{5} + \sqrt[3]{3})(\sqrt[3]{25} - \sqrt[3]{15} + \sqrt[3]{9})$

27. $(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})$

28. $(\sqrt[3]{x^2} + \sqrt[3]{xy})(\sqrt[3]{xy} - \sqrt[3]{y})$

29. $(\sqrt[3]{a^2} + \sqrt[3]{ab})(\sqrt[3]{a} - \sqrt[3]{b})$

30. $(\sqrt[3]{cd} - \sqrt[3]{c})(\sqrt[3]{cd^2} + \sqrt[3]{d})$

31. $\dfrac{3}{\sqrt{x-1} + 2}$

32. $\dfrac{2\sqrt{x+3}}{\sqrt{x+3} - 1}$

Factor completely over the set of polynomials with real coefficients.

33. $2x^2 - y^2$

34. $9a^2 - 5$

35. $x^2 + 2x\sqrt{3} + 3$

36. $5x^2 - 2xy\sqrt{10} + 2y^2$

37. $b^3 - 7$

38. $c^3 + 3$

C 39. Prove that the set of all numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers, is closed under multiplication.

40. Prove that the set of all numbers of the form $a + b\sqrt{2}$, where a and b are rational, is closed under division.

41. Factor $x^4 + 1$ over the set of polynomials with real coefficients. (*Hint:* $x^4 + 1 = (x^4 + 2x^2 + 1) - 2x^2$)

8-9 Equations Involving Radicals

To solve an equation in which one term contains a variable in a radicand, you first isolate that term on one side of the equality sign. Then you can raise both members to the power of the radical index, and solve the resulting equation.

EXAMPLE 1 Solve $y + \sqrt{y-2} - 4 = 0$.

SOLUTION

$$y + \sqrt{y-2} - 4 = 0$$

$$y - 4 = -\sqrt{y-2} \qquad (1)$$

$$(y-4)^2 = (-\sqrt{y-2})^2 \qquad (2)$$

$$y^2 - 8y + 16 = y - 2$$

$$y^2 - 9y + 18 = 0$$

$$(y - 6)(y - 3) = 0$$
$$y = 6,\ y = 3$$

Check:

For $y = 6$: $6 + \sqrt{6 - 2} - 4 \stackrel{?}{=} 0$
$$6 + 2 - 4 \neq 0$$

For $y = 3$: $3 + \sqrt{3 - 2} - 4 \stackrel{?}{=} 0$
$$3 + 1 - 4 = 0$$

$\therefore$ the solution set is $\{3\}$. Answer.

Can you explain why an extraneous value appeared, that is, why the squared Equation (2) is not equivalent to the given equation? Think of the equation $x = 3$ and the equation obtained by squaring both members, $x^2 = 9$; the latter has *two* real solutions, $x = 3$ and $x = -3$, while the former has only *one*, $x = 3$.

In general, for two functions $P(x)$ and $Q(x)$, and n a positive integer, the solution set of the equation $P(x) = Q(x)$ is a subset of the solution set of $[P(x)]^n = [Q(x)]^n$. The following theorem summarizes the facts.

Theorem. For n a positive integer and a and $b \in \mathcal{R}$:

1. If $a = b$, then $a^n = b^n$.
2. If $a^n = b^n$ and n is odd, then $a = b$.
3. If $a^n = b^n$ and n is even, then $a = \pm b$.

If more than one term in an equation contains a variable in a radicand, you may have to repeat the process of isolating a radical term.

EXAMPLE 2 Solve $\sqrt{x - 3} = \sqrt{2} - \sqrt{x}$ over $\mathcal{R}$.

SOLUTION

$$\sqrt{x - 3} = \sqrt{2} - \sqrt{x}$$
$$(\sqrt{x - 3})^2 = (\sqrt{2} - \sqrt{x})^2$$
$$x - 3 = 2 - 2\sqrt{2x} + x$$
$$-5 = -2\sqrt{2x}$$
$$(-5)^2 = (-2\sqrt{2x})^2$$
$$25 = 8x$$
$$\tfrac{25}{8} = x$$

Check:

$$\sqrt{\tfrac{25}{8} - 3} \stackrel{?}{=} \sqrt{2} - \sqrt{\tfrac{25}{8}}$$
$$\sqrt{\tfrac{1}{8}} \stackrel{?}{=} \sqrt{2} - 5\sqrt{\tfrac{1}{8}}$$
$$\tfrac{1}{4}\sqrt{2} + \tfrac{5}{4}\sqrt{2} \stackrel{?}{=} \sqrt{2}$$
$$\tfrac{3}{2}\sqrt{2} \neq \sqrt{2}$$

$\therefore$ the solution set is $\emptyset$. Answer.

Example 2 illustrates the fact that the solution set of $P(x) = Q(x)$ may be empty even though $[P(x)]^n = [Q(x)]^n$ has one or more real roots. Actually in this case you can tell by inspecting the given equation that there can be no real value of x for which $P(x) = Q(x)$.

Oral Exercises

In Oral Exercises 1–9, (a) state any restrictions on the variable; (b) explain how you would solve the equation, but do not solve it.

1. $\sqrt{x+2} = 9$
2. $3 - \sqrt{2-y} = 0$
3. $\sqrt[3]{6n} = 2$
4. $\sqrt[3]{2p+1} = 3$
5. $\sqrt{3d+1} - 6 = 4$
6. $-1 + \sqrt[4]{\frac{a}{2}} = 2$
7. $3 + \sqrt{4n-5} = 10$
8. $\frac{2}{3}\sqrt[3]{4m} = 4$
9. $\sqrt{b^2+16} - b = 2$

Written Exercises

Solve over $\mathcal{R}$.

A

1–9. Solve Oral Exercises 1–9.

10. $2\sqrt{c+9} = c+1$
11. $7 - \sqrt{3t} = -5$
12. $3 + \sqrt{7-r} = 4 - r$
13. $\sqrt[3]{d^2+2} - 3 = 0$
14. $2\sqrt[3]{7n-1} - 5 = 0$
15. $2x = \sqrt{8x-3}$
16. $3\sqrt{3y} - 3 = 2y$
17. $\sqrt{k+7} = \sqrt{k} + 1$
18. $\sqrt{z-8} - \sqrt{z} = 2$
19. $\sqrt{r-6} = 3 + \sqrt{r}$
20. $\sqrt{t-8} + 4 = \sqrt{t}$
21. $\sqrt{x+1} = \sqrt{x+6} - 1$
22. $\sqrt{x+5} + \sqrt{8-x} = 5$
23. $\sqrt{x+1} = \sqrt{2x+3} - 1$
24. $\sqrt{2x+7} - \sqrt{x+3} = 1$
25. $\sqrt{4x+1} - \sqrt{x+2} = 1$
26. $\sqrt{x+10} + \sqrt{2x+4} - 8 = 0$

B

27. $x\sqrt{x} = 8$
28. $x\sqrt{x+1} = x+3$
29. $\dfrac{3x-2}{\sqrt{x}-1} = 5\sqrt{x}$
30. $\dfrac{3x+1}{\sqrt{x}+3} = 2\sqrt{x} - 1$
31. $\dfrac{x-2}{\sqrt{2x}-3} = \sqrt{2x} + 2$
32. $\dfrac{2x-3}{\sqrt{2x}-2} = 1 + \sqrt{2x}$

C

33. Prove: If r and s denote real numbers and $r^3 = s^3$, then:
    a. r and s are both 0, both positive, or both negative.
    b. $r = s$
    [*Hint:* Use the result of part (a) and $r^3 - s^3 = (r-s)(r^2 + rs + s^2)$]
34. Prove: If r and s denote real numbers and $r^4 = s^4$, then:
    a. r and s are both 0 or both nonzero.
    b. $r = s$ or $r = -s$
    [*Hint:* Use the fact that $r^4 - s^4 = (r^2 - s^2)(r^2 + s^2)$]

programming in BASIC

Exercises

1. Write a program that will print the roots of a quadratic equation by using the quadratic formula. Be sure to include a special print-out if

$$b^2 - 4ac < 0.$$

2. Modify the preceding program to test whether or not $b^2 - 4ac$ is a perfect square. If it is not, print an intermediate step like this (for 3, 9, 1):

```
R1 = (-9 + / 69) / 6 = -.115563
R2 = (-9 - / 69) / 6 = -2.88444
```

(If you wish, you can mark over the first / to make a square-root sign.)

8-10 The Quadratic Formula

You learned in Section 6-4 how to solve a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$, in which the left-hand member can be factored over the integers. For example:

$$\begin{aligned} x^2 - 36 &= 0 \\ (x + 6)(x - 6) &= 0 \\ x = 6 \text{ or } x &= -6 \end{aligned} \qquad \begin{aligned} 2x^2 - 5x + 2 &= 0 \\ (2x - 1)(x - 2) &= 0 \\ x = \tfrac{1}{2} \text{ or } x &= 2 \end{aligned}$$

You can see by inspection that the quadratic equation $3y^2 + 9y + 1 = 0$ cannot be solved by factoring the left member over the integers. If, however, we could put the equation in the form

$$(x + d)^2 = f \qquad (d \text{ and } f \in \mathcal{R}, f \geq 0),$$

it could be easily solved. We rewrite the equation as $(x + d)^2 - f = 0$ and factor it as $[(x + d) - \sqrt{f}][(x + d) + \sqrt{f}] = 0$. The factored equation is easily solved as follows:

$$\begin{aligned} x + d &= \pm\sqrt{f} \\ x &= -d \pm \sqrt{f} \end{aligned}$$

EXAMPLE 1 Solve $(x - \sqrt{3})^2 = 12$.

SOLUTION

$$\begin{aligned} (x - \sqrt{3})^2 &= 12 \\ x - \sqrt{3} &= \pm\sqrt{12} \\ x &= \sqrt{3} \pm 2\sqrt{3} \\ x &= 3\sqrt{3} \text{ or } x = -\sqrt{3}. \end{aligned}$$

$\therefore$ the solution set is $\{3\sqrt{3}, -\sqrt{3}\}$. Answer.

To transform any quadratic equation

$$(1)\qquad ax^2 + bx + c = 0 \qquad (a,\ b,\ \text{and } c \in \mathcal{R},\ a \neq 0)$$

into the desired form $(x + d)^2 = f$, or

$$(2)\qquad x^2 + 2dx + d^2 = f,$$

let us first rewrite (1) equivalently as

$$(3)\qquad x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Next, by equating the coefficients of x in Equations (2) and (3), we can find an expression for d, and hence for d^2, in terms of a and b. We can then express the left member of Equation (3) in the desired form $(x + d)^2$; that is, we can **complete the square** in that member, as follows:

$$2d = \frac{b}{a}, \qquad \text{so} \qquad d = \frac{b}{2a} \qquad \text{and} \qquad d^2 = \frac{b^2}{4a^2}.$$

Add $\frac{b^2}{4a^2}$ to both members of Equation (3):

$$(4)\qquad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$(5)\qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$(6)\qquad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$(7)\qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$(8)\qquad \boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Equation (8) is known as the **quadratic formula.** You can see from the quadratic formula that for x to be a real number, it is necessary that we have $b^2 - 4ac \geq 0$.

EXAMPLE 2 Solve $3y^2 + 9y + 1 = 0$ by **(a)** completing the square, and **(b)** using the quadratic formula.

SOLUTION a. $3y^2 + 9y + 1 = 0$; $a = 3$, $b = 9$; $\frac{b^2}{4a^2} = \frac{81}{36} = \frac{9}{4}$.

$$y^2 + 3y = -\tfrac{1}{3}$$

$$y^2 + 3y + \tfrac{9}{4} = -\tfrac{1}{3} + \tfrac{9}{4}$$

$$(y + \tfrac{3}{2})^2 = \frac{-4 + 27}{12} = \tfrac{23}{12}$$

$$y + \tfrac{3}{2} = \pm\sqrt{\tfrac{23}{12}} = \pm\tfrac{1}{2}\sqrt{\tfrac{23}{3}} = \pm\tfrac{1}{6}\sqrt{69}$$
$$y = -\tfrac{3}{2} \pm \tfrac{1}{6}\sqrt{69}$$

$\therefore$ the solution set is $\{-\frac{3}{2} + \frac{1}{6}\sqrt{69}, -\frac{3}{2} - \frac{1}{6}\sqrt{69}\}$. Answer.

b. $a = 3$, $b = 9$, $c = 1$:

$$y = \frac{-9 \pm \sqrt{81 - 12}}{6}$$
$$= -\tfrac{3}{2} \pm \tfrac{1}{6}\sqrt{69}.$$

$\therefore$ the solution set is $\{-\frac{3}{2} + \frac{1}{6}\sqrt{69}, -\frac{3}{2} - \frac{1}{6}\sqrt{69}\}$. Answer.

Oral Exercises

State the number that must be added to both sides of the equation in order to complete the square. (See Example 2a.) # 1-6

1. $x^2 - 6x = -1$
2. $x^2 + 10x = 3$
3. $y^2 + 3y = 3$
4. $t^2 - \frac{2}{3}t = 4$
5. $r^2 + \frac{4}{5}r = -2$
6. $n^2 - \frac{3}{2}n = 2$

State the values of a, b, and c (for $a > 0$) that would be used in the quadratic formula.

7. $k^2 - 4k = 3$
8. $2y^2 = 6y + 3$
9. $5 = 4x - 2x^2$
10. $16v^2 = 1 - 20v$
11. $9z^2 = 2z$
12. $16x^2 - 1 = 24x$

Written Exercises

Solve by the indicated method; give irrational answers in simple radical form. In Exercises 1–9, solve by completing the square.

A

1–6. Solve the equations in Oral Exercises 1–6.

7. $2u^2 - 6u - 1 = 0$
8. $3v^2 + 2v - 4 = 0$
9. $4y^2 - 10y + 3 = 0$

Solve by using the quadratic formula.

10–15. Solve the equations in Oral Exercises 7–12.

16. $3 - 4k^2 = 6k$
17. $\frac{1}{9}n^2 = \frac{2}{3}n - 4$
18. $\frac{3}{2}(3t^2 - t) = 1$

Solve by any method.

19. $3(x + 1)^2 - 5 = 0$
20. $\frac{2}{3}x^2 = 6x - 4$
21. $5x^2 + 3x - \frac{1}{5} = 0$
22. $\dfrac{(2x - 1)^2}{3} = 4$

23 $\frac{7}{x+1} = 2x-5$

Solve by any method.

B 23. $\dfrac{7}{x+1} = 2x - 3$

24. $\dfrac{x+2}{x-1} = \dfrac{3x-2}{x-5}$

25. $\dfrac{2}{x-1} - \dfrac{4}{x} = 3$

26. $\dfrac{4}{x-2} + 3 = \dfrac{2}{x+2}$

27. $(x-1)^2 - 4(x-1) - 1 = 0$
(*Hint:* Solve for $x - 1$ first.)

28. $(2x+1)^2 - 8(2x+1) + 9 = 0$

C 29. $x^4 - 2x^2 - 2 = 0$

30. $4x^6 - 2x^3 - 1 = 0$

31. Factor $x^4 + 4$ over the real numbers. (*Hint:* See Written Exercise 41 in Section 8-8.)

32. Solve the equation: $\sqrt{2x - x^2} = (x-1)^2$
Hint: $2x - x^2 = 1 - (x^2 - 2x + 1)$

Problems

Express irrational results in simple radical form. Give *only* answers that are physically possible (e.g. distances cannot be negative).

A 1. One diagonal of a rhombus is 5 cm longer than the other. The area of the rhombus is 75 cm^2. Find the length of the shorter diagonal. (Recall: The diagonals of a rhombus are perpendicular and bisect one another.)

2. One leg of a right triangle is 3 cm longer than the other. The hypotenuse is 2 cm longer than the longer leg. What is the area of the triangle?

3. A rectangular box has the same height as a cube. The width of its base is 2 cm more than the side of the cube. The length of its base is 3 cm more than the side of the cube. If the box has a volume 45 cm^3 greater than that of the cube, find the length of a side of the cube.

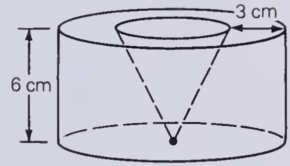
4. Two cylinders have the same volume. The radius of the second is 4 cm greater than the radius of the first, and the height of the second is half that of the first. What is the radius of the first cylinder? ($V_{\text{cylinder}} = \pi r^2 h$.)

5. The ratio of the volumes of 2 rectangular boxes with square bases and equal heights is 3:4. Each side of the base of the larger box is 2 cm longer than the side of each base of the smaller. What are the areas of the two bases?

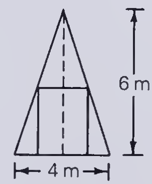
6. A rectangular lawn 40 m by 50 m is to have a rough ground cover planted along its outer two edges. If there is enough ground cover to plant 225 m^2, what should the width (x) of the strips planted with the ground cover be?

B 7. Ellen French can walk home from work by either of two routes. One takes her along two adjacent sides of a rectangular park whose length is 80 m more than its width. The other route is a short cut along the diagonal of the rectangle. If the diagonal route is 100 m shorter than the other, what are the dimensions of the park?

8. A machine part is to be made of metal in the shape of a cylinder with a conical depression in it. The depression will be of the same height as the cylinder. The volume of metal is to be 72π cm³, the height of the cylinder is to be 6 cm, and the upper surface of the part is to be a ring-shaped region of width 3 cm. What should the radius of the cylinder be? ($V_{\text{cone}} = \frac{1}{3}\pi r^2 h$)



9. A rectangle is inscribed in an isosceles triangle so that one side of the rectangle rests on the base of the triangle. If the triangle has a base of length 4 m and an altitude of 6 m, for what values of the height of the rectangle will its area be 5 m²?



C 10. The shorter base and the altitude of an isosceles trapezoid are equal, and the nonparallel sides are each of length 4 cm. For what values of the altitude will the area of the trapezoid be 12 cm²?

Self-Test 3

VOCABULARY rationalizing the denominator (p. 277); completing the square (p. 286); quadratic formula (p. 286)

Express in simple radical form.

1. $2\sqrt{24} - \sqrt{54} + 5\sqrt{150}$
2. $\sqrt[3]{16x^4y} - \sqrt[3]{54xy^4}$

Obj. 1, p. 276

3. $(2\sqrt{3} - 5)(\sqrt{3} + 2)$
4. $\dfrac{4 + \sqrt{5}}{4 - \sqrt{5}}$

Solve over $\mathcal{R}$.

5. $5 + \sqrt[3]{\dfrac{2x}{3}} = 9$
6. $\sqrt{7 - x} = \sqrt{2 - x} + 1$

Obj. 2, p. 276

7. Solve by completing the square: $2x^2 - 6x + 3 = 0$

Obj. 3, p. 276

8. Solve by using the quadratic formula: $3x^2 - 8x - 7 = 0$

Check your answers with those at the back of the book.

Chapter Summary

1. The graph of the *power function* defined by $p(x) = x^n$ with n an *even* positive integer, is *symmetric with respect to the vertical axis* and has a *minimum point* at the origin. If n is an *odd* positive integer, the graph contains the origin and is *symmetric with respect to the origin.*
2. Whenever a function is specified by an equation of the form $y = ax^n$, $a \neq 0$, we say that *y varies directly as x^n* or that *y is directly proportional to the nth power of x.*
3. For every positive integer n, any solution of $x^n = b$ is an *nth root of b.* The *radical* $\sqrt[n]{b}$ denotes the *principal nth root of b.* If n is odd, $\sqrt[n]{b^n} = b$; if n is even, $\sqrt[n]{b^n} = |b|$.
4. If a rational root of a *polynomial equation in simple form* with integral coefficients is expressed in lowest terms $\frac{p}{q}$, with $q \neq 0$, then p must be an integral factor of the constant term and q an integral factor of the leading coefficient. Any other real root of the equation is an *irrational number.*
5. The *property of density* of the real numbers asserts that between any two real numbers, there is always another real number.
6. A number can be represented by a terminating or a repeating decimal if and only if it is a rational number. The fraction in lowest terms equivalent to a repeating decimal may be found using the formula for the sum of an infinite series.
7. If n is a positive integer, and a, b, and $\sqrt[n]{b}$ denote real numbers, then:

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad \text{and} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \quad b \neq 0$$

   If m is also a positive integer, $\sqrt[n]{b^m} = (\sqrt[n]{b})^m$.
8. You can write the *sum or difference of radicals having the same index and the same radicand* as a single term by using the distributive property. You can write the *product or quotient of radicals having the same index* as a single term by applying the product or quotient property of radicals.
9. To solve an *irrational equation,* isolate a radical as one member and raise each member to the power corresponding to the root index.
10. The *quadratic formula,*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

   enables you to solve any quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$.

Chapter Review

1. For what value of a will the point $(-3, 54)$ lie on the graph of $y = ax^3$? 8-1

   a. 27 b. 2 c. -2 d. -27

2. Evaluate $-\sqrt{\frac{1}{16}}$. 8-2

   a. -4 b. $\frac{1}{4}$ c. $-\frac{1}{4}$ d. 4

3. Find all the rational roots of $x^3 - 2x - 4 = 0$ by trying each of the possible roots as a value for x. 8-3

   a. 2 b. -2 c. $-4, 4$ d. $1, -2$

4. Express $\frac{11}{40}$ as a decimal. 8-4

   a. 0.275 b. 0.25 c. $\overline{36}$ d. $0.27\overline{5}$

5. Express $0.\overline{481}$ as a fraction.

   a. $\frac{481}{1000}$ b. $\frac{13}{27}$ c. $\frac{481}{999}$ d. $\frac{15}{22}$

6. Express 0.00342 in standard notation. 8-5

   a. 3.42×10^{-3} b. $\dfrac{171}{50{,}000}$ c. 34.2×10^{-6} d. 342×10^{-4}

7. Express 4.657×10^2 in decimal notation.

   a. 4657 b. 465.7 c. 4.657 d. 46.57

8. Is 2.151515. . . a rational number or an irrational number? 8-6

   a. rational b. irrational c. neither

9. Which number is a rational number between 1.4 and $\sqrt{2}$?

   a. 2 b. 1.33 c. 1.8 d. 1.41

10. Express $\sqrt{48}$ in simplified form. 8-7

    a. $4\sqrt{3}$ b. 24 c. $16\sqrt{3}$ d. 8

11. Express $\sqrt{\dfrac{x^3}{b^2}}$ in simple radical form.

    a. $x\sqrt{\dfrac{x^2}{b^2}}$ b. $\dfrac{\sqrt{x^3}}{b}$ c. $\dfrac{x\sqrt{x}}{b}$ d. $\dfrac{b}{x}$

12. Simplify $8\sqrt{18} - 2\sqrt{50}$. 8-8

    a. $6\sqrt{34}$ b. $16\sqrt{2} - 10$ c. $6\sqrt{68}$ d. $14\sqrt{2}$

13. Solve $8 - \sqrt{3t} = 5$ over $\mathcal{R}$. 8-9

    a. $\{9\}$ b. $\{-3\}$ c. $\{-9\}$ d. $\{3\}$

14. Solve $x^2 - 3x - 2 = 0$ over $\mathcal{R}$. 8-10

    a. $\left\{\dfrac{3 + \sqrt{17}}{2}, \dfrac{3 - \sqrt{17}}{2}\right\}$ b. $\{1, 2\}$ c. $\{\frac{3}{2}\sqrt{17}, -\frac{3}{2}\sqrt{17}\}$

Chapter Test

1. If y varies as x^3 and y is 32 when x is 4, find y when x is 6. *8-1*
2. The power required to run a boat varies as the cube of its speed. If a certain boat requires 300 kW to run at a steady speed of 12 km/h, how many kilowatts are needed to run the boat at a speed of 24 km/h?
3. Solve $y^2 - 6.25 = 0$ over $\mathcal{R}$. *8-2*
4. Find any rational roots of $x^3 + x^2 - 5x - 2 = 0$. *8-3*
5. Express as a decimal: **a.** $\frac{7}{125}$; and **b.** $\frac{6}{7}$. *8-4*
6. Express $.\overline{327}$ as a fraction in lowest terms.
7. Express 0.001105 in standard notation. *8-5*
8. Find one-significant-digit estimate of $\dfrac{4.3107 \times 0.03}{5732}$.
9. Determine whether 6.75 is greater than or less than $\sqrt{48}$. *8-6*
10. Find **(a)** a rational number and **(b)** an irrational number between $1.4\overline{15}$ and $1.4\overline{3}$.
11. **a.** Simplify $\dfrac{5\sqrt{250}}{2}$. *8-7*

    b. Then, use Tables 3 and 4 in the Appendix to give an approximation correct to the nearest hundredth.
12. Express $\sqrt[3]{\dfrac{25x^3y^4}{u^4}}$ in simple radical form.
13. Simplify: **a.** $(\sqrt{2} - \sqrt{6})^2$; and **b.** $\dfrac{12}{\sqrt{6} - 3}$. *8-8*
14. Solve $x - 3 - \sqrt{6x + 9} = 0$ over $\mathcal{R}$. *8-9*
15. Solve $x^2 - 8x = -11$ by completing the square. *8-10*
16. Solve $x^2 - 4x + 1 = 0$ using the quadratic formula.

Cumulative Review *(Chapters 5–8)*

1. Which point lies in the yz-plane?

   a. $(1, 2, 3)$ **b.** $(1, 0, 3)$ **c.** $(0, 1, 3)$ **d.** $(1, 3, 0)$

2. Given the graph of $2x + 3y = 18$. Write a linear system of equations whose solution set is the trace of the given graph in the yz-plane.

   a. $y = 6$, $z = 0$ **b.** $y = \frac{2}{3}x$, $z = 0$ **c.** $x = 9$, $y = 0$ **d.** $y = 6$, $x = 0$

Review Item 3 refers to the following system of equations:

$$\begin{aligned} x \qquad\quad + 3z &= 8 \\ x + 2y \qquad\quad &= 0 \\ 6x - 10y - 9z &= -10 \end{aligned}$$

3. Which determinant is D_y?

a. $\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 6 & -10 & -9 \end{vmatrix}$ b. $\begin{vmatrix} 1 & 8 & 3 \\ 1 & 0 & 0 \\ 6 & -10 & -9 \end{vmatrix}$ c. $\begin{vmatrix} 0 & 3 & 1 \\ 2 & 0 & 1 \\ -10 & -9 & 6 \end{vmatrix}$

4. Write as an equivalent expression containing only positive exponents: $(3x^{-2}y^3)^{-2}(\frac{1}{6}x^5y^{-3}z^2)^{-3}$.

a. $\dfrac{24}{x^2y^5z}$ b. $\dfrac{x^{11}y^3}{1944z^6}$ c. $\dfrac{z^6}{2x^{11}y^{12}}$ d. $\dfrac{24y^3}{x^{11}z^6}$

5. Factor the polynomial $3x^2 + 18x - 120$ completely.

a. $3(x + 4)(x - 10)$ b. $3(x - 4)(x + 10)$ c. $(3x + 12)(x - 10)$

6. Simplify $\dfrac{x}{x^2 + 6x + 9} + \dfrac{3}{x + 3}$.

a. $\dfrac{4x + 9}{x^2 + 6x + 9}$ b. $\dfrac{4x + 1}{x^2 + 6x + 1}$ c. $\dfrac{13}{x + 15}$ d. $\dfrac{1}{x + 3}$

7. Find the value of a_5 of an arithmetic sequence with $a_2 = 7$ and $d = -5$.

a. -13 b. 22 c. 27 d. -8

8. Find the common ratio of a geometric sequence in which $a_1 = 36$ and $a_4 = -\frac{9}{2}$.

a. $-\frac{1}{4}$ b. $-\frac{1}{2}$ c. $\frac{1}{3}$ d. $\frac{1}{6}$

9. Find the sum S_5 for a geometric series in which $a_1 = 4$ and $r = 3$.

a. 484 b. -242 c. -484 d. 160

10. Find the value of $\sum_{n=1}^{8} 3(n - 2)$.

a. 70 b. 100 c. 60 d. 92

11. Express $0.\overline{235}$ as a fraction.

a. $\frac{47}{200}$ b. $\frac{235}{999}$ c. $\frac{23}{100}$ d. $\frac{17}{89}$

12. Express $\sqrt{120}$ in simplified form.

a. 60 b. $4\sqrt{15}$ c. $10\sqrt{6}$ d. $2\sqrt{30}$

13. Solve $x^2 - 7x + 11 = 0$ over $\mathcal{R}$.

a. $\{\frac{7}{2} + \sqrt{5}, \frac{7}{2} - \sqrt{5}\}$ b. $\{2, 5\}$ c. $\left\{\dfrac{7 + \sqrt{5}}{2}, \dfrac{7 - \sqrt{5}}{2}\right\}$ d. $\left\{\dfrac{7\sqrt{5}}{2}, \dfrac{-7\sqrt{5}}{2}\right\}$

The Shiva Laser system at the Lawrence Livermore Laboratory is a prototype of a new system of energy production.

9 Complex Numbers and Polynomial Functions

The Set of Complex Numbers

***OBJECTIVES** for Sections 9-1 through 9-5:*

1. ***Simplify a square-root radical whose radicand is a negative number.***
2. ***Find the sum, difference, product and quotient (divisor not zero) of two given complex numbers.***
3. ***Use the discriminant of a quadratic equation to determine the nature of its roots.***
4. ***Use relations between the roots and coefficients of a quadratic equation to determine the sum and product of the roots when the equation is given, and vice versa.***

9-1 Imaginary Numbers

Over the set of positive real numbers, the linear equation

$$x + 1 = 0$$

has no solution. When you extend the replacement set of x to contain negative as well as positive numbers, however, the given equation has a single solution, namely, -1.

Over the set $\mathcal{R}$ of all real numbers, the quadratic equation

$$x^2 + 1 = 0$$

has no solution. Can we extend the replacement set of x to contain new numbers which will satisfy this equation? About four hundred years ago, mathematicians proposed the introduction of a number, i, with the property that

$$i^2 + 1 = 0,$$

or

$$\mathbf{i^2 = -1.}$$

Thus, i is a solution of the equation $x^2 + 1 = 0$.

The fact that $i^2 = -1$ suggests that you write

$$\sqrt{-1} = i$$

and call i "a square root of -1."

By requiring that multiplication continue to have the commutative and associative properties, you can discover how to multiply real numbers and i. Study the following examples:

EXAMPLE 1

a. $3 \cdot i = 3i$

b. $4(5i) = (4 \cdot 5)i = 20i$

c. $2i(-3) = 2(-3)i = -6i$

d. $(7i)(2i) = (7 \cdot 2)(i \cdot i) = 14i^2 = 14(-1) = -14$

e. $(-i)(i) = -i^2 = -(-1) = 1$

f. $(3i)^2 = (3i)(3i) = 3^2i^2 = 9(-1) = -9.$

Since for any $r > 0$, $(\sqrt{r}\, i)^2 = (\sqrt{r})^2 i^2 = r(-1) = -r$,

it is natural to make this definition:

> For every positive real number r, $\sqrt{-r} = i\sqrt{r}$.

EXAMPLE 2 $\sqrt{-12} = \sqrt{12}\, i = 2\sqrt{3}\, i$, or $2i\sqrt{3}$.

The last form of the answer in Example 2 is often used to avoid the error of writing $\sqrt{3i}$ for $\sqrt{3}\, i$.

The preceding suggests the following fact: *For every nonzero real number b, bi is a number whose square is $-b^2$; that is, $(bi)^2 = -b^2$.* For $b \neq 0$, we call bi a **pure imaginary* number.** The number i is called the **imaginary unit.** We define $0 \cdot i$ to be 0.

*The term "imaginary" is an unfortunate relic of seventeenth-century uneasiness about these numbers; it does not imply any doubt about the existence of the numbers. They are in fact of great importance in many branches of mathematics.

Do you see that the product of a nonzero real number and a pure imaginary number is a pure imaginary number, but the product of two pure imaginary numbers is a real number? When you simplify successive powers of i, you find the values repeating in cycles of four, according to the pattern $i, -1, -i, 1$.

$$\begin{aligned} i^1 &= i & i^5 &= i^4 \cdot i = 1 \cdot i = i \\ i^2 &= -1 & i^6 &= i^4 \cdot i^2 = 1 \cdot -1 = -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i & i^7 &= i^4 \cdot i^3 = 1 \cdot -i = -i \\ i^4 &= i^2 \cdot i^2 = -1 \cdot -1 = 1 & i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1 \end{aligned}$$

Notice also that $i(-i) = (-i)i = 1$. Thus, i and $-i$ are reciprocals; that is,

$$\frac{1}{i} = -i \quad \text{and} \quad \frac{1}{-i} = i.$$

You can use this fact in computing a quotient in which the divisor is a pure imaginary number.

EXAMPLE 3 a. $\dfrac{14}{8i} = \dfrac{14}{8} \cdot \dfrac{1}{i} = \dfrac{7}{4} \cdot -i = -\dfrac{7}{4}i$

b. $\dfrac{12}{i^3} = \dfrac{12}{-i} = 12\,\dfrac{1}{-i} = 12 \cdot i = 12i$

> To simplify a square-root radical whose radicand is a negative number, take these steps:
>
> 1. Express the radical as the product of a real number and i.
> 2. Then use the properties of the *real roots of real numbers* (Chapter 8) to simplify this product.

EXAMPLE 4 $\sqrt{-25} + \sqrt{-20} = i\sqrt{25} + i\sqrt{20}$
$= 5i + 2\sqrt{5}\,i$
$= (5 + 2\sqrt{5})i.$ Answer.

Notice the use of the distributive property in Example 4.

EXAMPLE 5 $\sqrt{-2} \cdot \sqrt{-50} = i\sqrt{2} \cdot i\sqrt{50}$
$= \sqrt{2 \cdot 50} \cdot i^2 = \sqrt{100} \cdot (-1) = -10.$ Answer.

Notice that if you wrote $\sqrt{-2} \cdot \sqrt{-50} = \sqrt{-2 \cdot -50} = \sqrt{100} = 10$, you would be applying properties that have been proved only for radicals denoting *real numbers*, and you would in fact obtain an incorrect result. This is why it is important to follow the order of operations indicated above.

EXAMPLE 6 $\dfrac{3}{\sqrt{-12}} = \dfrac{3}{i\sqrt{12}} = -\dfrac{3i}{\sqrt{12}} = -\dfrac{3i}{2\sqrt{3}}$

$$= -\frac{3i\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}} = -\frac{3\sqrt{3}\,i}{2\cdot 3} = -\frac{\sqrt{3}\,i}{2}$$

$$= -\frac{\sqrt{3}}{2}i. \quad \text{Answer.}$$

Oral Exercises

Express each of the following as a real number or a pure imaginary number.

1. $3i^2$
2. $(-3i)^2$
3. $\sqrt{-16}$
4. $\sqrt{-11}$
5. $\sqrt{-18}$
6. $\sqrt{-27}$
7. $3i\sqrt{-3}$
8. $5\sqrt{-4}$
9. $i(-i)$
10. $\dfrac{8i}{-2i}$
11. $\dfrac{-3i}{-4i}$
12. $\dfrac{i}{2}\cdot\dfrac{i}{3}$
13. $(-4i)(-5i)$
14. $\sqrt{-4}\cdot\sqrt{-9}$
15. $(2i)(-7i)$
16. $\dfrac{-12i}{3}$
17. $\dfrac{2}{i}$
18. $\dfrac{6}{-i}$
19. $\dfrac{10}{-5i}$
20. $\dfrac{-15}{3i}$
21. i^5
22. i^9
23. i^{13}
24. i^{17}
25. State an expression for $f(n)$ that will make the equation $i^{f(n)} = i$ true for all positive integers n. (See Exercises 21–24 above.)

Written Exercises

Express each of the following in simplest form as a real number or a pure imaginary number.

A

1. i^{11}
2. i^{14}
3. i^9
4. i^{24}
5. $\sqrt{-72}$
6. $\sqrt{-75}$
7. $5\sqrt{-48}$
8. $-3\sqrt{-98}$
9. $\sqrt{-\frac{2}{9}}$
10. $\sqrt{-\frac{2}{5}}$
11. $-\sqrt{-\frac{3}{8}}$
12. $\sqrt{-3}\cdot\sqrt{-12}$
13. $2\sqrt{-6}\cdot\sqrt{-10}$
14. $3\sqrt{-15}\cdot 2\sqrt{-20}$
15. $\dfrac{\sqrt{-12}}{\sqrt{-3}}$
16. $\dfrac{2\sqrt{-50}}{\sqrt{-2}}$
17. $\dfrac{4}{\sqrt{-5}}$
18. $\dfrac{\sqrt{3}}{\sqrt{-7}}$
19. $\dfrac{2\sqrt{5}}{\sqrt{-20}}$
20. $\dfrac{\sqrt{3}}{-2i}$
21. $\dfrac{10i}{-2i}$
22. $\dfrac{3}{i^3}$
23. $\dfrac{-4}{i^6}$
24. $\dfrac{5i}{i^{10}}$

B 25. $2\sqrt{-18} + 3\sqrt{-2}$

26. $3\sqrt{-5} - 2\sqrt{-45}$

27. $\sqrt{-75} + \sqrt{-147}$

28. $i^6 + i^7 + i^8$

29. $i^{13} + i^{14} + i^{15} + i^{16}$

30. $\frac{1}{i^4} + \frac{2}{i^5} + \frac{3}{i^6}$

31. $\sqrt{-\frac{2}{3}} - \sqrt{-\frac{5}{6}}$

32. $2\sqrt{-\frac{1}{5}} + \sqrt{-\frac{3}{10}}$

33. $4\sqrt{-\frac{1}{12}} - 2\sqrt{-\frac{2}{27}}$

34. Prove that if p is an integer and $p, q, r, s, \ldots$ form an arithmetic sequence with common difference 4, then the sequence $i^p, i^q, i^r, i^s, \ldots$ is geometric with common ratio 1.

35. Prove that if p is an integer and $p, q, r, s, \ldots$ form an arithmetic sequence with common difference 2, then the sequence $i^p, i^q, i^r, i^s, \ldots$ is geometric with common ratio -1.

C 36. Give the solution set of the equation $x^2 + r^2 = 0$.

1-23 odd

p. 304

9-2 Complex Numbers; Addition and Subtraction

The real numbers together with the pure imaginary numbers form a set in which you can compute products (Section 9-1). But to be able to add two numbers in this set, such as 2 and $7i$ you have to invent another new number, namely, $2 + 7i$, which is neither a real number nor a pure imaginary number. In fact, to assign a sum to any real number a and any pure imaginary number bi, you have to invent a number

$$a + bi,$$

which is called a **complex number.** If $b \neq 0$, $a + bi$ is also called an **imaginary number.** You call a, the *real part,* and b, the *imaginary part,* of $a + bi$.

We shall use the letter $\mathcal{C}$ to refer to the set of all complex numbers. In $\mathcal{C}$, equality of numbers is defined as follows:

> If $a, b, c,$ and d are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Notice that $9 + 6i \neq 6 + 9i$, because $9 \neq 6$.

By identifying the real number a with the complex number $a + 0i$, you can say that every real number belongs to $\mathcal{C}$. Similarly, from the agreement $0 + bi = bi$, it follows that every pure imaginary number is also a complex number.

To define the sum

$$(a + bi) + (c + di),$$

we are guided by the requirement that the familiar properties of sums and products in $\mathcal{R}$ continue to be true in $\mathcal{C}$. For example, if the commutative and associative properties of addition are valid in $\mathcal{C}$ and if multiplication in $\mathcal{C}$ is distributive with respect to addition, then

$$\begin{aligned}(5 + 3i) + (4 + 2i) &= (5 + 4) + (3i + 2i)\\ &= (5 + 4) + (3 + 2)i\\ \therefore\ (5 + 3i) + (4 + 2i) &= 9 + 5i.\end{aligned}$$

The previous example suggests the following:

Definition of Addition in $\mathcal{C}$

If a, b, c, and d are real numbers, then

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

To simplify notation, the symbol $a + (-b)i$ is often written $a - bi$. Thus,

$$(3 - 5i) + (-1 + 2i) = (3 - 1) + (-5 + 2)i = 2 - 3i.$$

The following facts are true in $\mathcal{C}$. (See Exercises 27 and 28 page 302.)

1. The *additive identity element* is $0 + 0i$, or 0. For example, $(6 + 7i) + (0 + 0i) = 6 + 7i$.
2. For all real numbers c and d, the *negative* or *additive inverse*, of $c + di$ is $-c - di$; that is,

$$-(c + di) = -c - di.$$

For example, $-(2 - 5i) = -2 + 5i$.

Because the relationship between addition and subtraction (page 20) is preserved in $\mathcal{C}$, you can use fact 2 above to obtain the rule for subtracting one complex number from another. You have:

$$\begin{aligned}(a + bi) - (c + di) &= (a + bi) + (-c - di)\\ \therefore\ (a + bi) - (c + di) &= (a - c) + (b - d)i.\end{aligned}$$

Complex numbers such as $2 + 3i$ and $2 - 3i$ are called *complex conjugates*. Thus, for any real numbers a and b, the complex conjugate of $a + bi$ is $a - bi$; conversely, the complex conjugate of $a - bi$ is $a + bi$. We may use the notation $\bar{z}$ to denote the conjugate of complex number z.

EXAMPLE Find **(a)** the sum and **(b)** the difference of $7 + 5i$ and its conjugate.

SOLUTION Let $z = 7 + 5i$. Then its conjugate $\bar{z}$ is $7 - 5i$.

a. $z + \bar{z} = (7 + 5i) + (7 - 5i) = (7 + 7) + (5 - 5)i = 14$

b. $z - \bar{z} = (7 + 5i) - (7 - 5i) = (7 - 7) + (5 + 5)i = 10i$

This example suggests the following theorem. Its proof is left as Exercise 29, page 302.

> **Theorem.** For all real numbers a and b:
>
> $$(a + bi) + (a - bi) = 2a$$
> $$(a + bi) - (a - bi) = 2bi$$

Therefore, the sum of a complex number $a + bi$ and its complex conjugate is a real number; if $b \neq 0$, their difference is a pure imaginary number.

Oral Exercises

In Exercises 1–8, state:

a. the additive inverse of the number;
b. the complex conjugate of the number;
c. the sum of the given number and its complex conjugate; and
d. the difference of the given number and its complex conjugate.

1. $3 + 4i$ **2.** $2 - 5i$ **3.** $-3 + i$ **4.** $\sqrt{2} + 5i$

5. $-1 - 6i$ **6.** $\sqrt{3} - 4i$ **7.** 5 **8.** $3i$

State the given sum or difference as a complex number.

9. $(2 + 5i) + (4 - i)$ **10.** $(-3 + 2i) + (1 - 6i)$

11. $(4 + 7i) - (2 + 3i)$ **12.** $(8 - 5i) - (2 + i)$

Written Exercises

If $a = 5 - 8i$, $b = -7 + 3i$, $c = 2 + 8i$, $d = -4i$,
$e = \frac{2}{3} + \frac{1}{2}i$, $f = -\frac{1}{6} + \frac{3}{2}i$, $g = \frac{4}{3} - \frac{1}{4}i$, $h = -\frac{2}{9}$,
express each of the following as a complex number.

A **1.** $a + b$ **2.** $a + c$ **3.** $a - c$ **4.** $c - d$ **5.** $\bar{c} - b$

6. $d - \bar{a}$ **7.** $e + f$ **8.** $f - \bar{g}$ **9.** $h + g$ **10.** $\bar{e} - g$

11. $e + \bar{g}$ **12.** $e - f$ **13.** $a + \bar{a}$ **14.** $\bar{f} + \bar{f}$ **15.** $e - \bar{e}$

16. $g - \bar{g}$ **17.** $f - \bar{h}$ **18.** $\bar{g} - \bar{d}$ **19.** $\bar{b} + \bar{f}$ **20.** $\bar{c} + \bar{e}$

Let $z_1 = a + bi$ and $z_2 = c + di$. Prove each of the following statements.

B 21. $z_1 - \bar{z}_1$ is the complex conjugate of $-z_1 + \bar{z}_1$.

22. $z_1 + z_2$ is the complex conjugate of $\bar{z}_1 + \bar{z}_2$.

23. $z_1 - z_2$ is the complex conjugate of $\bar{z}_1 - \bar{z}_2$.

24. $z_1 - \bar{z}_2$ is the complex conjugate of $\bar{z}_1 - z_2$.

25. If $z_1 = \bar{z}_1$, then z_1 is a real number.

26. If $z_1 = -\bar{z}_1$, then z_1 is a pure imaginary number.

C 27. Prove: For all real numbers a and b,

$$(a + bi) + (0 + 0i) = a + bi \text{ and } (0 + 0i) + (a + bi) = a + bi.$$

28. Prove: For all real numbers a and b,

$$(a + bi) + (-a - bi) = 0 + 0i.$$

29. Prove the theorem stated on page 301.

30. Prove: The complex conjugate of the sum of two complex numbers is the sum of their complex conjugates.

9-3 Complex Numbers; Multiplication and Division

Granted that the commutative, associative, and distributive properties of multiplication are to hold in $\mathcal{C}$, you can compute products of complex numbers by following the pattern for products of binomials over $\mathcal{R}$ (page 177). Here is an example:

$$\begin{aligned}(5 + 3i)(4 + 2i) &= 5(4 + 2i) + 3i(4 + 2i)\\ &= (20 + 10i) + (12i + 6i^2)\\ &= (20 - 6) + (10i + 12i)\\ \therefore (5 + 3i)(4 + 2i) &= 14 + 22i.\end{aligned}$$

This example leads us to the following:

Definition of Multiplication in $\mathcal{C}$

If a, b, c, and d are real numbers, then

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

In particular, for $b = 0$, this definition gives you the following rule for multiplying a real and a complex number:

$$a(c + di) = ac + adi.$$

Thus, $1(c + di) = 1 \cdot c + 1 \cdot di = c + di$, so that 1, or $1 + 0i$, is the *multiplicative identity element* in $\mathcal{C}$.

Notice that, by the definition of multiplication in $\mathcal{C}$,

$$(a + bi)(a - bi) = (a^2 + b^2) + (-ab + ab)i = a^2 + b^2.$$

Hence we have the following result:

Theorem. For all real numbers a and b,

$$(a + bi)(a - bi) = a^2 + b^2.$$

Therefore the product of a complex number and its complex conjugate is a real number.

Because equality, addition, and multiplication of complex numbers have been defined so that the properties of equality, addition, and multiplication for the set of real numbers (Chapter 1) are also valid when restated for the set of complex numbers, concepts and methods based on these properties apply in working with complex numbers.

The following examples show how you can use the foregoing theorem to express the reciprocal of a complex number and the quotient of two complex numbers (divisor not zero) in the standard form, $a + bi$ (see Exercises 22–23, page 304).

EXAMPLE 1 Express the reciprocal of $2 - 5i$ in the form $a + bi$.

SOLUTION *Plan.* Multiply the numerator and denominator of $\frac{1}{2 - 5i}$ by the complex conjugate of the denominator.

$$\frac{1}{2 - 5i} = \frac{1}{(2 - 5i)} \cdot \frac{(2 + 5i)}{(2 + 5i)} = \frac{2 + 5i}{4 + 25} = \frac{2 + 5i}{29}$$

$$= \frac{2}{29} + \frac{5}{29}i. \quad \text{Answer.}$$

EXAMPLE 2 Express $\frac{2 + 3i}{7 + 4i}$ in standard form, $a + bi$.

SOLUTION

$$\frac{2 + 3i}{7 + 4i} = \frac{(2 + 3i)(7 - 4i)}{(7 + 4i)(7 - 4i)}$$

$$= \frac{(14 + 12) + (-8 + 21)i}{49 + 16} = \frac{26 + 13i}{65}$$

$$= \frac{2}{5} + \frac{1}{5}i. \quad \text{Answer.}$$

Oral Exercises

State the product in the form $a + bi$.

1. $(4 + 5i)(4 - 5i)$
2. $(-3 - 3i)(-3 + 3i)$
3. $(4 - i\sqrt{2})(4 + i\sqrt{2})$
4. $(-\sqrt{3} + 2i)(-\sqrt{3} - 2i)$
5. $-(4 + 2i)(4 - 2i)$
6. $(\sqrt{5} - i\sqrt{3})(\sqrt{5} + i\sqrt{3})$
7. $(-2 + i\sqrt{5})(-2 - i\sqrt{5})$
8. $(\sqrt{7} - 4i)(\sqrt{7} + 4i)$
9. $-(3 + i\sqrt{6})(3 - i\sqrt{6})$

Written Exercises

Express each of the following in the form $a + bi$. When possible, you may give your answer in the form a, bi, or 0.

A

1. $(4 - 7i)(3 + i)$
2. $(-6 + 2i)(7 - i)$
3. $(3 - 4i)^2$
4. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$
5. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
6. $(-1 + i\sqrt{5})^2$
7. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$
8. $\dfrac{3 + i}{4 - i}$
9. $\dfrac{8i}{1 + 3i}$
10. $\dfrac{2 - 7i}{5 + 2i}$
11. $\dfrac{3 - 6i}{-2 - 5i}$
12. $\dfrac{1}{3 - i\sqrt{2}}$

B

13. $\dfrac{5 + 2i}{2 - i} - \dfrac{1 - i}{3 + 4i}$
14. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3$
15. $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^4$

Using the theorem on page 303, factor each of the following over $\mathcal{C}$.

16. $x^2 + 4$
17. $9y^2 + 25$
18. $v^2 + 7$
19. $64a^2 + 12b^2$
20. $18x^2 + 10$
21. $3z^2 + 8$

C

22. Show that for all real numbers a and b, not both 0,

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

23. Show that for all real numbers a, b, c, d, where c and d are not both zero,

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

24. Show that the complex conjugate of the product of two complex numbers equals the product of their complex conjugates.

25. Factor $x^4 + 1$ completely over $\mathcal{C}$. (*Hint:* Use the theorem on page 303 first; then use the result of Written Exercise 4.)

9-4 The Nature of the Roots of a Quadratic Equation

The quadratic formula developed in Section 8-10 can be used to determine the soluton set of a quadratic equation over $\mathcal{C}$.

EXAMPLE 1 Solve $x^2 + 3x + 5 = 0$ over $\mathcal{C}$.

SOLUTION $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $a = 1$, $b = 3$, $c = 5$.

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-3 \pm \sqrt{-11}}{2}$$

$$= \frac{-3 \pm i\sqrt{11}}{2} = -\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$$

Check: Substitute $-\frac{3}{2} + \frac{\sqrt{11}}{2}i$ for x.

$$\left(-\frac{3}{2} + \frac{\sqrt{11}}{2}i\right)^2 + 3\left(-\frac{3}{2} + \frac{\sqrt{11}}{2}i\right) + 5$$

$$= \left(\frac{9}{4} - \frac{11}{4}\right) - \frac{3\sqrt{11}}{2}i - \frac{9}{2} + \frac{3\sqrt{11}}{2}i + 5 = 0$$

Substitute $-\frac{3}{2} - \frac{\sqrt{11}}{2}i$ for x.

$$\left(-\frac{3}{2} - \frac{\sqrt{11}}{2}i\right)^2 + 3\left(-\frac{3}{2} - \frac{\sqrt{11}}{2}i\right) + 5$$

$$= \left(\frac{9}{4} - \frac{11}{4}\right) + \frac{3\sqrt{11}}{2}i - \frac{9}{2} - \frac{3\sqrt{11}}{2}i + 5 = 0$$

$\therefore$ the solution set is $\left\{-\frac{3}{2} + \frac{\sqrt{11}}{2}i,\ -\frac{3}{2} - \frac{\sqrt{11}}{2}i\right\}$. Answer.

The number $b^2 - 4ac$, which is named under the radical sign in the quadratic formula, is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$. We denote it by D.

If a, b, and c are real numbers, the discriminant indicates the nature of the roots of the equation.

Theorem. For all real numbers b and c, and all nonzero real numbers a, the quadratic equation $ax^2 + bx + c = 0$ has:

(1) two different real roots if $b^2 - 4ac > 0$;

(2) one double real root if $b^2 - 4ac = 0$;

(3) two imaginary complex conjugate roots if $b^2 - 4ac < 0$.

The discriminant also enables you to tell whether the roots of a quadratic equation with *rational coefficients* are rational numbers. If a, b, and c are rational numbers, $a \neq 0$, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ denotes a rational number if and only if $\sqrt{b^2 - 4ac}$ is rational (Section 8-3). But $\sqrt{b^2 - 4ac}$ is rational if and only if $b^2 - 4ac$ is the square of a rational number. Thus:

> A quadratic equation with rational coefficients has rational roots if and only if its discriminant is the square of a rational number.

EXAMPLE 2 Determine the nature of the roots of $3x^2 + 2x - 6 = 0$.

SOLUTION $3x^2 + 2x - 6 = 0$; $a = 3$, $b = 2$, $c = -6$.
$D = b^2 - 4ac = 2^2 - 4(3)(-6) = 76$.

Since $D > 0$, there are two unequal real roots.
Since D is not the square of a rational number, the roots are irrational numbers. Answer.

In the domain of a function f, any value of x which satisfies the equation $f(x) = 0$ is said to be a **zero** of f. Notice that "zeros of the function f," and "roots of the equation $f(x) = 0$," are just different ways of referring to the same numbers.

EXAMPLE 3 What is the nature of the zeros of

$$\{(x, y)\colon y = x^2 - 2x\sqrt{5} + 5\}?$$

SOLUTION $x^2 - 2x\sqrt{5} + 5 = 0$; $a = 1$, $b = -2\sqrt{5}$, $c = 5$.
$D = b^2 - 4ac = (-2\sqrt{5})^2 - 4 \cdot 1 \cdot 5 = 20 - 20 = 0$.

Since $D = 0$, there is one real zero (a double zero).
Although D is the square of a rational number $(0 = 0^2)$, the real zero is an irrational number because b is irrational. Answer.

If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac} > 0$. Thus, $ax^2 + bx + c = 0$ has two different real roots because

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \neq \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

But if $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$, and hence the roots of $ax^2 + bx + c = 0$ are real and equal:

$$\frac{-b + \mathbf{0}}{2a} = \frac{-b - \mathbf{0}}{2a} = -\frac{b}{2a}$$

$r_1 + r_2 = -\frac{b}{a}$
$r_1 r_2 = \frac{c}{a}$

We call $-\frac{b}{2a}$ a **double** root of the equation.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is a pure imaginary number. In this case,

$$\sqrt{b^2 - 4ac} = \sqrt{|b^2 - 4ac|}\, i,$$

so that the roots of $ax^2 + bx + c = 0$ are the complex conjugates:

$$-\frac{b}{2a} + \frac{\sqrt{|b^2 - 4ac|}}{2a} i \quad \text{and} \quad -\frac{b}{2a} - \frac{\sqrt{|b^2 - 4ac|}}{2a} i.$$

The roots of the equation in Example 1 on page 305 are complex conjugates.

Oral Exercises

Find the value of the discriminant D. Use the value of D to determine the number of real and complex roots of the equation.

1. $x^2 - 5x + 4 = 0$
2. $3x^2 + 7x + 4 = 0$
3. $4x^2 + 8x + 5 = 0$
4. $x^2 + 5x + 9 = 0$
5. $5x^2 + 13x = 0$
6. $x^2 - 8x + 16 = 0$

Written Exercises

For each of the following find the value of the discriminant D. On the basis of D, tell how many real and how many imaginary roots the equation has. If it has any real roots, tell whether these are rational.

A

1. $x^2 - 2x + 2 = 0$
2. $2x^2 + 7x + 4 = 0$
3. $5x^2 - 7x + 2 = 0$
4. $17x^2 - 11x = 0$
5. $3x^2 + 4x - 5 = 0$
6. $-2x^2 + 8x - 9 = 0$
7. $-3x^2 - 12 = 0$
8. $7x^2 - 9x + 2 = 0$
9. $9x^2 + 42x + 49 = 0$
10. $\frac{5}{2}x^2 + x - 8 = 0$

Solve over the set $\mathbb{C}$ of complex numbers.

11. $5x^2 + 2x + 1 = 0$
12. $2x^2 - x + 8 = 0$
13. $3x^2 - 9x + 5 = 0$
14. $-x^2 + 7x - 13 = 0$
15. $x^2 2\sqrt{2} + 4x - 3\sqrt{2} = 0$
16. $2x^2 - x\sqrt{7} + 2 = 0$
17. $9x^2 - x6\sqrt{2} + 2 = 0$
18. $x^2\sqrt{3} - 7x + 2\sqrt{3} = 0$

Determine the value(s) of k for which the given equation will have exactly one real root.

B

19. $4x^2 - 8x + 3k = 0$
20. $3x^2 + kx + 6 = 0$
21. $x^2 - kx + k + 8 = 0$
22. $x^2 + kx + 2k - 1 = 0$

Exercises 23–25 refer to a quadratic equation of the form $ax^2 + bx + c = 0$.

C **23.** Prove: If a and b are nonzero rational numbers and $c = 0$, then the equation has two rational roots.

24. Prove: If a and b are nonzero rational numbers and c is irrational, then the equation has no rational roots. (*Hint:* Assume the equation has a rational root and show that this leads to a contradiction.)

25. Prove: If b and c are nonzero rational numbers and a is irrational, then the equation has no rational roots.

programming in BASIC

Exercises

1. Write a program that will test values of discriminants and give print-outs similar to the following:

```
RUN

INPUT A(<>0), B, C?1,1,-6
( 1)X↑2 + ( 1)X + (-6) = 0 HAS TWO DIFFERENT REAL ROOTS.

RUN

INPUT A(<>0), B, C?1,2,1
( 1)X↑2 + ( 2)X + ( 1) = 0 HAS ONE DOUBLE REAL ROOT.

RUN

INPUT A(<>0), B, C?1,3,5
( 1)X↑2 + ( 3)X + ( 5) = 0 HAS TWO IMAGINARY COMPLEX
CONJUGATE ROOTS.
```

2. Modify the program you wrote for the exercises on page 285 to print the roots like this if $D < 0$:

```
R1 = -.5 + .866025I
                        for 1, 1, 1
R2 = -.5 - .866025I
```

3. Use a computer to check your work in Exercises 1–10 on page 307.

9-5 Roots and Coefficients of a Quadratic Equation

Do you recall (page 186) that the compound sentence

$$x - 3 = 0 \qquad \text{or} \qquad x + 2 = 0$$

is equivalent to the equation $(x - 3)(x + 2) = 0$? In general, if any real or complex numbers r_1 and r_2 are the roots of a quadratic equation in x, then the quadratic equation must be equivalent to $(x - r_1)(x - r_2) = 0$;

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

By transforming the equation $ax^2 + bx + c = 0$, $a \neq 0$, to

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

you can deduce (Exercise 23, page 310) the following property:

Property of the Sum and Product of the Roots of a Quadratic Equation

The solution set of the equation $ax^2 + bx + c = 0$, $a \neq 0$, is $\{r_1, r_2\}$ if and only if $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$.

EXAMPLE 1 Is $\left\{\frac{1 + \sqrt{2}}{2}, \frac{1 - \sqrt{2}}{2}\right\}$ the solution set of $4x^2 - 4x - 1 = 0$?

SOLUTION $4x^2 - 4x - 1 = 0$; $a = 4$, $b = -4$, $c = -1$; $-\frac{b}{a} = 1$, $\frac{c}{a} = -\frac{1}{4}$.

$$r_1 + r_2 = \frac{1 + \sqrt{2}}{2} + \frac{1 - \sqrt{2}}{2} = 1 = -\frac{b}{a}$$

$$r_1 r_2 = \frac{1 + \sqrt{2}}{2} \times \frac{1 - \sqrt{2}}{2} = -\frac{1}{4} = \frac{c}{a}$$

$\therefore \left\{\frac{1 + \sqrt{2}}{2}, \frac{1 - \sqrt{2}}{2}\right\}$ is the solution set of $4x^2 - 4x - 1 = 0$. Answer.

EXAMPLE 2 Find a quadratic equation whose roots are $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$.

SOLUTION $(1 + i\sqrt{3}) + (1 - i\sqrt{3}) = 2 = -\frac{b}{a}$

$(1 + i\sqrt{3})(1 - i\sqrt{3}) = 1 + 3 = 4 = \frac{c}{a}$

Let $a = 1$; then $b = -2$ and $c = 4$. $\quad \therefore x^2 - 2x + 4 = 0$ Answer.

Check: To show that the roots of $x^2 - 2x + 4 = 0$ are $1 \pm i\sqrt{3}$ by using the quadratic formula is left to you.

Oral Exercises

State the sum and product of the given numbers.

1. $3\sqrt{2},\ -3\sqrt{2}$
2. $5 + \sqrt{2},\ 5 - \sqrt{2}$
3. $-4 + 3i,\ -4 - 3i$
4. $2 + \sqrt{3},\ 2 - \sqrt{3}$
5. $3 + i\sqrt{5},\ 3 - i\sqrt{5}$
6. $1 - 2i\sqrt{3},\ 1 + 2i\sqrt{3}$

State the sum and product of the roots of the given equation.

7. $x^2 + 4x + 2 = 0$
8. $x^2 - 5x - 3 = 0$
9. $2x^2 - 7x = 0$
10. $3x^2 + x - 4 = 0$
11. $4x^2 - 3x + 9 = 0$
12. $8x^2 - 7 = 0$

Written Exercises

Write a quadratic equation having the given solution set.

A

1. $\{3, -5\}$
2. $\{-2, -7\}$
3. $\{\frac{5}{2}\}$
4. $\{1 + \sqrt{3}, 1 - \sqrt{3}\}$
5. $\{4\sqrt{3}, -4\sqrt{3}\}$
6. $\{2 + i, 2 - i\}$
7. $\{4 + \sqrt{2}, 4 - \sqrt{2}\}$
8. $\{3 + 5\sqrt{2}, 3 - 5\sqrt{2}\}$
9. $\{2 + 6i, 2 - 6i\}$
10. $\left\{\dfrac{2 + i\sqrt{7}}{4}, \dfrac{2 - i\sqrt{7}}{4}\right\}$
11. $\{1 + i, 2 - 2i\}$
12. $\left\{\dfrac{5 + \sqrt{3}}{2}, 5 - \sqrt{3}\right\}$

In Exercises 13–21, determine a value for k so that the given conditions are satisfied.

B

13. One root of $6x^2 - x + k = 0$ is $\frac{2}{3}$.
14. One root of $x^2 - 2x + k = 0$ is $1 - \sqrt{7}$.
15. One root of $4x^2 + kx - 15 = 0$ is $\frac{3}{4}$.
16. One root of $4x^2 + kx + 13 = 0$ is $\dfrac{2 + 3i}{2}$.
17. One root of $kx^2 + 2x + 8 = 0$ is twice the other.
18. One root of $3x^2 - 6x + k = 0$ is the reciprocal of the other.

C

19. One root of $kx^2 + x\sqrt{2} - 1 = 0$ is the reciprocal of the other.
20. One root of $2kx^2 + kx - 2 = 0$ is $-\frac{2}{3}$.
21. One root of $6x^2 + kx + (k + 1) = 0$ is $\frac{4}{3}$.
22. Without solving the equation, find the sum of the reciprocals of the roots of $2x^2 + 6x - 3 = 0$.
23. Use the quadratic formula to prove that if r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, then $r_1 + r_2 = -\dfrac{b}{a}$ and $r_1 r_2 = \dfrac{c}{a}$.

Self-Test 1

VOCABULARY pure imaginary number (p. 296)
imaginary unit (p. 296)
complex number (p. 299)
imaginary number (p. 299)
imaginary part (p. 299)
complex conjugate (p. 300)
discriminant of $ax^2 + bx + c = 0$ (p. 305)
zero of a function (p. 306)
double root (p. 307)

Express as a real or pure imaginary number.

1. $3\sqrt{-64}$ 2. $2\sqrt{-\frac{4}{3}}$ *Obj. 1, p. 295*

Express in the form $a + bi$.

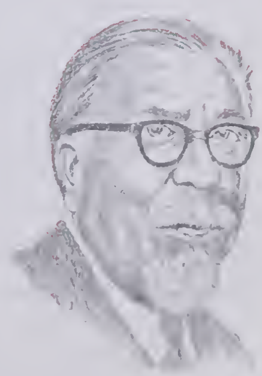
3. $(3 - 2i) + (-5 + 3i)$ 4. $7i - (3 + 4i)$ *Obj. 2, p. 295*
5. $(5 - 2i)^2$ 6. $(3i)(1 + i\sqrt{2})^{-1}$

In Test Items 7 and 8, give the value of the discriminant and tell whether the roots are real or imaginary. If real, tell whether they are rational.

7. $13x^2 + 6x + 1 = 0$ 8. $2x^2 - 11x + 5 = 0$ *Obj. 3, p. 295*
9. Give (a) the sum and (b) the product of the roots of the equation $3x^2 - 9x + 2 = 0$. *Obj. 4, p. 295*
10. Give an equation of the form $ax^2 + bx + c = 0$ whose roots are $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$.

Check your answers with those at the back of the book.

Norbert Wiener
1894–1964



Norbert Wiener was one of the outstanding mathematicians and scientists of the United States. Interrelationships among the various branches of thought had become very complicated, and Wiener's creation of a new subject, called cybernetics, deals with these matters. Cybernetics is the theoretical study of control processes in elecronics and mechanical and biological systems, especially the mathematical analysis of the flow of information in such systems.

Wiener showed great promise at an early age. He completed his undergraduate studies at 14, and obtained his doctorate at 18 from Harvard University. His thesis was in mathematical logic. At 25 he joined the faculty of the Massachusetts Institute of Technology where he remained until his death.

Quadratic Functions and Their Graphs

OBJECTIVES for Sections 9-6 through 9-8:

1. *Find an equation of the axis of symmetry and the coordinates of the vertex of the graph of an equation of the form $y = a(x - h)^2 + k$, and determine whether the vertex is a maximum or a minimum point.*
2. *Sketch the graph of a given quadratic function.*
3. *Solve extreme-value problems involving quadratic functions.*
4. *Solve quadratic inequalities.*

9-6 The Graph of $y = a(x - h)^2 + k$

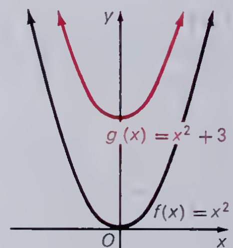


Figure 1

x	$f(x)$
-1	1
0	0
1	1

x	$g(x)$
-1	4
0	3
1	4

A function f with domain $\mathcal{R}$ and values given by a quadratic polynomial, that is,

$$f = \{(x, y)\colon y = ax^2 + bx + c,\ a,\ b,\ c, \text{ and } x \in \mathcal{R},\ a \neq 0\},$$

is called a *quadratic function,* or a *polynomial function of degree two,* over $\mathcal{R}$. The graph of a quadratic function is called a **parabola.** (See Section 10-4 for an extended discussion of parabolas.)

The parabolas shown in Figure 1 are the graphs of the two quadratic functions

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = x^2 + 3.$$

Both are symmetric with respect to the y-axis. Notice that the *minimum* (lowest) *point* of the graph of g, (0, 3), is 3 units above that of f, (0, 0).

Now compare the graphs of f and g with those of the quadratic functions F and G in Figure 2, where

$$F(x) = -\tfrac{1}{2}x^2 \qquad \text{and} \qquad G(x) = -\tfrac{1}{2}x^2 - 4.$$

Again, both parabolas are symmetric with respect to the line $x = 0$ (the y-axis). But in this case, the graph of G has a *maximum* (highest) point, that is, a point with greatest ordinate, which is 4 units below the maximum of F.

In general, the graphs of

$$f(x) = ax^2 \qquad \text{and} \qquad g(x) = ax^2 + k$$

are both symmetric with respect to the line $x = 0$. Moreover, if $k > 0$, each point of the graph of g is k units above the corresponding point of the graph of f; and if $k < 0$, it is $|k|$ units below.

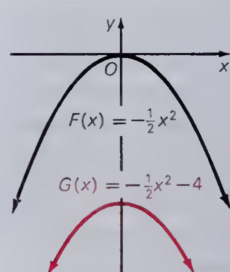


Figure 2

x	$F(x)$
-2	-2
0	0
2	-2

x	$G(x)$
-2	-6
0	-4
2	-6

Next let us compare the graph of $\{(x, y)\colon f(x) = ax^2\}$ with those of two other general quadratic functions, of the form

$$p(x) = a(x - h)^2, a \neq 0,$$

and

$$q(x) = a(x - h)^2 + k, a \neq 0.$$

If we let $a = \frac{1}{2}$, $h = 2$, $k = 3$, we obtain the graphs below.

x	$f(x)$
-2	2
0	0
2	2

x	$p(x)$
0	2
2	0
4	2

x	$q(x)$
0	5
2	3
4	5

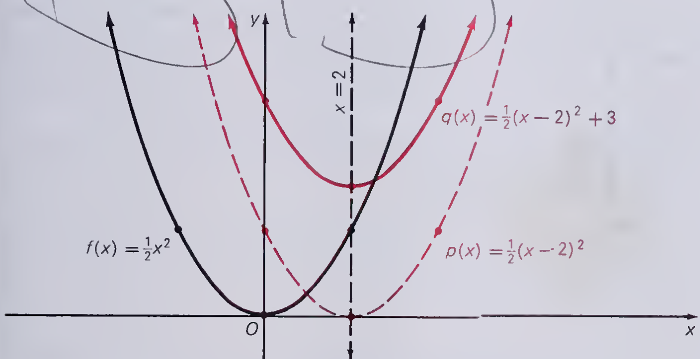


Figure 3

By observing these three graphs, we can summarize the facts concerning the **axis of symmetry** and the **vertex** (the point of the graph that lies on the axis of symmetry) of the graph of $y = a(x - h)^2 + k$.

> The graph of the function
>
> $$\{(x, y)\colon y = a(x - h)^2 + k, a \neq 0\}$$
>
> over $\mathcal{R}$ is a parabola with the line $x = h$ as an axis of symmetry and the point (h, k) as vertex. If $a > 0$, (h, k) is a minimum point, and the parabola opens upward; if $a < 0$, (h, k) is a maximum point, and the parabola opens downward.

Oral Exercises

For the graph of each of the following functions state (a) an equation of the axis of symmetry and (b) the coordinates of the vertex; (c) state whether the graph opens upward or downward.

1. $\{(x, y)\colon y = -2(x - 3)^2 + 1\}$

2. $\{(x, y)\colon y = 5(x + 1)^2 - 7\}$

3. $f(x) = -4x^2 + 6$

4. $g(x) = \frac{1}{2}(x + 3)^2$

In Exercises 5–10, the first pair of coordinates given is the vertex of a parabola; the second pair of coordinates is another point on the parabola. Give the equation of the axis of symmetry and name a third point on the parabola.

5. $(0, -2)$; $(3, 0)$
6. $(0, 4)$; $(5, -1)$
7. $(3, 1)$; $(-1, -7)$
8. $(\frac{1}{2}, 0)$; $(3, 6)$
9. (a, b); $(a + c, d)$
10. $(a + b, c)$; (b, d)

Written Exercises

In Exercises 1–8, sketch the graphs of all three of the given functions on one set of axes. Be sure to label each graph.

A

1. $\{(x, y)\colon y = x^2\}$; $\{(x, y)\colon y = x^2 + 2\}$; $\{(x, y)\colon y = x^2 - 3\}$
2. $\{(x, y)\colon y = 2x^2\}$; $\{(x, y)\colon y = 2(x + 2)^2\}$; $\{(x, y)\colon y = 2(x - 3)^2\}$
3. $\{(x, y)\colon y = -(x - 2)^2 + 4\}$; $\{(x, y)\colon y = -(x - 2)^2 - 3\}$; $\{(x, y)\colon y = -(x - 2)^2\}$
4. $f(x) = \frac{1}{2}(x + 3)^2 - 1$; $g(x) = \frac{1}{2}x^2 - 1$; $h(x) = \frac{1}{2}(x - 4)^2 - 1$
5. $f(x) = -(x - 3)^2 + 2$; $g(x) = \frac{1}{2}(x - 3)^2 + 2$; $h(x) = -2(x - 3)^2 + 2$
6. $f(x) = 2(x + 1)^2 + 4$; $g(x) = -2(x - 2)^2 + 4$; $h(x) = 2(x - 4)^2 + 4$
7. $f(x) = -(x + \frac{3}{2})^2 + 2$; $g(x) = -(x + \frac{3}{2})^2 - 2$; $h(x) = -(x - \frac{3}{2})^2 + 2$
8. $f(x) = 2(x - 1)^2 + 1$; $g(x) = -\frac{1}{2}(x - 1)^2 + 11$; $h(x) = -\frac{1}{4}(x - 1)^2 + 10$

In Exercises 9–22 find the function of the form $y = a(x - h)^2 + k$ whose graph satisfies the given conditions.

9. Has vertex $(2, 1)$ and passes through the point $(5, 19)$.
10. Has vertex $(-3, 4)$ and passes through the point $(-2, 9)$.
11. Has vertex $(6, 1)$ and passes through the point $(0, -8)$.
12. Has vertex $(-\frac{1}{4}, -\frac{1}{2})$ and passes through the point $(-1, -5)$.
13. Has $x = -3$ as its axis of symmetry and passes through the point $(-2, 8)$, and $a = 1$.
14. Has $x = \frac{1}{2}$ as its axis of symmetry and passes through the point $(2, -1)$, and $a = -2$.
15. Passes through the points $(-3, 4)$, $(0, -2)$, and $(3, 4)$.
16. Passes through the points $(-2, 2)$, $(1, 5)$, and $(4, 2)$.

B

17. Has an equation of the form $y = (x + h)^2 + 5$ and passes through the point $(3, 9)$. (There is more than one answer.)
18. Has an equation of the form $y = \frac{1}{2}(x + h)^2 - 10$ and passes through the point $(-2, 22)$.

19. Has $x = 2$ as its axis of symmetry and passes through the points (1, 8) and (0, 17). (*Hint:* Solve a system of equations in a and k.)

20. Has $x = -1$ as its axis of symmetry and passes through the points (1, 1) and (3, −5).

21. Has the y-axis as its axis of symmetry and passes through the points (1, −5) and (−2, 1).

22. Has $x = \frac{1}{2}$ as its axis of symmetry and passes through the points $(\frac{3}{2}, 1)$ and (2, 6).

C 23. Show that the point $(h + r, s)$ is on the graph of $y = a(x - h)^2 + k$, $a \neq 0$, if and only if the point $(h - r, s)$ is also on the graph.

24. Show that the point (r, s) is on the graph of $y = a(x - h)^2 + k, a \neq 0$, if and only if the point $(2h - r, s)$ is also on the graph.

25. Show that if the two distinct points $(r_1, 0)$ and $(r_2, 0)$ lie on the graph of $y = a(x - h)^2 + k$, then $\dfrac{r_1 + r_2}{2} = h$.

26. Graph the function $f(x) = |x^2 - 4|$.

9-7 The Graph of a Quadratic Function

To graph a general quadratic function over $\mathcal{R}$ of the form

$$\{(x, y): y = ax^2 + bx + c, a \neq 0\}, \qquad (1)$$

it would be helpful if we could express it equivalently in the form described in the preceding section:

$$\{(x, y): y = a(x - h)^2 + k, a \neq 0\}. \qquad (2)$$

We could then specify the vertex, (h, k), and the axis of symmetry, $x = h$, in terms of the coefficients, a, b, c, in (*1*). Then by locating two or three more points we could draw the graph with reasonable accuracy. First let us consider an example.

EXAMPLE 1 Specify the vertex and the axis of symmetry, and draw the graph, of the function defined by $y = 3x^2 + 6x + 1$.

SOLUTION 1. Rewrite the given equation in the form $\dfrac{y - c}{a} = \dfrac{ax^2 + bx}{a}$.

$$y = 3x^2 + 6x + 1$$

$$\frac{y - 1}{3} = \frac{3x^2 + 6x}{3}$$

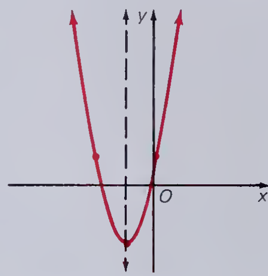
$$= x^2 + \tfrac{6}{3}x$$

(*Solution continued on page 316.*)

2. Add to both members the number $\frac{b^2}{4a^2}$ in order to complete the square in the right-hand member.

$$\frac{y-1}{3} + 1 = x^2 + 2x + 1$$

$$\frac{y-1}{3} = (x+1)^2 - 1$$



3. Then transform the equation into the desired form (2), on page 315.

$$y = 3[(x+1)^2 - 1] + 1$$
$$y = 3(x+1)^2 - 2$$
$$y = 3(x-(-1))^2 + (-2)$$

Thus the axis of symmetry is $x = -1$, and the vertex is $(-1, -2)$. Substituting 0 and -2 for x in the original equation, you find that $(0, 1)$ and $(-2, 1)$ also are on the graph, which is sketched at the right. Answer.

Using the method of Example 1 you can find the vertex and an equation of the axis of symmetry for the general case of the function

$$f = \{(x, y)\colon y = ax^2 + bx + c, a \neq 0\}.$$

First rewrite the equation as

$$\frac{y-c}{a} = \frac{ax^2 + bx}{a} = x^2 + \frac{b}{a}x.$$

Completing the square, you obtain:

$$\frac{y-c}{a} + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$y - c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right]$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$y = a\left(x - \left(\frac{-b}{2a}\right)\right)^2 + \left(-\frac{b^2 - 4ac}{4a}\right) \qquad (3)$$

Comparing Equation (3) with $y = a(x-h)^2 + k$,

you have for equation $x = h$ of the axis of symmetry: $x = -\frac{b}{2a}$; and for coordinates (h, k) of the vertex: $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$. The vertex is a minimum or a maximum point according as $a > 0$ or $a < 0$.

EXAMPLE 2 Specify the axis of symmetry and the vertex of $f(x) = 2x^2 - 4x + 1$.

SOLUTION $a = 2$, $b = -4$, $c = 1$. Equation of axis of symmetry:

$$x = -\frac{b}{2a} = \frac{4}{4}, \qquad \text{or} \qquad x = 1.$$

Vertex: $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$, or $\left(1, -\frac{16 - 8}{8}\right) = (1, -1)$. Answer.

The fact that the quadratic function $f(x) = ax^2 + bx + c$ has exactly one *extreme value* (minimum for $a > 0$, maximum for $a < 0$) is useful in practical applications. The ordered pair $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ specifies the value of $(x, f(x))$ at the extreme point of the graph of f. Thus the ordinate, $-\frac{b^2}{4a} + c$, at the extreme point is the extreme value of the function, $f\left(-\frac{b}{2a}\right)$.

EXAMPLE 3 Find two real numbers whose sum is 18 and whose product is as great as possible.

SOLUTION Let x be one number, and $18 - x$ the other. We want to find the value of x for which the function specified by the product of x and $18 - x$,

$$f(x) = x(18 - x), \qquad \text{or} \qquad f(x) = -x^2 + 18x,$$

assumes its maximum value. Here $a = -1$ and $b = 18$. The maximum occurs when $x = -\frac{b}{2a} = -\frac{18}{-2}$, or 9. Then the other number is $18 - x = 18 - 9$, or 9. $\therefore$ the two numbers are 9 and 9. Answer.

Oral Exercises

Tell whether the graph of the equation will have a minimum or maximum point.

1. $y = x^2 - 6x$
2. $y = x^2 + 5x$
3. $y = 2x^2 - 8x$
4. $y = -x^2 + 3x$
5. $y = 2x^2 - 4$
6. $y = \frac{1}{2}x^2 - 3$

Written Exercises

In Exercises 1–6, give the equation of the axis of symmetry and the coordinates of the vertex of the graph of each function and sketch the graph.

A **1–6.** Use the equations in Oral Exercises 1–6.

In Exercises 7–12, give the equation of the axis of symmetry and the coordinates of the vertex of the graph of each function and sketch the graph.

7. $f(x) = x^2 - 8x + 15$
8. $f(x) = x^2 + 6x + 7$
9. $f(x) = 2x^2 - 4x - 1$
10. $g(x) = 3x^2 - 12x + 8$
11. $f(x) = -2x^2 + 8x - 5$
12. $g(x) = \frac{1}{3}x^2 - 2x - 3$

Find the quadratic function f containing the given ordered pairs. (*Hint:* Determine a, b, and c so that the given ordered pairs satisfy the equation $y = ax^2 + bx + c$.)

x y

B 13. $(-1, 0)$, $(2, -3)$, $(3, -8)$
14. $(0, -1)$, $(1, 3)$, $(2, 8)$
15. $(-1, -6)$, $(2, 3)$, $(3, 10)$
16. $(-2, -3)$, $(-1, -1)$, $(1, -3)$

For Exercises 17–20, put each equation in the form $y = a(x - h)^2 + k$, and give the equation of the axis of symmetry and the coordinates of the vertex of the graph in terms of r and s.

17. $y = r(x^2 - s)$
18. $y = rx^2 + 2rx$
19. $y = rx^2 - 2rsx - 2s^2$
20. $y = \frac{s^2}{2} + rsx - rx^2$

C 21. Prove that if the graphs of two quadratic functions have the same vertex and one other common point, then they can be defined by the same equation.

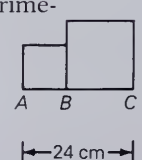
22. Prove that if $\left(\frac{-b}{2a} + r, s\right)$ is a point on the graph of $ax^2 + bx + c = y$, then so is $\left(\frac{-b}{2a} - r, s\right)$.

Problems

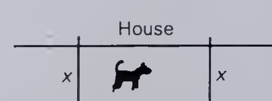
A 1. Find two numbers whose sum is 16 and whose product is as great as possible.

2. Find the dimensions of the rectangle of greatest area whose perimeter is 20 cm.

3. In the diagram, the left square is constructed on $\overline{AB}$ and the right square is constructed on $\overline{BC}$. Determine the length of $\overline{AB}$ so that the length of $\overline{AC}$ will be 24 cm and the sum of the areas of the squares will be a minimum.



4. A dog kennel is to be constructed alongside a house from 60 m of fencing. Determine x so that the greatest area possible will be enclosed.



B 5. If an object is projected vertically upward with a velocity of 19.6 m/s from a height of 20 m, its height h above the ground in meters after t s is given by the equation $h = 20 + 19.6t - 4.9t^2$. What is the maximum height the object will reach?

6. In a 110 V circuit having a resistance of 11 Ω, the power W in watts when a current I is flowing is given by $W = 110I - 11I^2$. Determine the maximum power that can be delivered in this circuit.

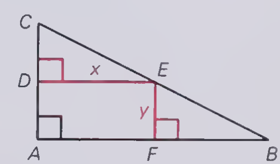
7. A real estate firm estimates that the monthly profit p in dollars from a building s stories high is given by $p = -20s^2 + 880s$. According to this formula, what height building would be the most profitable?

C 8. The velocity v of the object in Written Exercise 5 after t s is given by $v = 19.6 - 9.8t$. Solve this equation for t, and by substituting in the equation in Exercise 5, show that h is a maximum when $v = 0$.

9. A wire 40 cm long is cut into two pieces, and each piece is bent into a square. Where should the wire be cut if the total area of the two squares is to be a minimum?

10. A limousine shuttle service operating between an airport and the center of a city charges a fare of \$10 and carries 300 persons per day. The firm estimates that business will decrease by 15 passengers per day for each increase of \$1 in the fare. Find the most profitable fare to charge for the service.

11. In the diagram at the right, $AFED$ is a rectangle, $AC = 6$, and $AB = 10$.
   a. Use similar triangles to find an expression for y in terms of x.
   b. Give an expression for the area of rectangle $AFED$ in terms of x alone.
   c. Find x so that this area will be a maximum.



9-8 Quadratic Inequalities

Figure 4 shows the graph of the function

$$\{(x, y): y = x^2 - 4x + 1\}.$$

From the graph, you can see that the points with coordinates $2 - \sqrt{3}$ and $2 + \sqrt{3}$ (the zeros of the function) separate the other points on the x-axis into three sets:

$$A = \{x: x < 2 - \sqrt{3}\}$$
$$B = \{x: 2 - \sqrt{3} < x < 2 + \sqrt{3}\}$$
$$C = \{x: x > 2 + \sqrt{3}\}$$

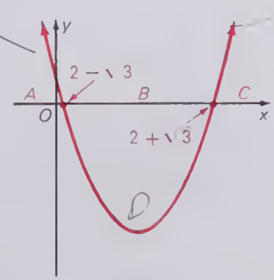


Figure 4

Do you see that over any one of these subsets the value of $x^2 - 4x + 1$ (that is, the value of y) is *always positive* (the graph is above the x-axis) or *always negative* (the graph is below the x-axis)?

$x \in A$	$x \in B$	$x \in C$
$x^2 - 4x + 1 > 0$	$x^2 - 4x + 1 < 0$	$x^2 - 4x + 1 > 0$

Thus, to test whether $x^2 - 4x + 1$ denotes a positive number or whether it denotes a negative number for *every* value of x in one of these subsets, you need only determine the sign of $x^2 - 4x + 1$ for *any one* value of x in that subset.

EXAMPLE Find the solution set over $\mathcal{R}$ of $-x^2 + 3x + 5 > 0$.

SOLUTION 1. Determine the roots of $-x^2 + 3x + 5 = 0$. $a = -1$, $b = 3$, $c = 5$.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(-1)(5)}}{2(-1)} = \frac{-3 \pm \sqrt{29}}{-2} = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

2. From each subset into which these numbers separate the set of real numbers, choose a number for which to evaluate $-x^2 + 3x + 5$.

Subset	Particular Number	$-x^2 + 3x + 5$
$x < \frac{3}{2} - \frac{\sqrt{29}}{2}$	-2	-5, negative
$\frac{3}{2} - \frac{\sqrt{29}}{2} < x < \frac{3}{2} + \frac{\sqrt{29}}{2}$	0	5, positive
$x > \frac{3}{2} + \frac{\sqrt{29}}{2}$	5	-5, negative

$$\{x\colon -x^2 + 3x + 5 > 0\} = \left\{x\colon \frac{3}{2} - \frac{\sqrt{29}}{2} < x < \frac{3}{2} + \frac{\sqrt{29}}{2}\right\}.$$ Answer.

Oral Exercises

State whether the given value of x satisfies the given inequality.

1. $x^2 - 4x + 5 \leq 0$; 2
2. $2x^2 + 3x - 8 > 0$; 4
3. $x^2 - \frac{3}{4} > 0$; $\frac{\sqrt{3}}{2}$
4. $x^2 - 7x \geq -5$; -3
5. $4x - 8 > x^2$; $\sqrt{2}$
6. $0 < x^2 - 8x + 16$; -2

Written Exercises

In Exercises 1–9:

a. rewrite the inequality as one in which 0 is the right-hand member;
b. find the roots of the related equation;
c. sketch the graph of the related quadratic function; and
d. solve the inequality by noting which part(s) of the graph lie above or below the x-axis.

A

1. $x^2 - 9 > 0$
2. $4x^2 - 25 \leq 0$
3. $3x - x^2 < 0$
4. $2x^2 - 5x < 0$
5. $x^2 - x \geq 6$
6. $2x^2 + 3 \leq 7x$

B

7. $x^2 - 4x + 2 \leq 0$
8. $3x^2 \geq x + 1$
9. $x^2 + 2x > 2$

In Exercises 10–11, write the inequality as a conjunction of simpler inequalities. Then sketch the graphs of the related quadratic functions to determine the solution set of the original inequality.

C

10. $-3 < x^2 - 4 < 0$
11. $0 \geq x^2 - 4x \geq -3$

Self-Test 2

VOCABULARY parabola (p. 312)
vertex (p. 313)
axis of symmetry (p. 313)

Test Items 1–3 refer to the equation $y = 4 - (x - 3)^2$.

1. Find an equation for the axis of symmetry and the coordinates of the vertex. *Obj. 1, p. 312*
2. State whether the vertex is a maximum or minimum point.
3. Sketch the graph of the equation. *Obj. 2, p. 312*
4. Rewrite $y = 2x^2 - 8x + 5$ in the form $y = a(x - h)^2 + k$, and sketch its graph.
5. Find the area of the right triangle of largest area such that the sum of the lengths of its legs is 12 cm. *Obj. 3, p. 312*
6. Solve the inequality $x^2 - 5x + 4 > 0$ by drawing a sketch of the related quadratic equation. *Obj. 4, p. 312*

Check your answers with those at the back of the book.

Polynomial Functions and Equations

***OBJECTIVES** for Sections 9-9 through 9-12:*

1. *Use synthetic substitution to find the value of a given polynomial function at a given domain value.*
2. *Use synthetic division to find the partial quotient and the remainder when a polynomial is divided by* x − c, *where* c ∈ $\mathcal{C}$.
3. *Use synthetic division and depressed equations to factor polynomials and to find rational zeros of polynomial functions.*
4. *Use the fact that imaginary roots of polynomial equations with real coefficients occur in complex conjugate pairs to solve such equations.*
5. *Use synthetic substitution and linear interpolation to approximate the real roots of a polynomial equation.*

9-9 Values of Polynomial Functions

Functions such as

$$\{(x, y)\colon y = 3x - \sqrt{2}\},\ \{(x, y)\colon y = x^2 - 2ix + 3\}$$
$$\{(x, y)\colon y = 4x^3 + 2x^2 - 1\},$$

whose values are given by polynomials, are called **polynomial functions.** You know how to evaluate a polynomial function by direct substitution. For example, if a_0, a_1, a_2, a_3 denote complex numbers and P is the function

$$\{(x, P(x))\colon P(x) = a_0x^3 + a_1x^2 + a_2x + a_3\},$$

then to evaluate P at 7, you write

$$P(7) = a_0(7^3) + a_1(7^2) + a_2(7) + a_3.$$

To find the value of $P(7)$, you might compute:

(1) 7^2
(2) 7^3
(3) $a_0(7^3)$
(4) $a_1(7^2)$
(5) $a_2(7)$
(6) $a_0(7^3) + a_1(7^2) + a_2(7) + a_3$

This pattern of computation, however, is not the most efficient one to use as a basis for a *program* for a computer. Each product would have to be stored until needed (using valuable space) and the final addition would have to be performed by taking two addends at a time, making additional steps.

To discover a more efficient way to compute $P(7)$, study the following sequence of operations.

1. Multiply a_0 by 7: $a_0 \cdot 7$
2. Add a_1: $a_0 \cdot 7 + a_1$
3. Multiply the result of Step 2 by 7: $(a_0 \cdot 7 + a_1) \cdot 7$
4. Add a_2: $(a_0 \cdot 7 + a_1) \cdot 7 + a_2$
5. Multiply the result of Step 4 by 7: $[a_0 \cdot 7 + a_1) \cdot 7 + a_2] \cdot 7$
6. Add a_3: $[(a_0 \cdot 7 + a_1) \cdot 7 + a_2] \cdot 7 + a_3$

By simplifying the expression in Step 6, you can verify that

$$[(a_0 \cdot 7 + a_1) \cdot 7 + a_2] \cdot 7 + a_3 = a_0 \cdot 7^3 + a_1 \cdot 7^2 + a_2 \cdot 7 + a_3 = P(7).$$

Notice that in this sequence of steps, each result is computed directly from the preceding result. No values need to be saved or "stored."

You can use this second method even if you do not have a computer. Steps 1–6 can be arranged conveniently as shown below. The circled numerals designate each of the steps.

$$\begin{array}{r|llll}
7 & a_0 & a_1 & a_2 & a_3 \\
 & & (1)\ a_0 \cdot 7 & (3)\ (a_0 \cdot 7 + a_1) \cdot 7 & (5)\ [(a_0 \cdot 7 + a_1) \cdot 7 + a_2] \cdot 7 \\
\hline
 & a_0 & a_0 \cdot 7 + a_1 & (a_0 \cdot 7 + a_1) \cdot 7 + a_2 & [(a_0 \cdot 7 + a_1) \cdot 7 + a_2] \cdot 7 + a_3 \\
 & & (2) & (4) & (6)
\end{array}$$

If $P(x) = 5x^3 - 12x^2 - 20x + 1$, you can find $P(7)$ by following Steps 1–6, using 5, -12, -20, and 1 in place of a_0, a_1, a_2, and a_3, respectively:

$$\begin{array}{r|rrrr}
7 & 5 & -12 & -20 & 1 \\
 & & 35 & 161 & 987 \\
\hline
 & 5 & 23 & 141 & 988 \\
 & & & & P(7)
\end{array}$$

Thus $P(7) = 988$. This process, called **synthetic substitution,** applies to polynomials of any degree. Notice that $P(x)$ must be written in descending powers of x. Also, if a power is missing, 0 must be written in the corresponding place.

EXAMPLE If $Q(x) = 2x^4 - x^3 + 2x - 1$, find $Q(3)$ and $Q(3i)$.

SOLUTION Write the coefficients of $Q(x)$ in order, using 0 where necessary. Then, use synthetic substitution

$$\begin{array}{r|rrrrr}
3 & 2 & -1 & 0 & 2 & -1 \\
 & & 6 & 15 & 45 & 141 \\
\hline
 & 2 & 5 & 15 & 47 & 140
\end{array}$$

$$\begin{array}{r|rrrrr}
3i & 2 & -1 & 0 & 2 & -1 \\
 & & 0 + 6i & -18 - 3i & 9 - 54i & 162 + 33i \\
\hline
 & 2 & -1 + 6i & -18 - 3i & 11 - 54i & 161 + 33i
\end{array}$$

$\therefore\ Q(3) = 140;\ Q(3i) = 161 + 33i$. Answer.

#1-5

Oral Exercises

Find the requested values of $P(x) = 7x^3 - 3x^2 + 7x - 8$ for the given values by considering the equation in the rewritten form shown:

$$P(x) = [(7x - 3)x + 7]x - 8.$$

1. $P(0)$ 2. $P(1)$ 3. $P(-1)$ 4. $P(2)$ 5. $P(4)$

Written Exercises

In Exercises 1–20 use synthetic substitution to find the given values over $\mathcal{C}$ of the polynomial given directly above. If the substituted value is a zero, so state.

$P(x) = x^3 - 3x^2 - 4x + 12$

A 1. $P(2)$ 2. $P(-1)$ 3. $P(-2)$ 4. $P(3)$ 5. $P(-3)$

$Q(x) = x^4 - x^3 + 2x^2 - 3x - 10$

6. $Q(1)$ 7. $Q(-1)$ 8. $Q(2)$ 9. $Q(3)$ 10. $Q(-2)$

$R(x) = 2x^5 - x^3 - 4x^2 - 3x - 4$

11. $R(2)$ 12. $R(i)$ 13. $R(-i)$ 14. $R(-2)$ 15. $R(2i)$

$S(x) = x^4 - 3x^3 - 12x + 16$

odd

16. $S(4)$ 17. $S(2)$ 18. $S(2i)$ 19. $S(-2i)$ 20. $S(3i)$

B 21. Determine m so that $F(2) = 1$ for $F(x) = 3x^3 - 8x^2 + 6x + m$.

22. Determine m so that $G(-1) = 3$ for $G(x) = 3x^4 + mx - 5$.

23. Determine m so that 2 is a root of $H(x) = 3x^3 - x^2 + mx - 12$.

24. Determine m so that -3 is a root of $K(x) = 2x^3 + mx^2 - x + 6$.

25. Determine a and b so that $T(-3) = 49$ and $T(5) = -39$ for $T(x) = -x^3 + 3x^2 + ax + b$.

26. Determine a and b so that $T(1) = 2$ and $T(2) = 17$ for $T(x) = x^4 - x^2 + ax + b$.

C 27. Use synthetic substitution to show that if $P(x) = ax^3 + bx^2 + cx + d$, then $P(0) = d$.

28. Use synthetic substitution to show that if $P(x) = ax^3 + bx^2 + cx + d$ has 2 as one of its roots, then $8a + 4b + 2c + d = 0$.

29. Let $P(x) = ax^3 + bx^2 + cx + d$ with a, b, c, and d all real numbers.
   a. Show that if i is a root of $P(x)$, then $a = c$ and $b = d$.
   b. Show that if $a = c$ and $b = d$, then $-i$ is a root of $P(x)$.
   c. What conclusion can be drawn from (a) and (b)?

ON THE CALCULATOR

To evaluate a polynomial using the calculator we may rewrite the polynomial to suggest a sequence of steps on the calculator. The general third degree polynomial may be rewritten as follows. (See Section 9-9.)

$$ax^3 + bx^2 + cx + d = [(ax + b)x + c]x + d$$

EXAMPLE Evaluate $4x^3 + 2x^2 - 5x + 1$ when $x = 7$.

SOLUTION Rewrite the polynomial as $[(4x + 2)x - 5]x + 1$. When $x = 7$, use these steps:

4 $\boxed{\times}$ 7 $\boxed{+}$ 2 $\boxed{=}$ $\boxed{\times}$ 7 $\boxed{-}$ 5 $\boxed{=}$ $\boxed{\times}$ 7 $\boxed{+}$ 1 $\boxed{=}$ 1436. Answer.

Exercises

Evaluate the polynomial for the given value of the variable.

1. $6x^2 + 3x + 1$; 4
2. $x^2 - 3x + 8$; 2
3. $5x^2 - 4x - 7$; -3
4. $3x^3 + 5x^2 - 2x + 1$; 2
5. $x^3 - 4x^2 + 2x - 1$; 4
6. $2x^3 + 3x^2 - 7x + 16$; -2
7. $x^3 - 12x^2 + 5x$; 6
8. $x^4 - 2x^2 + 17$; -2
9. $3x^3 - 8x + 12$; 4
10. $x^4 - x^3 + x^2 - x$; -2
11. $x^4 - x^3 + x^2 - x$; 2
12. $2x^4 + 3x^3 - 5x + 4$; -3

programming in BASIC

Exercise

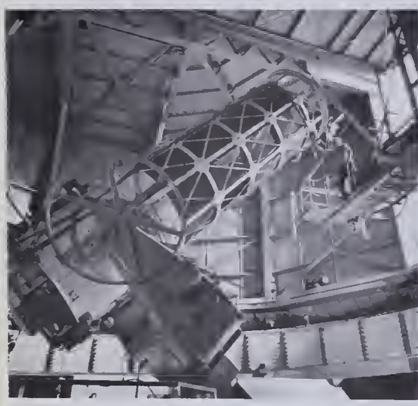
Write a program that will print out the numbers you obtain by synthetic substitution. For example, the program should give a print-out something like this for $P(x) = 5x^3 - 12x^2 - 20x + 1$ shown on page 323.

```
RUN

DEGREE OF POLYNOMIAL?3
INPUT COEFFICIENTS IN DESCENDING ORDER.
?5
?-12
?-20
?1
INPUT VALUE.
?7
5   23   141   988 = P( 7)
ANOTHER VALUE (1,YES; 0,NO)?0
```

Careers

in Astronomy



Large reflecting telescopes, such as this, are useful in spectroscopic research.

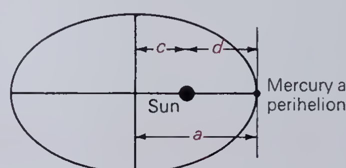
Information radioed by satellite is used to create the image of the sun on the screen.

Astronomers collect and analyze data on the sun, the planets and their moons, stars, and galaxies. They attempt to determine the motions of these bodies, their chemical composition, surface temperatures, sizes, and shapes, as well as their history and probable future.

In making observations, astronomers use a variety of equipment. Photographic devices are usually attached to telescopes to record observations. Measuring devices are also placed on balloons, rockets, and satellites. Astronomers observe not only the visible light radiated from stars and reflected from planets, but also x-rays, radio waves, and infra-red radiation.

One area of astronomy deals with the motion of the bodies of the solar system. It can be shown that the orbit about the sun of every object in the solar system is a conic section with the sun at one focus. The 17th-century astronomer Johannes Kepler deduced that the orbits of all of the planets are ellipses (see Section 10-5).

EXAMPLE How far is Mercury from the sun at perihelion, the point in the orbit of the planet that is nearest the sun?



SOLUTION The planet is closest to the sun when it is on the major axis of the ellipse representing the path of its orbit. We are trying to find the distance d in the figure. The data available are: the distance a from the center of the orbit to the planet, which for Mercury is 58×10^6 km, and the *eccentricity* of the orbit, which gives a measure of how the shape of an ellipse differs from that of a circle. The eccentricity is the ratio c/a, and has the value 0.206 for the orbit of Mercury.

$$\text{eccentricity} = \frac{c}{a} = \frac{\text{distance center of orbit to sun}}{\text{distance center of orbit to planet}}$$

For Mercury: $0.206 = \dfrac{c}{58 \times 10^6}$

$$c = 0.206 \times (58 \times 10^6) \approx 12 \times 10^6 \text{ km}$$

From the figure you can see that $d = a - c$:

$$d \approx (58 \times 10^6) - (12 \times 10^6) = 46 \times 10^6 \text{ km}$$

9-10 Remainder and Factor Theorems

By dividing $2x^2 - 4x - 5$, $x^2 - x - 6$, $x^3 - 4x^2 + 5x - 6$, and $x^3 + 2x^2 - 7x - 4$, respectively, by $x - 3$, you obtain the *partial quotients* (Q) and *remainders* (R) shown in the following chart.

$P(x)$	$Q(x)$	R	$P(3)$
$2x^2 - 4x - 5$	$2x + 2$	1	1
$x^2 - x - 6$	$x + 2$	0	0
$x^3 - 4x^2 + 5x - 6$	$x^2 - x + 2$	0	0
$x^3 + 2x^2 - 7x - 4$	$x^2 + 5x + 8$	20	20

Notice that in every case the remainder R equals $P(3)$.

Let us now take a more formal approach and divide the general third degree polynomial by $x - 7$ using these steps:

$$\begin{aligned} P(x) &= a_0x^3 + a_1x^2 + a_2x + a_3, \\ P(7) &= a_0(7^3) + a_1(7^2) + a_2(7) + a_3 \\ P(x) - P(7) &= a_0(x^3 - 7^3) + a_1(x^2 - 7^2) + a_2(x - 7) + (a_3 - a_3) \\ &= a_0(x - 7)(x^2 + 7x + 49) + a_1(x - 7)(x + 7) + a_2(x - 7) \\ &= (x - 7)[a_0(x^2 + 7x + 49) + a_1(x + 7) + a_2] \\ &= (x - 7)[a_0x^2 + [a_0(7) + a_1]x + [a_0(7)^2 + a_1(7) + a_2]]. \end{aligned}$$

Thus, if you let $Q(x) = a_0x^2 + [a_0(7) + a_1]x + [a_0(7^2) + a_1(7) + a_2]$, you have

$$P(x) = (x - 7)Q(x) + P(7), \qquad \text{or} \qquad \frac{P(x)}{x - 7} = Q(x) + \frac{P(7)}{x - 7}.$$

You can use this argument to prove the following theorem for any polynomial of any positive degree in x and any divisor $x - r$.

Remainder Theorem

For every polynomial $P(x)$ of positive degree n over the set of complex numbers, and for every complex number r, there exists a polynomial $Q(x)$ of degree $n - 1$, such that

$$P(x) = (x - r)Q(x) + P(r).$$

Did you recognize the coefficients of $Q(x)$ in the discussion preceding the Remainder Theorem? They are the first three expressions in the last line of the substitution process shown on page 323. Because for any polynomial $P(x)$ you can use the synthetic-substitution process to find the partial quotient $Q(x)$ and the remainder $P(r)$ that is obtained on dividing $P(x)$ by $(x - r)$, synthetic substitution is often called **synthetic division.**

EXAMPLE 1 Use synthetic division to divide $x^3 - 2x^2 + 3x - 6$ by $x - 2$.

SOLUTION

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array} \qquad Q(x) = x^2 + 3;\ R = 0$$

$$\therefore \frac{x^3 - 2x^2 + 3x - 6}{x - 2} = x^2 + 3 + \frac{0}{x - 2} = x^2 + 3. \quad \text{Answer.}$$

A corollary of the Remainder Theorem is the following:

> **Factor Theorem**
>
> Over the set of complex numbers, $x - r$ is a factor of a polynomial $P(x)$ if and only if r is a root of $P(x) = 0$.

PROOF

If r is a root of $P(x) = 0$, then by the definition of root, $P(r) = 0$. Therefore, by the Remainder Theorem,

$$P(x) = (x - r)Q(x) + P(r) = (x - r)Q(x) + 0 = (x - r)Q(x)$$

and $(x - r)$ is a factor of $P(x)$. Conversely, if $(x - r)$ is a factor of $P(x)$, then

$$P(x) = (x - r)Q(x), \text{ so that } P(r) = (r - r)Q(r) = 0 \cdot Q(r) = 0.$$

This theorem can help you identify factors of polynomials and zeros of polynomial functions.

EXAMPLE 2 Is $x - 5$ a factor of $P(x) = x^4 - 3x^3 - 11x^2 + 3x + 10$?

SOLUTION If $P(5) = 0$, then $x - 5$ is a factor. Use synthetic substitution.

$$\begin{array}{r|rrrrr} 5 & 1 & -3 & -11 & 3 & 10 \\ & & 5 & 10 & -5 & -10 \\ \hline & 1 & 2 & -1 & -2 & 0 \end{array}$$

$\therefore$ $(x - 5)$ is a factor. Answer.

EXAMPLE 3 Find the zeros of the function P, where $P(x) = 2x^3 + x^2 - 6x - 3$.

SOLUTION *Plan:* Solve the equation $P(x) = 0$.

1. Because the coefficients are integers, use the theorem on page 262 to identify possible rational roots, $\frac{p}{q}$. $\qquad 2x^3 + x^2 - 6x - 3 = 0$

   $\frac{p}{q} \in \{\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, 1, -1, 3, -3\}$

2. Use the Factor Theorem and synthetic substitution to test each possibility. By mentally doing the addition steps in the process, you can arrange the work conveniently, as shown.

x				$P(x)$
	2	1	-6	-3
$\frac{1}{2}$	2	2	-5	$-\frac{11}{2}$
$-\frac{1}{2}$	2	0	-6	0

$\therefore P(x) = (x + \frac{1}{2})(2x^2 - 6) = 0.$

3. Solve the *depressed* equation, $2x^2 - 6 = 0$:

$$x^2 = 3;\ x = \sqrt{3} \text{ or } x = -\sqrt{3}$$

$\therefore$ the set of zeros of P is $\{-\frac{1}{2}, -\sqrt{3}, \sqrt{3}\}$. Answer.

Whenever r is a root of the polynomial equation $P(x) = 0$, you find the remaining roots by solving the **depressed equation** $P(x) \div (x - r) = 0$.

Oral Exercises

Exercises 1–7 refer to the following synthetic division problem in which $P(x)$ is divided by $x - r$.

$$\begin{array}{r|rrrr} -2 & 3 & 4 & 0 & -11 \\ & & -6 & 4 & -8 \\ \hline & 3 & -2 & 4 & -19 \end{array}$$

1. State $P(x)$ as a polynomial.
2. State the value of r and $x - r$ as a polynomial.
3. State the partial quotient.
4. State the remainder.
5. State the value of $P(r)$.
6. State the division as an equation in the form $P(x) = (x - r)Q(x) + P(r)$.
7. State the division as an equation with $\dfrac{P(x)}{x - r}$ as its left member.

Written Exercises

Use synthetic division to write the given polynomial in the form $P(x) = (x - r)Q(x) + P(r)$ for the given values of r.

A

1. $P(x) = 2x^3 - 5x^2 - 3x + 2$: a. $r = 2$ b. $r = 3$
2. $P(x) = 4x^3 + 7x^2 - 5x + 6$: a. $r = 1$ b. $r = -3$
3. $P(x) = 4x^3 - 4x^2 + 5x - 8$: a. $r = -2$ b. $r = \frac{1}{2}$
4. $P(x) = 6x^3 + 7x^2 + x + 6$: a. $r = -\frac{3}{2}$ b. $r = i$

Use synthetic division to divide the given polynomial $P(x)$ by the given polynomials $x - r$. Express the result as an equation whose left-hand member is $\frac{P(x)}{x - r}$.

5. $P(x) = 3x^3 - 7x^2 - 20x + 3$: a. $x + 2$ b. $x - 4$
6. $P(x) = 2x^3 - 6x^2 + 3x - 8$: a. $x - 3$ b. $x - i$
7. $P(x) = x^4 - x^3 - 4x - 16$: a. $x + 3$ b. $x - 2i$
8. $P(x) = 4x^4 - 8x^3 + 7x^2 - 5$: a. $x - \frac{3}{2}$ b. $x + \frac{1}{2}$

Use the Factor Theorem to decide whether or not the binomial given in the form $x - r$ is a factor of the polynomial given as $P(x)$. If it is not, give the remainder when $P(x)$ is divided by $x - r$.

9. $x + 3$; $P(x) = x^4 + 3x^3 - 2x^2 - x + 15$
10. $x - 2$; $P(x) = x^4 - 5x^2 + 5x - 6$
11. $x - i$; $P(x) = x^4 + 5x^2 - 2$
12. $x + 2i$; $P(x) = 2x^3 + 3x^2 + 8x + 12$
13. $x + \frac{1}{2}$; $P(x) = 4x^3 + 12x^2 + 7x - 1$
14. $x - \frac{1}{3}$; $P(x) = 6x^4 - 5x^3 + 10x^2 - 1$

For each of the polynomials below, one root is given. Find the other roots.

15-27 ODD

15. $x^3 - 5x^2 - 2x + 24$; $r_1 = 3$
16. $x^3 + 2x^2 - 3x - 6$; $r_1 = -2$
17. $2x^3 - 3x^2 - 2x + 3$; $r_1 = -1$
18. $x^3 - 5x^2 + 4x - 20$; $r_1 = 5$
19. $2x^3 - 7x^2 + 2x + 3$; $r_1 = -\frac{1}{2}$
20. $2x^3 - 10x^2 + 9x - 4$; $r_1 = 4$

B 21. $x^4 - 2x^3 - 7x^2 + 8x + 12$; $r_1 = 2$
22. $x^4 - x^3 - x^2 - x - 2$; $r_1 = -1$

Find the zeros of each function.

23. $P(x) = x^3 - x^2 - 9x + 9$
24. $P(x) = x^3 - 2x^2 - 5x + 6$
25. $P(x) = x^3 - 7x^2 + 10x + 6$
26. $P(x) = x^4 - x^3 + 2x^2 - 4x - 8$
27. $P(x) = 2x^3 - 3x^2 - 18x - 8$
28. $P(x) = 3x^4 + 5x^3 + x^2 + 5x - 2$
29. Find m so that $x - 2$ will be a factor of $x^3 - 5x^2 + mx - 2$.
30. Find m so that -3 will be a zero of $x^3 + 2x^2 - 8x + m$.
31. Find m so that $x^3 - 2x^2 + mx + 4$ will leave a remainder of -2 when divided by $x - 3$.

C 32. Find m so that $x + 1$ will be a factor of $x^{97} + mx - 5$. (*Hint:* Use the Factor Theorem.)
33. Show that if $x - i$ is a factor of $2x^{18} + mx^2 - 1$, then so is $x + i$.
34. Show that if $P(x) = ax^3 + bx^2 + cx + d$, with a, b, c, and d real numbers, then $P(ki)$ is the complex conjugate of $P(-ki)$, for any real number k.

programming in BASIC

You may use the theorem on page 262 and the Factor Theorem on page 328 to help you write a program that will determine the rational roots of a polynomial equation with integral coefficients.

Exercise

Write a program that will help you find the rational roots of a polynomial equation. Study this print-out for solving $x^3 - 7x - 6 = 0$.

```
RUN

DEGREE OF POLYNOMIAL EQUATION?3
INPUT COEFFICIENTS IN DESCENDING ORDER.
?1
?0
?-7
?-6
TRY FACTORS OF -6 / 1 AS VALUES.
INPUT VALUE (TYPE 9E+10 TO END)?6
 1  6  29  168 = P( 6)
INPUT VALUE (TYPE 9E+10 TO END)?3
 1  3  2  0 = P( 3)
 3 IS A ROOT.
DEPRESSED EQUATION--DEGREE 2:
TRY FACTORS OF 2 / 1 AS VALUES.
INPUT VALUE (TYPE 9E+10 TO END)?2
 1  5  12 = P( 2)
INPUT VALUE (TYPE 9E+10 TO END)?1
 1  4  6 = P( 1)
INPUT VALUE (TYPE 9E+10 TO END)?-1
 1  2  0 = P(-1)
-1 IS A ROOT.
DEPRESSED EQUATION--DEGREE 1:
TRY FACTORS OF 2 / 1 AS VALUES.
INPUT VALUE (TYPE 9E+10 TO END)?-2
 1  0 = P(-2)
-2 IS A ROOT.
```

This program ends if n roots are found (n is the degree of the equation). If not all roots are rational, this program can be ended by typing 9E + 10 as indicated.

9-11 The Fundamental Theorem of Algebra

Over the set of real numbers a polynomial equation may have no solution. For example, $x^2 + 1 = 0$ has no *real* root. But over the set of complex numbers it has two roots, namely i and $-i$. The German mathematician C. F. Gauss in 1799 first proved that *every polynomial equation with complex coefficients has at least one root.* This result, called the **Fundamental Theorem of Algebra**, leads to the following assertion, which we will accept without proof.

> **Theorem.** Every polynomial equation with complex coefficients and positive degree n has exactly n complex roots.

In applying this theorem, you may have to count the same number as a root more than once. For example, 5 is a *double* root of the equation $x^2 - 10x + 25 = 0$.

You recall (page 305) that the imaginary roots of a quadratic equation with real coefficients occur in conjugate pairs. Thus, the fact that $2 - 3i$ is a root of $x^2 - 4x + 13 = 0$ implies that $2 + 3i$ is also a root. This property is typical of all polynomial equations with *real* coefficients.

> **Theorem.** If a polynomial equation with real coefficients has $a + bi$ as a root (a and b real, $b \neq 0$), then $a - bi$ is also a root.

The proof of this theorem is indicated in Exercises 17–19, page 333.

If you know that

$$P(x) = x^4 + 4x^3 + 12x^2 + 28x + 35 = 0$$

has the root $-2 + i$, the preceding theorem together with the Factor Theorem enables you to solve the equation. Since the equation has real coefficients, *both* $-2 + i$ and $-2 - i$ are roots. Therefore, both $x - (-2 + i)$ and $x - (-2 - i)$ are factors of the left-hand member. Since neither of these polynomials is a factor of the other, their product

$$[(x + 2) - i][(x + 2) + i],$$

or $x^2 + 4x + 5$, must be a factor of $P(x) = 0$. But

$$P(x) \div (x^2 + 4x + 5) = x^2 + 7.$$

The roots of the depressed equation are $i\sqrt{7}$ and $-i\sqrt{7}$; hence, the solution set of $P(x) = 0$ is

$$\{-2 + i, -2 - i, i\sqrt{7}, -i\sqrt{7}\}.$$

Oral Exercises

1. Two roots of $x^3 - 2x^2 + 3x - 6 = 0$ are 2 and $i\sqrt{3}$. What is the other root?
2. Two roots of $x^3 - 5x^2 + 7x + 13 = 0$ are -1 and $3 + 2i$. What is the other root?
3. A cubic equation with real coefficients has roots 7 and $2 - i$. What is the third root?
4. A cubic equation with real coefficients has roots -2 and $1 - 3i$. What is the third root? Find the equation.
5. One root of a quadratic equation is $-3i$. What is the other root? Find the equation.
6. One root of a quadratic equation is $2 + i$. What is the other root? Find the equation.

Written Exercises

Find a polynomial equation having the given roots.

A 1. $2, -2, 3$ 2. $\sqrt{3}, -\sqrt{3}, -1$ 3. $2i, -2i, 5$

4. $1, -1, i\sqrt{6}, -i\sqrt{6}$ 5. $-2, 3 + i, 3 - i$ 6. $4, 1 + i\sqrt{2}, 1 - i\sqrt{2}$

In Exercises 7–14, you are given an equation and one or more of its roots. Find the other root(s) and factor the equation completely.

7. $x^3 + x^2 - x + 15 = 0;\ 1 - 2i, -3$
8. $x^3 + 5x^2 + 5x - 11 = 0;\ 1, -3 + i\sqrt{2}$
9. $x^3 + 2x^2 + 9x + 18 = 0;\ 3i$
10. $x^4 + 9x^2 + 20 = 0;\ -i\sqrt{5}$
11. $x^3 - x^2 - 7x + 15 = 0;\ 2 + i$
12. $x^4 - 6x^3 + 8x^2 + 12x - 20 = 0;\ 3 - i$
13. $x^4 - 2x^3 + 7x^2 - 10x + 10 = 0;\ 1 + i$
14. $x^4 - 2x^3 + 13x^2 - 6x + 30 = 0;\ -i\sqrt{3}$

B 15. Prove that every polynomial of *odd* degree with real coefficients has at least one real root. (*Hint:* Use the theorems on page 332.)

16. Prove that every polynomial of *even* degree with real coefficients has an even number of real roots or no real roots.

17. For any positive integer k and real numbers a, b, and c, show that $c(a + bi)^k$ is the conjugate of $c(a - bi)^k$. (*Hint:* Use the fact that the conjugate of a real number is itself, in conjunction with the results of Exercise 24 on page 304.)

C 18. Show that if $P(x)$ is a polynomial with real coefficients, then $P(a + bi)$ is the conjugate of $P(a - bi)$, for any real numbers a and b. (*Hint:* Use Exercise 17 above, in conjunction with Exercise 30 on page 302.)

19. Show that if $P(x)$ is a polynomial with real coefficients and $P(a + bi) = 0$, then $P(a - bi) = 0$, for any real numbers a and b.

programming in BASIC

Exercise

BASIC provides a method of defining special functions. This is done by a statement of the form DEF FNA(X), where A may be replaced by any letter and X is a variable.

1. This is the beginning of a program which uses a defined function in locating integral roots from -10 to 10 of the equation $x^3 - 2x^2 + 4x - 8 = 0$. Complete the program and run it.

```
10  DEF FNA(X) = X↑3 - 2*X↑2 + 4*X - 8
20  FOR X = -10 TO 10
30  IF FNA(X) = 0 THEN 60
```

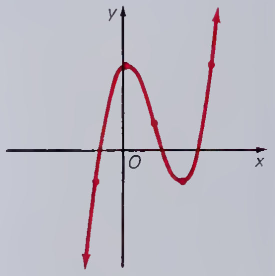
9-12 Estimates of Real Roots

Estimates of the real roots of a polynomial equation (or the real zeros of a polynomial function) can be found by several methods. First, you can consider the graphs of polynomial functions. You have sketched the graphs of many first- and second-degree polynomial functions over $\mathcal{R}$, as well as of some power functions (Section 8-1). The following example suggests how to get some idea of the shape of the graph of a polynomial function of higher degree over $\mathcal{R}$.

EXAMPLE Graph P if $P(x) = x^3 - 3x^2 + 3$.

SOLUTION Use synthetic substitution to find some values for $P(x)$. When these points are connected by a smooth curve, you obtain the graph shown at the right.

x	$P(x)$
0	3
1	1
2	-1
3	3
-1	-1



In drawing the graphs of polynomial functions as smooth unbroken curves, you assume the following:

Property of Continuity

If P is a polynomial function with real coefficients, and if m is any number between $P(a)$ and $P(b)$, then there is at least one number c between a and b for which $P(c) = m$.

In other words, P takes on every value between any two of its values.

In the preceding example, since $P(2) = -1$ and $P(3) = 3$, so that $P(2) < 0 < P(3)$, there must be a value of x between 2 and 3 for which $P(x) = 0$. By inspecting the diagram, you can see that the graph crosses the x-axis at a point whose x-coordinate is approximately 2.5. Thus, an estimate of one root of $P(x) = 0$ is 2.5.

You can also see that since $P(1) = 1$ and $P(2) = -1$, there must be a root between 1 and 2 (approximately 1.4), and since $P(-1) = -1$ and $P(0) = 3$, there must be a root between -1 and 0 (approximately -0.9). Thus, the equation $x^3 - 3x^2 + 3 = 0$ has three real roots.

You can obtain a closer estimate of a root by computation. By direct or synthetic substitution, you can show that

$$P(2.5) = -0.125 < 0 \qquad \text{and} \qquad P(2.6) = 0.296 > 0,$$

so that $2.5 < r_1 < 2.6$. Now look at Figure 5, which shows the part of the graph of P over the interval $2.5 \leq x \leq 2.6$. Notice that the line segment joining the points $A(2.5, -0.125)$ and $B(2.6, 0.296)$ of the graph crosses the x-axis at C, which is near the point where the graph itself crosses. This suggests that the x-coordinate of C *is a fairly good approximation of* r_1. Denoting the coordinates of C by $(2.5 + h, 0)$, you have

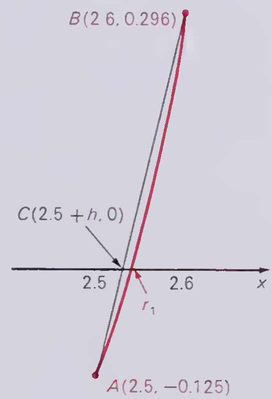


Figure 5

$$\text{slope of } \overline{AC} = \text{slope of } \overline{AB}$$

$$\frac{0.125}{h} \approx \frac{0.421}{0.1} = 4.21$$

$$\therefore h \approx \frac{0.125}{4.21} \approx 0.03.$$

Thus, $r_1 \approx 2.5 + 0.03$, or 2.53. As a matter of fact, $P(2.53) \approx -0.008423$, which is fairly close to 0. This process of approximating a value of $P(x)$ by using a line segment is called **linear interpolation.**

You can check that $P(2.54) \approx 0.032264$, so that $2.53 < r_1 < 2.54$. To obtain an even better approximation of r_1, you can repeat the interpolation over this shorter interval. The process can be repeated as many times as desired, until you have an approximation of a root to any number of decimal places. Such repetitive procedures can be programmed for a computer.

Oral Exercises

For each of the following, state $P(a)$, $P(b)$, and a value c such that $P(c) = m$.

1. $P(x) = 4x$; $a = 1$, $b = 3$, $m = 8$
2. $P(x) = 3 - x$; $a = 1$, $b = -1$, $m = 3$
3. $P(x) = x^2 - 2$; $a = 0$, $b = 3$, $m = 6$
4. $P(x) = x^2 + 1$; $a = 0$, $b = 3$, $m = 6$

Written Exercises

In Exercises 1–10, estimate to the nearest half unit those zeros of the function lying between -3 and 3 inclusive.

A

1. $P(x) = x^3 - 4x$
2. $P(x) = x^3 - 3x^2$
3. $P(x) = x^3 + x$
4. $P(x) = x^3 + 2x^2$
5. $P(x) = x^3 - x^2 - 6$
6. $P(x) = x^3 - 2x^2 - 4x + 8$
7. $P(x) = x^4 - 10x^2 + 9$
8. $P(x) = x^3 - 3x + 1$
9. $P(x) = x^3 - 2x^2 - 4x + 7$
10. $P(x) = x^3 - 3x^2 - x + 2$

In Exercises 11–12:

a. estimate to the nearest hundredth the real roots of the polynomial between 1 and 2; and
b. check your answers by using synthetic substitution.

B

11. $y = x^3 - 2x - 2$
12. $y = x^3 - 3x^2 - 2x + 6$

For each of the following, estimate to the nearest hundredth from the graph of $y = P(x)$ a value c, between the given a and b, for which $P(c) = m$.

13. $P(x) = x^2 - 4$; $a = 1$, $b = 3$, $m = 3$
14. $P(x) = x^3 + x$; $a = 0$, $b = 2$, $m = 5$

C

15. Write a polynomial of the third degree with integral coefficients that has three real zeros all of which are greater than 0.3 and less than 1.

Self-Test 3

VOCABULARY

polynomial function (p. 322)
synthetic substitution (p. 323)
synthetic division (p. 327)
depressed equation (p. 329)
Fundamental Theorem of Algebra (p. 332)
Property of Continuity (p. 334)
linear interpolation (p. 335)

1. If $P(x) = 2x^4 + 6x^3 - x^2 + 7x + 4$, use synthetic substitution to find: **a.** $P(-2)$; and **b.** $P(i)$. *Obj. 1, p. 322*
2. Use synthetic division to find the partial quotient and remainder when $x^3 - x^2 - 17x - 12$ is divided by $x + 3$. *Obj. 2, p. 322*
3. Find all the roots of $2x^3 - 7x^2 + x + 10 = 0$. *Obj. 3, p. 322*
4. Find all the roots of $x^3 - 3x + 52 = 0$ given that one root is $2 - 3i$. *Obj. 4, p. 322*
5. Estimate to the nearest hundredth the zero of $P(x) = 10x^2 - 8x - 9$ between 1 and 2 by linear interpolation. *Obj. 5, p. 322*

Check your answers with those at the back of the book.

Chapter Summary

1. A *complex number, $a + bi$*, where a and b are real numbers and i, or $\sqrt{-1}$, is the *imaginary unit*, is a real number if $b = 0$, is an *imaginary number* if $b \neq 0$, and is a *pure imaginary number* if $a = 0$, $b \neq 0$. If a, b, c, and d are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$.

2. Complex numbers may be added and also multiplied: For all real numbers a, b, c, and d,

$$(a + bi) + (c + di) = (a + c) + (b + d)i;$$

and

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The closure, associative, commutative, and distributive properties hold for the set $\mathcal{C}$ of complex numbers.

The *additive identity element* in $\mathcal{C}$ is $0 + 0i$, or 0; for all real numbers c and d, the *additive inverse* of $c + di$ is $-c - di$. The *multiplicative identity element* in $\mathcal{C}$ is $1 + 0i$, or 1.

3. From the value of the *discriminant* $b^2 - 4ac$ of a quadratic equation $ax^2 + bx + c = 0$ with real coefficients, $a \neq 0$, you can tell whether its roots are real roots or imaginary, complex conjugate roots and whether its real roots are rational or irrational, and equal or unequal.

4. To solve a *quadratic inequality* such as $ax^2 + bx + c > 0$, $a \neq 0$, you first find roots, r_1 and r_2, $r_1 \leq r_2$, of $ax^2 + bx + c = 0$, and then consider the intervals $x < r_1$, $r_1 < x < r_2$, and $x > r_2$.

5. For any polynomial function P, *synthetic substitution* (or *synthetic division*) may be used in finding values of $P(x)$ for given values of x.

6. The *Remainder Theorem* states that for every polynomial $P(x)$, of degree n ($n \geq 1$), and every complex number r, there is a polynomial $Q(x)$, of degree $n - 1$, such that $P(x) = (x - r)Q(x) + P(r)$. This leads to the *Factor Theorem*, which states that $x - r$ is a factor of $P(x)$ if and only if r is a root of $P(x) = 0$. The coefficients of $Q(x)$, as well as $P(r)$, can be determined by synthetic division.

7. From the *Fundamental Theorem of Algebra* it can be proved that every polynomial equaton of degree n ($n \geq 1$) with complex coefficients has exactly n complex roots.

8. If a polynomial with real coefficients has $a + bi$ as a root (a and b real, $b \neq 0$), then $a - bi$ is also a root.

9. The *Property of Continuity* states that if P is a polynomial function, and if m is any number between $P(a)$ and $P(b)$, then there is a number c between a and b for which $P(c) = m$.

10. If a and b are real numbers and you find $P(a) > 0$ and $P(b) < 0$ or $P(a) < 0$ and $P(b) > 0$, then there is at least one root between a and b.

Chapter Review

1. Express $\sqrt{-81}$ as a real or pure imaginary number. 9-1
   a. -9 b. $9i$ c. $-9i$ d. 9

In Review Items 2–5, express the indicated sum, difference, product, or quotient in the form $a + bi$.

2. $(4 - 3i) + (7 + 2i)$ 9-2
   a. $11 + 5i$ b. $6 + 4i$ c. $11 - i$ d. $21 - 8i$
3. $(-2 + 2i) - (-8 - 3i)$
   a. $6 - 5i$ b. $5 - 6i$ c. $-6 - 5i$ d. $6 + 5i$
4. $(2 + 3i)(7 - 2i)$ 9-3
   a. $14 + 6i$ b. $20 + 17i$ c. $4 - 21i$ d. $20 - 17i$
5. $\dfrac{4}{1 - i}$
   a. $2 - 2i$ b. $4i - 4$ c. $2 + 2i$ d. $-2 - 2i$
6. What is the nature of the roots of the equation $2x^2 - 4x + 7 = 0$? 9-4
   a. real, irrational b. imaginary c. real, rational
7. Give the sum of the roots of the equation $2x^2 + 6x - 3 = 0$. 9-5
   a. $1\frac{1}{2}$ b. 3 c. -2 d. -3
8. Give the product of the roots of the equation $2x^2 + 10x - 5 = 0$.
   a. $\frac{5}{2}$ b. -5 c. $-\frac{5}{2}$ d. -2
9. Give an equation for the axis of symmetry of the graph of $y = \frac{1}{2}(x + 4)^2 + 7$. 9-6
   a. $x = 2$ b. $x = -2$ c. $x = 4$ d. $x = -4$
10. Find the vertex of the function $y = x^2 - 6x + 9$. 9-7
    a. $(0, 9)$ b. $(3, 0)$ c. $(6, 9)$ d. $(1, 4)$
11. Find the roots of the equation related to $5x \geq x^2 - 6$. 9-8
    a. $\{-1, 6\}$ b. $\{-1\frac{1}{2}, 1\frac{1}{2}\}$ c. $\{1, -6\}$ d. $\{1, -1\}$

In Review Items 12–13, let $P(x) = 3x^3 - 2x^2 - 4x - 3$.

12. Use synthetic substitution to find P(2). 9-9
    a. -8 b. 5 c. 3 d. 2
13. Use synthetic division to write $P(x)$ in the form $(x - 2)Q(x) + P(2)$. 9-10
    a. $(x - 2)(3x^2 + 4x - 4) + 5$ b. $(x - 2)(3x^2 + 4x + 4) + 5$
    c. $(x - 2)(3x^2 + 4x - 4) + 3$ d. $(x - 2)(3x^2 - 4x + 4) + 3$

14. Given that $-2 + i$ is a solution of $x^3 + 3x^2 + x - 5 = 0$, find its solution set. 9-11

a. $\{-2 + i, -2 - i, 1\}$ b. $\{-2 + i, 2 - i, 1\}$ c. $\{-2 + i, 2 + i, -1\}$

15. Graph $P(x) = x^3 - 3x^2 - 2x + 6$ and estimate to the nearest half unit any zeros between 4 and -4. 9-12

a. $\{-3, -1.5, 1.5\}$ b. $\{-1.5, 1.5, 3\}$ c. $\{-3, -1.5, 3\}$

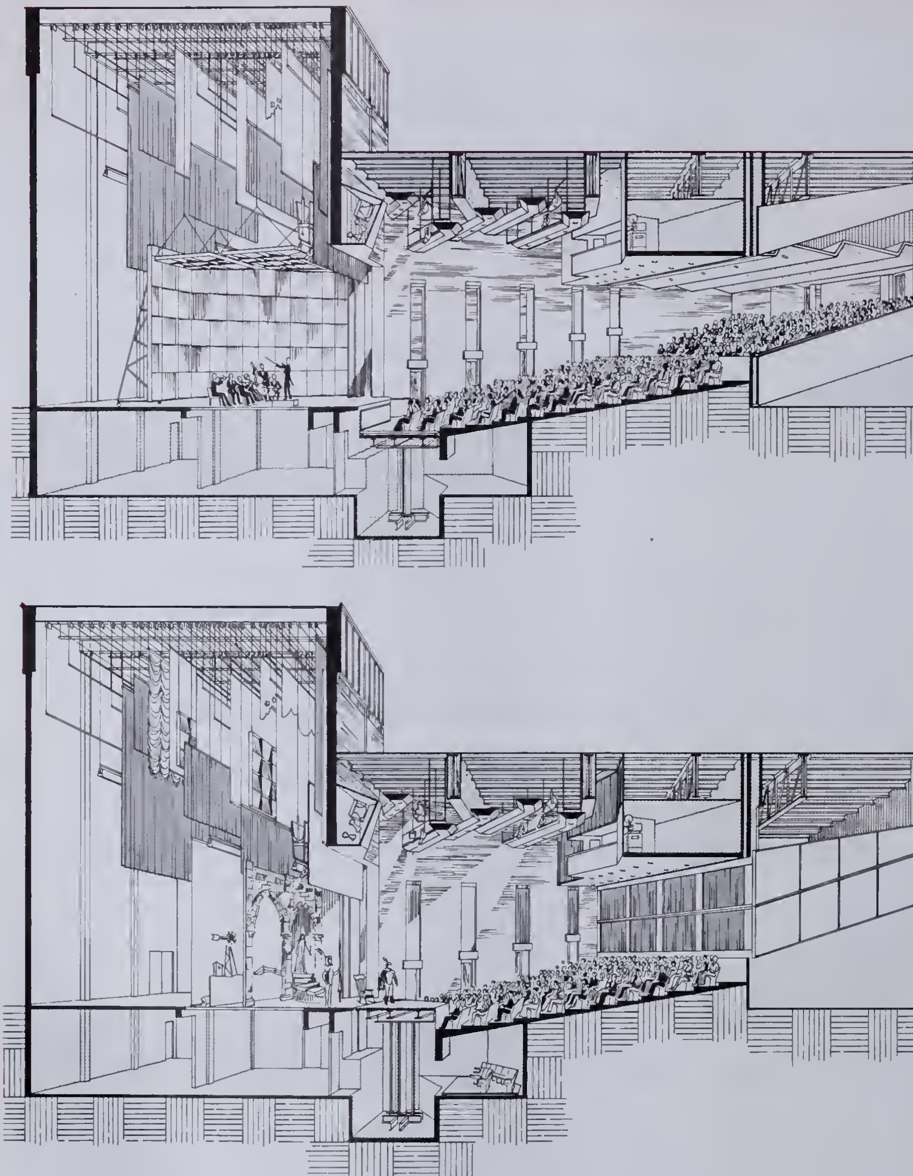
Chapter Test

In Test Items 1–6, simplify the indicated expression.

1. $\sqrt{-32}$
2. $\dfrac{\sqrt{-24}}{\sqrt{-8}}$ 9-1
3. $(7 + 2i) + (8 - 3i)$
4. $(-3 + 7i) - (8 - i)$ 9-2
5. $(2 + 3i)(4 + i)$
6. $\dfrac{2 + i}{1 + i}$ 9-3
7. How many real or complex roots does the equation $x^2 + 5x + 2 = 0$ have? Explain whether any real roots are rational or irrational. 9-4
8. Solve the equation $x^2 - 4x + 6 = 0$ over $\mathbb{C}$.
9. Give the sum and product of the roots of $3x^2 + 18x + 2 = 0$. 9-5
10. Write a quadratic equation with integral coefficients whose solution set is $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$.
11. Find a function in the form $y = a(x - h)^2 + k$ whose graph has vertex (3, 7) and passes through the point (2, 9). 9-6
12. Give the equation of the axis of symmetry and the coordinates of the vertex of $x^2 + 4x - 5 = y$, and sketch the graph of the equation. 9-7
13. a. Sketch the graph of the equation related to $x^2 - 8x < 15$. 9-8
    b. Give the solution set of the inequality.
14. For $P(x) = x^3 + 3x^2 - 5x + 2$, use synthetic substitution to find (a) $P(-1)$ and (b) $P(2i)$. 9-9
15. For $R(x) = 3x^3 - 7x^2 - 20x + 3$, use synthetic division to divide $R(x)$ by $x - 3$. Express the result as an equation whose left member is $\dfrac{R(x)}{x - 3}$. 9-10
16. Solve $x^4 - 5x^3 + 8x^2 - 10x + 12 = 0$ given $i\sqrt{2}$ is a root. 9-11

In Test Items 17–18, answer a. True or b. False.

17. The function $f(x) = x^3 - 5x^2 + 2x + 8$ has a zero between 3 and 5. 9-12
18. The function $g(x) = 2x^3 + 5x^2 - 6x - 9$ has no zeros between 0 and 4.



This multiple-use auditorium is shown in its concert mode (above) and drama theater mode (below). In the concert mode, audience seating is expanded, and an orchestra shell is used on stage. In the theater mode, a thrust stage extends into the audience, and sets are hung from above.

10
Quadratic Relations and Systems

Coordinates and Distances in a Plane

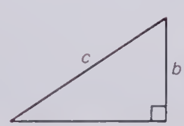
OBJECTIVES for Sections 10-1 and 10-2:
1. ***Determine the distance between two points in a plane.***
2. ***Determine the midpoint of a line segment.***
3. ***Determine an equation of the line perpendicular to a given line and passing through a given point.***

10-1 Distance between Points

On page 58, you saw that the distance between the points having coordinates a and b on the number line is $|b - a|$. To determine the distance between *any* two points in a coordinate plane with equal units on both axes, you use the following familiar theorem:

Pythagorean Theorem

In a right triangle, the square of the length c of the hypotenuse is equal to the sum of the squares of the lengths a and b of the other two sides: $c^2 = a^2 + b^2$.



Of course, you should recall that the converse is also true.

Converse of the Pythagorean Theorem

If a, b, and c are the lengths of the sides of a triangle, and if $c^2 = a^2 + b^2$, then the triangle is a right triangle with hypotenuse of length c.

Figure 1 shows two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, where segment $\overline{P_1P_2}$ is not parallel to a coordinate axis. Then you can construct a right triangle such as the one shown, where T, the third vertex, has coordinates (x_2, y_1). Since $\overline{P_1T}$ and $\overline{P_2T}$ are parallel to the coordinate axes, their lengths are $|x_2 - x_1|$ and $|y_2 - y_1|$, respectively. Then, by the Pythagorean Theorem, $d(P_1, P_2)$ (read "the distance from P_1 to P_2") satisfies

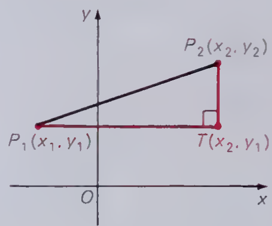


Figure 1

$$[d(P_1, P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Since distance is a nonnegative number, you have the following:

Distance Formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 1 Find the distance between $P_1(-2, 6)$ and $P_2(3, 7)$.

SOLUTION $d(P_1, P_2) = \sqrt{[3 - (-2)]^2 + (7 - 6)^2} = \sqrt{5^2 + 1^2} = \sqrt{26}.$

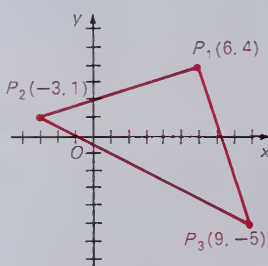
EXAMPLE 2 Find the perimeter of the triangle whose vertices are $(6, 4)$, $(-3, 1)$, and $(9, -5)$. Determine whether the triangle is isosceles and whether it is a right triangle.

SOLUTION First, make a sketch showing the triangle. Next, use the distance formula to find the lengths of the sides.

$$d(P_2, P_1) = \sqrt{[6 - (-3)]^2 + (4 - 1)^2} = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$$

$$d(P_1, P_3) = \sqrt{(9 - 6)^2 + (-5 - 4)^2} = \sqrt{3^2 + (-9)^2} = \sqrt{90} = 3\sqrt{10}$$

$$d(P_2, P_3) = \sqrt{[9 - (-3)]^2 + (-5 - 1)^2} = \sqrt{12^2 + (-6)^2} = \sqrt{180} = 6\sqrt{5}$$



Then the perimeter is $3\sqrt{10} + 3\sqrt{10} + 6\sqrt{5} = 6\sqrt{10} + 6\sqrt{5}$. Since $d(P_2, P_1) = d(P_1, P_3)$, the triangle is isosceles.

Also, since $(3\sqrt{10})^2 + (3\sqrt{10})^2 = (6\sqrt{5})^2$, the converse of the Pythagorean Theorem assures us that the triangle is a right triangle.

$\therefore$ the perimeter is $6\sqrt{10} + 6\sqrt{5}$, and the triangle is an isosceles right triangle. Answer.

An immediate consequence of the distance formula is the following formula for the midpoint of any line segment.

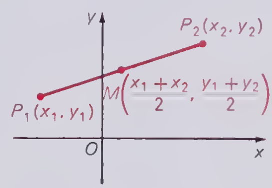
Midpoint Formula

The midpoint (M) of the segment with endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

You can prove this formula as follows:

If $d(P_1, M) = d(M, P_2) = \frac{1}{2}d(P_1, P_2)$, then M is the midpoint of $\overline{P_1P_2}$. Using the distance formula you find that:



$$d(P_1, M) = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$
$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$
$$= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(M, P_2) = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} = \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Since $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, you have

$$d(P_1, M) = \tfrac{1}{2}d(P_1, P_2) \quad \text{and} \quad d(M, P_2) = \tfrac{1}{2}d(P_1, P_2).$$

$\therefore$ M is the midpoint of $\overline{P_1P_2}$.

Oral Exercises

Give the coordinates of the midpoint of the line segment with the given endpoints.

1. (2, 0), (4, 6) **2.** (−3, 1), (3, 9) **3.** (2, −2), (3, 1)

4. Given that (0, 0) is the midpoint of a segment one endpoint of which has coordinates (a, b), find the coordinates of the other endpoint.

Written Exercises

Find the length of the line segments with the given endpoints. Express all radicals in simple form.

A 1. $(3, 5), (-2, 5)$ 2. $(4, -7), (4, -1)$ 3. $(2, 4), (5, 8)$

4. $(-5, 4), (7, -1)$ 5. $(-1, 4), (-2, 6)$ 6. $(3, -4), (-1, 2)$

7. $(\frac{1}{2}, 2)(-\frac{1}{4}, 3)$ 8. $(-\sqrt{3}, -4), (3\sqrt{3}, -3)$ 9. $(a, b), (b, a)$

10–12. Verify that the midpoints calculated in Oral Exercises 1–3 are equidistant from the endpoints of the segments.

In Exercises 13–16:
a. sketch the triangle whose coordinates are given;
b. state whether the triangle is isosceles and, if so, give the lengths of the equal sides;
c. state whether the triangle is a right triangle and, if so, write an equation relating the lengths of the legs to that of the hypotenuse.

odd

13. $A(0, 0), B(4, 3), C(1, -1)$ 14. $A(-5, 3), B(-1, -3), C(5, 1)$

15. $A(-2, 1), B(4, -1), C(5, 2)$ 16. $A(-\frac{5}{2}, \frac{1}{2}), B(2, -1), C(-\frac{1}{2}, \frac{7}{2})$

Find the coordinates of F if M is the midpoint of $\overline{FG}$.

17. $M(3, 2);\ G(5, 1)$ 18. $M(-1, 3);\ G(2, -2)$

19. $M(\frac{1}{2}, -\frac{5}{2});\ G(4, -7)$ 20. $M(a, b);\ G(a + c, b - d)$

In $\triangle XYZ$, whose vertices are given, find the length of the median to side $\overline{XY}$. Recall that the median to a side of a triangle is the segment joining the midpoint of the side to the opposite vertex.

B 21. $X(-3, 5);\ Y(9, 7);\ Z(-2, -4)$ 22. $X(2, -5);\ Y(3, -3);\ Z(-3, 2)$

23. Show that the distance between $P_1(a, c)$ and $P_2(b, c)$ is $|a - b|$.

24. Show that the distance between $P_1(a, 0)$ and $P_2(b, 2\sqrt{ab})$ is $|a + b|$.

25. Show that if $M(p, q)$ is the midpoint of $\overline{AB}$, with $A(r, s)$, then the coordinates of B are $(2p - r, 2q - s)$.

26. Find all values of x such that the distance between $A(1, -5)$ and $B(x, 7)$ is 13 units.

27. Find an equation whose solution set consists of those points (x, y) equidistant from $(-2, 3)$ and $(4, -1)$.

C 28. The coordinates of the vertices of a parallelogram are $A(0, 0)$, $B(a, 0)$, $C(a + b, c)$ and $D(b, c)$. Show that the diagonals of the parallelogram bisect each other.

29. Show that if $A(r, mr + b)$, $B(s, ms + b)$, and $C(t, mt + b)$ are three points on the graph of $y = mx + b$, with $r < s < t$, then $d(A, B) + d(B, C) = d(A, C)$.

10-2 Perpendicular Lines

You know, of course, that two lines intersecting at right angles are called **perpendicular lines.** For example, every horizontal line in a plane, such as the graph of $y = 2$ in Figure 2, is perpendicular to each vertical line, as, for example, the graph of $x = -3$.

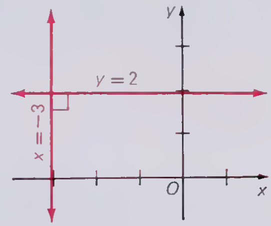


Figure 2

If neither of two perpendicular lines is vertical, you can use the Pythagorean Theorem and its converse to establish an interesting relationship between their slopes. Figure 3 shows two lines L_1 and L_2, with equations

$$y = m_1 x + b_1,$$
$$y = m_2 x + b_2,$$

and intersecting at $P(x_1, y_1)$. Since $P(x_1, y_1)$ lies on both lines, the points

$$T_1(x_1 + 1, y_1 + m_1)$$

and

$$T_2(x_1 + 1, y_1 + m_2)$$

must lie on L_1 and L_2, respectively. The points P, T_1, and T_2 are then the vertices of a triangle. If, now, the lines L_1 and L_2 are perpendicular, triangle PT_1T_2 is a right triangle, with right angle at P. Thus, by the Pythagorean Theorem,

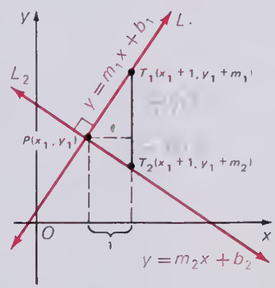


Figure 3

$$[d(T_1, T_2)]^2 = [d(P, T_1)]^2 + [d(P, T_2)]^2$$
$$(m_1 - m_2)^2 = (1 + m_1^2) + (1 + m_2^2)$$
$$m_1^2 - 2m_1m_2 + m_2^2 = 2 + m_1^2 + m_2^2$$
$$-2m_1m_2 = 2$$
$$m_1m_2 = -1$$

Conversely, suppose that, for nonvertical lines L_1 and L_2, the slopes m_1 and m_2 are such that $m_1m_2 = -1$. In this case, L_1 and L_2 cannot be parallel, because for parallel lines, $m_1 = m_2$, and $m_1m_2 = -1$ would imply $m_1^2 = -1$, a statement that is false for every $m_1 \in \mathfrak{R}$. Hence, L_1 and L_2 must intersect at some point $P(x_1, y_1)$. If points T_1 and T_2 are now determined as above,

$$[d(T_1, T_2)]^2 = m_1^2 - 2m_1m_2 + m_2^2 = m_1^2 - 2(-1) + m_2^2$$
$$= m_1^2 + 2 + m_2^2 = (1 + m_1^2) + (1 + m_2^2).$$
$$\therefore\ [d(T_1, T_2)]^2 = [d(P, T_1)]^2 + [d(P, T_2)]^2.$$

By the converse of the Pythagorean Theorem, then, L_1 and L_2 must be perpendicular.

Summarizing the foregoing arguments, you have the theorem at the top of page 346.

> **Theorem.** Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Notice that $m_1 m_2 = -1$ implies that

$$m_1 = -\frac{1}{m_2} \quad \text{and} \quad m_2 = -\frac{1}{m_1}.$$

The slopes of perpendicular lines are *negative reciprocals* of each other.

EXAMPLE Find an equation of the line passing through $(-2, 5)$ that is perpendicular to the graph of $x - 3y = 7$.

SOLUTION Transform the given equation to slope-intercept form (page 86),

$$y = \tfrac{1}{3}x - \tfrac{7}{3},$$

from which the slope of its graph is seen to be $\frac{1}{3}$. Any line perpendicular to the graph of the given equation will then have slope $-\frac{1}{\frac{1}{3}}$, or -3. Use the slope-intercept form $y = mx + b$. Let $m = -3$, $x = -2$, and $y = 5$ to solve for b.

$$\begin{aligned} 5 &= -3(-2) + b \\ b &= -1 \end{aligned}$$

$\therefore$ the equation of the line is $y = -3x - 1$. Answer.

Oral Exercises

State the slope of a line perpendicular to the line whose equation is given.

1. $y = 2x - 3$
2. $y = -\frac{2}{3}x + 5$
3. $y = \frac{4}{5}x + 7$
4. $x + 2y = 5$
5. $4x - y = 5$

Written Exercises

Find an equation of the line passing through the given point and perpendicular to the line with the given equation.

A

1. $2y = x - 6$; $(4, -1)$
2. $3y + 2x = 5$; $(-2, 5)$
3. $5x + 3 = 2y$; $(10, 3)$
4. $2y = 3x + 1$; $(-\frac{2}{5}, -\frac{4}{5})$

Find an equation of the line containing C and perpendicular to the line passing through points A and B.

5. $A(5, 3)$, $B(-3, 1)$, $C(-2, 2)$
6. $A(2, 1)$, $B(-6, 7)$, $C(-3, -5)$
7. $A(-4, -5)$, $B(-1, 3)$, $C(4, \frac{3}{2})$
8. $A(\frac{3}{4}, \frac{1}{3})$, $B(-\frac{5}{4}, \frac{5}{3})$, $C(\frac{4}{3}, -7)$

Find an equation for the perpendicular bisector of the line segment with the given endpoints. Recall that the perpendicular bisector of a segment is a line passing through the midpoint of the segment and perpendicular to the line containing the segment.

9. $(3, -7), (1, 9)$

10. $(4, -3), (8, -11)$

11. $(4, 5), (-1, 2)$

12. $(\frac{1}{2}, -\frac{4}{3}), (\frac{5}{6}, \frac{4}{3})$

Prove that the quadrilateral with vertices A, B, C, and D is a rectangle by showing that its opposite sides are of equal length and that one pair of adjacent sides is perpendicular.

B **13.** $A(-3, 0)$, $B(3, 3)$, $C(1, 7)$, $D(-5, 4)$

14. $A(-6, -1)$, $B(0, -5)$, $C(2, -2)$, $D(-4, 2)$

15. Consider the parallelogram with vertices $(0, 0)$, $(\sqrt{a^2 + b^2}, 0)$, $(\sqrt{a^2 + b^2} + b, a)$, and (b, a).

a. Show that it is a rhombus by showing that its sides are of equal length.

b. Show that its diagonals are perpendicular.

16. Find an equation of the perpendicular bisector of the line segment joining $(2a, 0)$ and $(0, 2b)$.

C **17.** Show that the median to side $\overline{BC}$ in the triangle with vertices $A(a, a)$, $B(a + 2b, a + 2c)$, and $C(a + 2c, a + 2b)$ is perpendicular to $\overline{BC}$.

18. Show that if the triangle with vertices $A(a, 0)$, $B(-a, 0)$, and $C(r, s)$, $a > 0$, is a right triangle with right angle at C, then $d((0, 0), C) = a$. (*Hint:* Use the slope relationship for perpendicular lines to show that $r^2 + s^2 = a^2$.)

Self-Test 1

VOCABULARY Pythagorean Theorem (p. 341)
distance formula (p. 342)
midpoint formula (p. 343)

1. Find the distance between $P_1(4, -2)$ and $P_2(7, 3)$. *Obj. 1, p. 341*
2. Find the midpoint of the segment with endpoints $(-2, 4)$ and $(-4, -8)$. *Obj. 2, p. 341*
3. Find an equation of the line containing $(-1, 3)$ and perpendicular to the line through the points $(2, 0)$ and $(3, -2)$. *Obj 3, p. 341*

Check your answers with those at the back of the book.

Graphing Quadratic Relations

OBJECTIVES for Sections 10-3 through 10-7:

1. ***Sketch graphs for second-degree sentences in two variables.***
2. ***Write equations for circles, parabolas, ellipses, and hyperbolas, given appropriate properties of these curves.***
3. ***Find a set of values in an inverse variation, given appropriate information.***
4. ***Apply inverse variations to solve simple word problems.***

10-3 Circles

Can you find an equation for the circle with center at (3, 2) and radius 5? You can if you recall that in a plane, a *circle* is the set of all points at a given distance, the *radius*, from a given point, called the *center* of the circle.

By the distance formula, for each point (x, y) on the circle (Figure 4), you have

$$\sqrt{(x-3)^2+(y-2)^2}=5, \qquad (1)$$

or

$$(x-3)^2+(y-2)^2=25. \qquad (2)$$

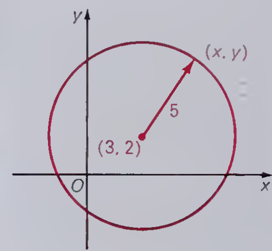


Figure 4

Conversely, if (*2*) is satisfied then so is (*1*), and the distance from (3, 2) to (x, y) is 5. Hence (*2*) is an equation of the circle.

In general, you have:

> An equation of the circle with center (h, k) and radius $r (r > 0)$ is
>
> $$(x-h)^2+(y-k)^2=r^2.$$
>
> If the center is the origin, then an equation of the circle is
>
> $$x^2+y^2=r^2.$$

Notice that $(x-3)^2+(y-2)^2=25$ is equivalent to

$$x^2+y^2-6x-4y-12=0.$$

This is an example of the fact that $(x-h)^2+(y-k)^2=r^2$ is equivalent to an equation of the form

$$x^2+y^2+ax+by+c=0,$$

where a, b, and c are real-number constants. If you are given an equation of a circle in the latter form, you can transform it to an equivalent equation in the form $(x-h)^2+(y-k)^2=r^2$ by completing the square (page 286) twice, once for x and once for y.

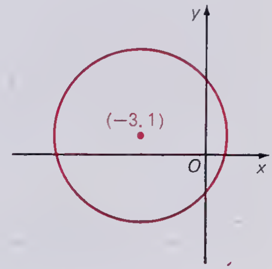
EXAMPLE Sketch the graph of $\{(x, y)\colon x^2 + y^2 + 6x - 2y - 6 = 0\}$.

SOLUTION Add 6 to each member and group the terms involving x and y.

$$(x^2 + 6x \qquad) + (y^2 - 2y \qquad) = 6$$

Complete the square in x by adding $(\frac{6}{2})^2 = 9$ to each member and the square in y by adding $(-\frac{2}{2})^2 = 1$ to each member.

$$\underbrace{(x^2 + 6x + 9)}_{(x+3)^2} + \underbrace{(y^2 - 2y + 1)}_{(y-1)^2} = \underbrace{6 + 9 + 1}_{16}$$



From the resulting equation, it is evident by inspection that the graph is a circle with center $(-3, 1)$ and radius 4, as shown at the right. Answer.

Oral Exercises

Find the radius and the center of a circle with the given equation.

1. $(x - 2)^2 + (y - 3)^2 = 7^2$
2. $(x - 5)^2 + (y - 3)^2 = 16$
3. $x^2 + (y - 8)^2 = 81$
4. $x^2 + (y + 2)^2 = 8$
5. $x^2 + y^2 = 49$
6. $(x + 4)^2 + (y + 7)^2 = 2$

Written Exercises

Find an equation of the form $x^2 + y^2 + ax + by + c = 0$ for the circle with the given center and radius.

A

1. $(0, 0)$; 7
2. $(3, 1)$; 2
3. $(-2, 5)$; 4
4. $(-3, -1)$; 5
5. $(\frac{1}{2}, -\frac{3}{2})$; 2
6. $(-\frac{1}{2}, 1)$; $\frac{3}{2}$
7. (a, b); a
8. $(-a, -b)$; $a + b$

Write the given equation in the form $(x - h)^2 + (y - k)^2 = r^2$, give the center and radius of the circle defined by the equation, and sketch its graph.

9. $2x^2 + 2y^2 = 18$
10. $x^2 + y^2 - 4x = 0$
11. $x^2 + y^2 + 6y = 0$
12. $x^2 + y^2 - 6x - 8y = 0$
13. $x^2 + y^2 + 2x - 4y - 4 = 0$
14. $x^2 + y^2 + 6x + 2y + 6 = 0$
15. $2x^2 + 2y^2 + 2x - 6y - 3 = 0$
16. $4x^2 + 4y^2 - 16x - 12y + 21 = 0$

Find an equation of the form $x^2 + y^2 + ax + by + c = 0$ for the circle with center C and passing through the point P.

B

17. $C(2, -1)$; $P(5, 3)$
18. $C(-2, 3)$; $P(-2, 7)$
19. $C(k, 0)$; $P(2k, 0)$
20. $C(r, s)$; $P(0, 0)$

Find an equation for a circle with a diameter having the given endpoints. (*Hint:* The midpoint of a diameter is the center of any circle.)

21. $(3, 5), (3, 1)$ **22.** $(-2, 2), (4, 2)$ **23.** $(-2, 1), (6, 7)$ **24.** $(-3, 2), (1, -6)$

C **25.** Find an equation for the circle with center at $(3, -2)$ and tangent to the y-axis. (*Hint:* The tangent line to a circle is perpendicular to the radius drawn to the point of tangency.)

26. A circle with center at the origin passes through $(-3, 4)$. Find an equation of the tangent line at this point.

27. The circle defined by $x^2 + y^2 + 4x - 21 = 0$ passes through $(1, 4)$ and $(3, 0)$.

a. Prove that the chord with the given endpoints is perpendicular to the radius that passes through the midpoint of the chord.

b. Prove that the radius perpendicular to the chord passes through its midpoint.

28. Find an equation of the circle passing through $(4, 0)$, $(-4, 0)$ and $(0, 8)$, using the general equation $x^2 + y^2 + ax + by + c = 0$.

29. Show that the set of all points (x, y) whose distance from the point $(8, 0)$ is twice as great as their distance from $(2, 0)$ is a circle, and give the center and radius of the circle.

10-4 Parabolas

In the preceding section, you saw that an equation can be found for the set of points called a *circle*. Equations can be found for other sets of points. Consider, for example, the set consisting of every point P whose distance from a fixed point, called the **focus,** is equal to the perpendicular distance from P to a line, called the **directrix,** that does not contain the focus. A curve consisting of a set of points satisfying these conditions is called a **parabola.**

To find an equation for the parabola with focus $F(0, 4)$ and directrix the line L with equation $y = -2$ (Figure 5), you have

$$d(F, P) = \sqrt{(x - 0)^2 + (y - 4)^2},$$
$$d(P, L) = |y - (-2)|,$$

from which:

$$\begin{aligned}
|y + 2| &= \sqrt{x^2 + (y - 4)^2} \\
|y + 2|^2 &= \left(\sqrt{x^2 + (y - 4)^2}\right)^2 \\
(y + 2)^2 &= x^2 + (y - 4)^2 \\
y^2 + 4y + 4 &= x^2 + y^2 - 8y + 16 \\
12y &= x^2 + 12 \\
y &= \tfrac{1}{12}x^2 + 1
\end{aligned}$$

Thus, the set of points described above is the graph of the quadratic function

$$\{(x, y)\colon y = \tfrac{1}{12}x^2 + 1\}.$$

Using the methods of Section 9-6, you can plot its graph as pictured in Figure 5.

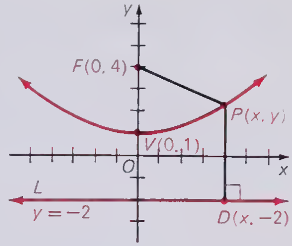


Figure 5

x	y
−4	$2\frac{1}{3}$
−2	$1\frac{1}{3}$
0	1
2	$1\frac{1}{3}$
4	$2\frac{1}{3}$

The table at the right suggests that the graph is symmetric with respect to the line with equation $x = 0$, that is, the y-axis. In fact, since $(-r, t)$ satisfies the equation of the function whenever (r, t) does (recall page 256), the y-axis is the **axis of symmetry,** or simply the **axis,** of this parabola. The point $\boldsymbol{V(0, 1)}$, where the parabola intersects the axis is called the **vertex** of the parabola.

Similarly, you can show that an equation for the parabola with focus $F(3, 2)$ and directrix the line L with equation $x = -1$ is

$$x = \tfrac{1}{8}(y - 2)^2 + 1,$$

whose graph is shown in Figure 6. The vertex of this parabola is the point $\boldsymbol{V(1, 2)}$, and the axis is the line with equation $\boldsymbol{y = 2}$, as suggested by the table below Figure 6.

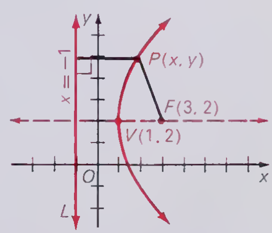


Figure 6

x	y
3	6
$1\frac{1}{2}$	4
1	2
$1\frac{1}{2}$	0
3	−2

From the definition of a parabola and Figures 5 and 6, can you explain why *the vertex is the midpoint of the segment of the axis between the focus and the directrix?*

The parabola with vertex $V(h, k)$ and the line L with equation $y = k - m$, $m \neq 0$, as the directrix is the graph of the relation

$$y - k = \frac{1}{4m}(x - h)^2.$$

Similarly, the parabola with vertex $V(h, k)$ and the line L with equation $x = h - m$, $m \neq 0$, as directrix is the graph of the relation

$$x - h = \frac{1}{4m}(y - k)^2.$$

(See Exercises 21–23 on page 353.)

In general, a parabola whose equation is of the form

$$y = a(x - h)^2 + k \qquad \text{or} \qquad x = a(y - k)^2 + h$$

has vertex $\boldsymbol{V(h, k)}$ and axis of symmetry

$$x = h \qquad \text{or} \qquad y = k,$$

respectively. If $a > 0$, the graph opens upward or to the right; if $a < 0$, the graph opens downward or to the left.

If you are given an equation of the form

$$y = ax^2 + bx + c \qquad \text{or} \qquad x = ay^2 + by + c,$$

you can sketch the curve more readily by completing the square in the squared variable and comparing the resulting equation with the information above.

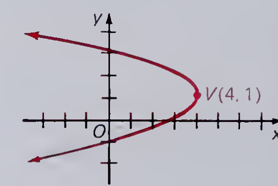
EXAMPLE Sketch the graph of $\{(x, y): x - 3 = 2y - y^2\}$.

SOLUTION Complete the square in y:

$$\begin{aligned} -x + 3 &= y^2 - 2y \\ -x + 3 + 1 &= y^2 - 2y + 1 \\ -x + 4 &= (y - 1)^2 \\ x &= -(y - 1)^2 + 4 \end{aligned}$$

x	y
0	-1
3	0
4	1
3	2
0	3

Comparing this with relations on page 351, you see that the vertex is $V(4, 1)$, and the axis is the graph of $y = 1$. Since $a = -1$, and $-1 < 0$, the equation has a graph that is a parabola opening to the left.



Oral Exercises

Find the vertex and the axis of symmetry of the given equation.

1. $y = -2(x - 3)^2 + 3$
2. $y = 7(x + 4)^2 + 4$
3. $y = 3x^2 - 2$
4. $x = 4(y - 1)^2 + 2$
5. $x = -3y^2 - 8$
6. $x = 5(y + 3)^2 + 7$

Written Exercises

Rewrite each equation in one of the standard forms $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$, and sketch its graph.

A

1. $y = \frac{1}{4}(x^2 - 4)$
2. $x = 2(2 - y^2)$
3. $x = y^2 - 6y$
4. $y = -2x^2 - 2x$
5. $y = \frac{1}{2}x^2 - 3x$
6. $x = -\frac{1}{4}y^2 + 2y$
7. $y = 2x^2 - 12x + 14$
8. $x = 3y^2 + 6y - 2$
9. $y = -\frac{1}{2}x^2 + 4x - 9$
10. $4x = y^2 - 6y + 1$

B

11. $x = 2 + \sqrt{y - 3}$
12. $y = -3 + \sqrt{x + 1}$

(*Hint:* In Exercises 11–12, you must restrict one of the variables.)

Find an equation of the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ for the parabola with focus F and directrix having the given equation.

13. $F(0, 3)$; $y = -1$
14. $F(2, 4)$; $y = 2$
15. $F(2, 3)$; $x = -2$
16. $F(-5, 2)$; $x = -1$
17. $F(2, \frac{3}{2})$; $y = \frac{1}{2}$
18. $F(\frac{3}{4}, 3)$; $x = \frac{5}{4}$

C **19.** Find an equation of the parabola with vertex $(\frac{1}{2}, -\frac{1}{4})$ and directrix having the equation $y = -\frac{1}{2}$.

20. Find an equation of the parabola with vertex $(\frac{5}{4}, -\frac{3}{2})$ and focus $(1, -\frac{3}{2})$.

21. Verify that the parabola with focus $F(0, m)$ and directrix with equation $y = -m$ is the graph of $y = \frac{1}{4m}x^2$.

22. Verify that the parabola with focus $F(m, k)$ and directrix with equation $x = -m$ is the graph of $x = \frac{1}{4m}(y - k)^2$.

23. Verify that the parabola with vertex (h, k) and directrix with equation $y = k - m$ is the graph of $y = \frac{1}{4m}(x - h)^2 + k$. (This shows that if m is the distance between the focus and the vertex or between the vertex and the directrix of a parabola, then $a = \frac{1}{4m}$.)

24. Show that the parabola with focus $(h, k + m)$ and directrix with equation $y = k - m$ passes through the points $(h + 2m, k + m)$ and $(h - 2m, k + m)$, and that the line segment joining these points has length $4|m|$.

10-5 Ellipses

The path traversed by a planet as it revolves about the sun is a plane curve called an *ellipse,* which is defined as follows: In the plane, the set of points for each of which the sum of the distances from two fixed points is a given constant is an **ellipse.** Each of the fixed points is a **focus** (plural: **foci**) of the ellipse, and the distances from these points to a point P on the curve are called **focal radii** of P (Figure 7). The point C bisecting $\overline{F_1F_2}$ is called the **center** of the ellipse. This definition suggests that a sketch of an ellipse may be made by fastening a piece of string at the points F_1 and F_2, stretching it taut with a pencil at point P, and drawing the curve.

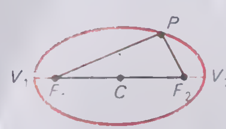


Figure 7

To find an equation for an ellipse with foci $F_1(-4, 0)$ and $F_2(4, 0)$ and $d(P, F_1) + d(P, F_2) = 10$ (Figure 8 on page 354), you have:

$$d(P, F_1) = \sqrt{[x - (-4)]^2 + (y - 0)^2} = \sqrt{(x + 4)^2 + y^2}$$
$$d(P, F_2) = \sqrt{(x - 4)^2 + (y - 0)^2} = \sqrt{(x - 4)^2 + y^2}$$

Since $d(P, F_1) + d(P, F_2) = 10$, it follows that

$$\sqrt{(x + 4)^2 + y^2} + \sqrt{(x - 4)^2 + y^2} = 10,$$

or

$$\sqrt{(x + 4)^2 + y^2} = 10 - \sqrt{(x - 4)^2 + y^2}.$$

Squaring each member and simplifying, you obtain

$$(x + 4)^2 + y^2 = 100 - 20\sqrt{(x - 4)^2 + y^2} + (x - 4)^2 + y^2,$$
$$4x - 25 = -5\sqrt{(x - 4)^2 + y^2}.$$

Again squaring and simplifying, you find that:

$$16x^2 - 200x + 625 = 25[(x - 4)^2 + y^2]$$
$$9x^2 + 25y^2 = 225$$
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

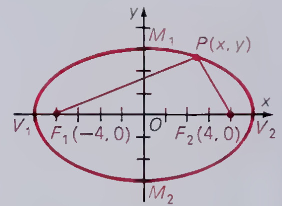


Figure 8

Thus, the ellipse described is the graph of the relation

$$\left\{(x, y)\colon \frac{x^2}{25} + \frac{y^2}{9} = 1\right\}.$$

You can verify that the equation is satisfied by the coordinates of these points shown in Figure 8: $V_1(-5, 0)$, $V_2(5, 0)$, $M_1(0, 3)$, $M_2(0, -3)$. Thus,

the x-intercepts are -5 *and* 5;
the y-intercepts are 3 *and* -3.

Also, since $(r, -t)$ and $(-r, t)$ satisfy the equation whenever (r, t) does,

the curve is symmetric with respect to both coordinate axes.

The graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ can be sketched by plotting the intercepts and a few additional points obtained from $y = \pm\frac{3}{5}\sqrt{25 - x^2}$, as shown in the adjoining table. The graph is pictured in Figure 8.

x	y
3	$\pm\frac{12}{5}$
4	$\pm\frac{9}{5}$
-3	$\pm\frac{12}{5}$
-4	$\pm\frac{9}{5}$

Do you see that the curve contains only points for which $|x| \leq 5$ and $|y| \leq 3$?

In general, you can show (Exercise 21, page 356) that:

> The ellipse with center at the origin, foci $(c, 0)$ and $(-c, 0)$, and the sum of the focal radii for each of its points the constant $2a$, where $a > c$, is the graph of the relation
>
> $$\left\{(x, y)\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \qquad (1)$$
>
> where $b^2 = a^2 - c^2$. It has x-intercepts a and $-a$ and y-intercepts b and $-b$.

If the foci are on the y-axis, you can verify (Exercise 22, page 356) the following.

The ellipse with center at the origin, foci $(0, c)$ and $(0, -c)$, and the sum of the focal radii for each of its points the constant $2a$, where $a > c$, is the graph of the relation

$$\left\{(x, y): \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\right\}, \qquad (2)$$

where again $b^2 = a^2 - c^2$. Here the x-intercepts are b and $-b$, and the y-intercepts are a and $-a$.

In each case, the ellipse is symmetric with respect to both the x-axis and the y-axis. Notice also that $a > b$.

As you can verify, for an ellipse with an equation of either form (*1*) or form (2) above, $2a$ and $2b$ represent the distances between intercepts. The segments of length $2a$ and $2b$ cut off on the axes by the ellipse are called the **major axis** and **minor axis** of the ellipse, respectively. In Figure 8, the major axis is $\overline{V_1V_2}$, and the minor axis is $\overline{M_1M_2}$.

EXAMPLE Sketch the graph of $\{(x, y): x^2 + 9y^2 = 36\}$.

SOLUTION Divide both members of $x^2 + 9y^2 = 36$ by 36 to obtain

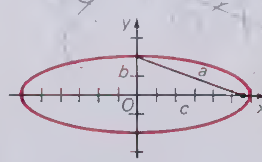
$$\frac{x^2}{36} + \frac{y^2}{4} = 1.$$

Since $36 > 4$, this is of the form (1). Then, by inspection, the graph is an ellipse, and:

1. It is symmetric with respect to both axes.
2. The x-intercepts are 6 and -6.
3. The y-intercepts are 2 and -2.

x	y
0	2
2	$\frac{4\sqrt{2}}{3}$
4	$\frac{2\sqrt{5}}{3}$
6	0

From $y = \frac{1}{3}\sqrt{36 - x^2}$, you can find the first-quadrant points shown in the table. By symmetry, you can find corresponding points in the other quadrants and sketch the graph as shown.



In the preceding example, the foci are on the x-axis. Since you know that $c^2 = a^2 - b^2 = 36 - 4 = 32$, it follows that $c = \sqrt{32} = 4\sqrt{2}$, and the foci are $(4\sqrt{2}, 0)$ and $(-4\sqrt{2}, 0)$. The foci are not used in sketching the graph. A right triangle with sides a, b, and c is shown in the figure for the example above.

Oral Exercises

Find the coordinates of the foci of the ellipse whose equation is given.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
3. $x^2 + \frac{y^2}{4} = 1$
4. $\frac{x^2}{9} + y^2 = 1$
5. $\frac{x^2}{25} + \frac{y^2}{169} = 1$
6. $\frac{x^2}{169} + \frac{y^2}{144} = 1$

Written Exercises

Graph each relation and give the coordinates of the foci.

A

1–6. Use the relations in Oral Exercises 1–6.

7. $25x^2 + 4y^2 = 100$
8. $9x^2 + y^2 = 36$
9. $x^2 + 9y^2 = 225$
10. $9x^2 + 16y^2 = 144$
11. $25x^2 + 16y^2 = 100$
12. $25x^2 + 9y^2 = 25$

In Exercises 13–18, find an equation for the ellipse with axes on the coordinate axes and with the given characteristics.

EXAMPLE Major axis of length 10; foci at (3, 0) and (−3, 0).

SOLUTION Since the major axis has length $2a$, you have $2a = 10$, or $a = 5$. Since one focus is at (3, 0), you have $c = 3$. Then $b^2 = a^2 - c^2$; so $b^2 = 25 - 9$, and $b = 4$.

$\therefore \frac{x^2}{25} + \frac{y^2}{16} = 1.$ Answer.

B

13. Foci at (8, 0) and (−8, 0); y-intercepts 6 and −6.
14. Major axis of length 22; x-intercepts 7 and −7.
15. Minor axis of length 8; foci at (0, 3) and (0, −3).
16. Sum of focal radii 16; y-intercepts 5 and 5.
17. Sum of focal radii 12; foci at $(0, \sqrt{11})$ and $(0, -\sqrt{11})$.
18. Major axis of length 20; foci at $(5\sqrt{3}, 0)$ and $(-5\sqrt{3}, 0)$.

C

19. Find the equation of the ellipse with foci (−4, 0) and (4, 0) that passes through the point $(3, \sqrt{15})$. (*Hint:* Find the sum of the focal radii.)
20. Consider the set of all points (x, y) whose distance from the line $y = 8$ is twice their distance from the point (0, 2). Show that this set is an ellipse by finding a relation expressed in the standard form for an ellipse that defines this set.
21. Use the method shown on pages 353 and 354 to derive an equation of the ellipse with foci $(c, 0)$ and $(-c, 0)$ and sum of focal radii $2a$.
22. Repeat Exercise 21 for the ellipse with foci $(0, c)$ and $(0, -c)$ with sum of focal radii $2a$.

Careers
in Architecture

Almost every modern building is planned and designed by architects. In designing a structure, architects must make sure that, in addition to meeting their client's requirements, the structure will be safe and attractive. The architects first make preliminary drawings of the floor plan and the exterior and interior details of the building. They then meet with their client to decide on a final design for the structure. This final design is used to prepare working drawings. Consulting engineers usually prepare detailed drawings of the plumbing, electrical connections, and climate-control systems. The architects advise and represent their client in dealing with the building contractor. They also visit the construction site periodically to see that the design and specifications are being followed. Architects are involved in all the stages of creating a structure, from the first sketches to the completed building.

The architect and engineer (above) work on the design of a new building. A model (below) shows a design concept in three dimensions.

EXAMPLE An architect is designing a house which will be 10 m long and 8 m wide. A scale drawing of the floor plan is 30 cm by 24 cm. In the drawing, find the dimensions representing a 4.2 m $\times$ 5.5 m living room.

SOLUTION 1 m = 100 cm, so 10 m = 1000 cm. Using the proportion

$$\frac{\text{actual size}}{\text{drawing size}} = \frac{1000}{30},$$

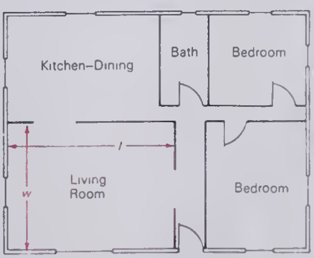
we have

$$\frac{420}{w} = \frac{1000}{30} \quad \text{and} \quad \frac{550}{l} = \frac{1000}{30}.$$

Solving:

$$w = \frac{420 \cdot 30}{1000} = 12.6 \text{ cm}$$

$$l = \frac{550 \cdot 30}{1000} = 16.5 \text{ cm}$$



10-6 Hyperbolas

Just as in Section 10-5 you used the sum of distances between points to define an ellipse, so do you use the *difference* of such distances to define another curve. Consider the set of points in the plane such that for each point, the absolute value of the difference of its distances, called the **focal radii,** from two fixed points, called the **foci,** is a constant. Such a set of points is a two-branched curve called a **hyperbola.**

To obtain an equation for the hyperbola with foci at $F_1(-5, 0)$ and $F_2(5, 0)$ and with focal radii differing by 8 (Figure 9), you can begin by expressing the fact that for any point $P(x, y)$ on the hyperbola, either

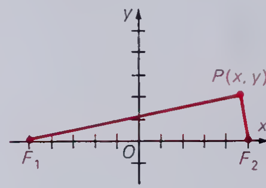


Figure 9

$$\sqrt{[x-(-5)]^2+y^2}-\sqrt{(x-5)^2+y^2}=8$$

or

$$\sqrt{[x-(-5)]^2+y^2}-\sqrt{(x-5)^2+y^2}=-8.$$

In either case, the squaring process you used to obtain the equation for an ellipse (page 353) can be applied to obtain

$$\frac{x^2}{16}-\frac{y^2}{9}=1.$$

By inspection, you can see that the graph of this equation:

(1) Is symmetric with respect to both axes.
(2) Has x-intercepts 4 and -4.

It has no y-intercepts, since if you solve the equation for y in terms of x, you obtain

$$y=\pm\tfrac{3}{4}\sqrt{x^2-16},$$

from which it is evident that the curve has no points for which $|x|<4$.

x	$\frac{3}{4}\sqrt{x^2-16}$	y
4	$\frac{3}{4}\sqrt{16-16}$	0
5	$\frac{3}{4}\sqrt{25-16}$	2.3
6	$\frac{3}{4}\sqrt{36-16}$	3.3

Using the foregoing facts, and constructing the brief table of first-quadrant values shown, you can sketch the hyperbola in Figure 10.

As you can see in Figure 10, the graph lies entirely within two of the regions determined by the diagonals of the rectangle that is bounded by segments of the lines with equations

$$x=-4, \qquad x=4, \qquad y=3, \qquad \text{and} \qquad y=-3.$$

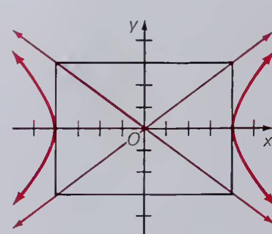


Figure 10

These diagonals are called **asymptotes** of the hyperbola and have equations

$$y=\tfrac{3}{4}x \qquad \text{and} \qquad y=-\tfrac{3}{4}x.$$

For a given value, $x_1>0$, of x, the difference between the ordinate of a

point on the asymptote in Quadrant I and the ordinate of the corresponding point on the curve is given by

$$\begin{aligned}\tfrac{3}{4}x_1 - \tfrac{3}{4}\sqrt{x_1^2 - 16} &= \frac{(\frac{3}{4}x_1 - \frac{3}{4}\sqrt{x_1^2 - 16})(\frac{3}{4}x_1 + \frac{3}{4}\sqrt{x_1^2 - 16})}{\frac{3}{4}x_1 + \frac{3}{4}\sqrt{x_1^2 - 16}} \\ &= \frac{\frac{9}{16}x_1^2 - \frac{9}{16}(x_1^2 - 16)}{\frac{3}{4}x_1 + \frac{3}{4}\sqrt{x_1^2 - 16}} \\ &= \frac{36}{3x_1 + 3\sqrt{x_1^2 - 16}}.\end{aligned}$$

As you can see, the greater x_1, the less the difference in ordinates, and for increasing values of x_1, the curve approaches closer and closer to the asymptote. By symmetry you can see that the corresponding situation holds in each of the remaining quadrants.

In general, you can show (Exercise 23, page 362) that:

> If $(-c, 0)$ and $(c, 0)$ are foci of the hyperbola for which the absolute value of the difference of the focal radii is the constant $2a > 0$, then the hyperbola is the graph of the relation
>
> $$\left\{(x, y): \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right\}, \qquad (1)$$
>
> where $b^2 = c^2 - a^2$. The equations for the asymptotes are
>
> $$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

Similarly, you can show (Exercise 24, page 362) that:

> If $(0, -c)$ and $(0, c)$ are foci of the hyperbola for which the absolute value of the difference of the focal radii is the constant $2a > 0$, then the hyperbola is the graph of the relation
>
> $$\left\{(x, y): \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1\right\}, \qquad (2)$$
>
> where $b^2 = c^2 - a^2$. The equations for the asymptotes are
>
> $$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$

EXAMPLE Sketch the graph of $\{(x, y): x^2 - 4y^2 = 36\}$.

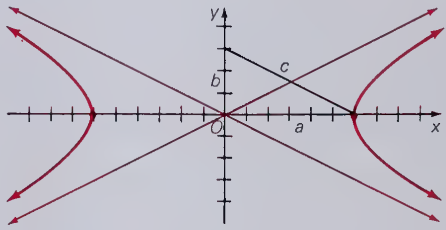
SOLUTION Divide each member by 36 to obtain

$$\frac{x^2}{36} - \frac{y^2}{9} = 1.$$

Comparing this with Equation (*1*) above, you see that $a = 6$ and $b = 3$. Then, by inspection, you can determine that:

(1) The graph is symmetric with respect to both axes.
(2) The x-intercepts are 6 and -6. There are no y-intercepts.
(3) The asymptotes are the graphs of $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$.

First sketching the asymptotes and identifying the intercepts, you have the graph shown at the right. A right triangle with sides measuring a, b, and c is also shown in the figure.



For hyperbolas with equations of the form (*1*) or (*2*) above, the origin is called the **center** of the hyperbola, the points of intersection of the branches of the curve with a coordinate axis are the **vertices,** the segment of length $2a$ of a coordinate axis between the intercepts is the **transverse axis,** and the segment of length $2b$ of the remaining coordinate axis with each endpoint a distance b from the origin is the **conjugate axis** (Figure 11).

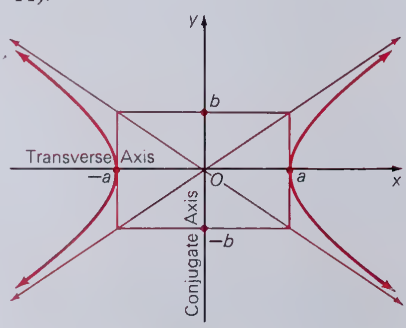


Figure 11

The circle, ellipse, parabola, and hyperbola are called **conic sections** because each can be formed as the intersection of a plane with a *conical surface of two nappes* (see Figure 12). (It is assumed that such a conical surface extends indefinitely.) A point, a line, and a pair of intersecting lines are sometimes called **degenerate conic sections.**

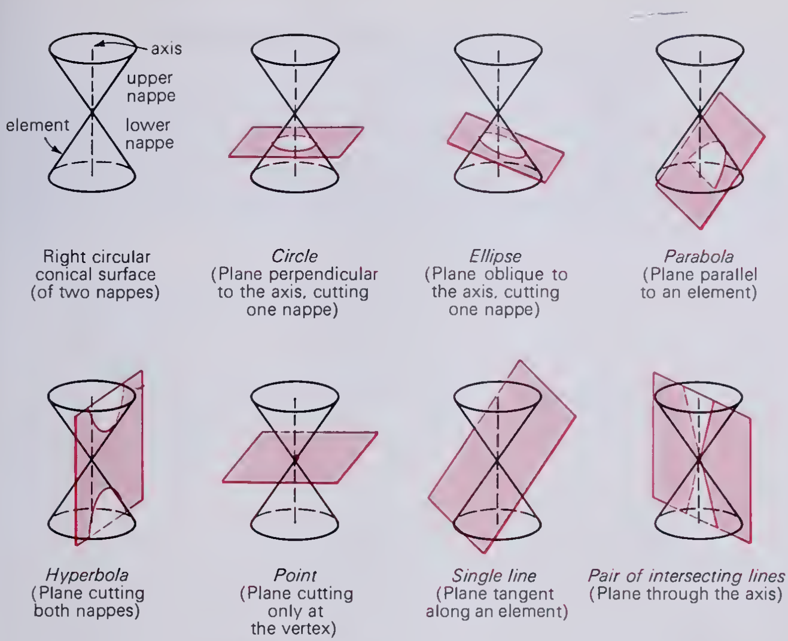


Figure 12

Oral Exercises

Give the equations of the asymptotes to the hyperbola whose equation is given.

1. $\frac{x^2}{64} - \frac{y^2}{36} = 1$
2. $\frac{y^2}{9} - \frac{x^2}{16} = 1$
3. $\frac{y^2}{16} - \frac{x^2}{4} = 1$
4. $\frac{x^2}{4} - y^2 = 1$
5. $\frac{y^2}{25} - \frac{x^2}{25} = 1$
6. $x^2 - \frac{y^2}{16} = 1$

Written Exercises

In Exercises 1–12,

a. draw the rectangle whose diagonals are the asymptotes of the hyperbola and draw the asymptotes themselves;
b. sketch the hyperbola; and
c. give the coordinates of the foci.

A **1–6.** Use the relations in Oral Exercises 1–6.

7. $3x^2 - 3y^2 = 48$
8. $4y^2 - 25x^2 = 100$
9. $16x^2 - 9y^2 = 9$
10. $16y^2 - 9x^2 = 324$
11. $5x^2 - 20y^2 = 100$
12. $3y^2 - 75x^2 = 225$

In Exercises 13–18, find an equation for the hyperbola with axes on the coordinate axes and with the given characteristics.

EXAMPLE Foci at (4, 0) and (−4, 0); absolute value of difference of focal radii 6.

SOLUTION Since the absolute value of the difference of the focal radii is $2a$, you have $2a = 6$, or $a = 3$. Since the foci are at (4, 0) and (−4, 0), $c = 4$. Then

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 4^2 - 3^2 \\ &= 16 - 9 = 7. \end{aligned}$$

$\therefore \frac{x^2}{9} - \frac{y^2}{7} = 1.$ Answer.

B 13. Foci at (0, 5) and (0, −5); one vertex at (0, 4).

14. Length of transverse axis 4; foci at (7, 0) and (−7, 0).

15. Conjugate axis of length 10; one y-intercept is 8.

16. Foci at (11, 0) and (−11, 0); absolute value of the difference of focal radii 14.

17. Endpoints of the conjugate axis $(4\sqrt{2}, 0)$ and $(-4\sqrt{2}, 0)$; foci at (0, 8) and (0, −8).

18. One vertex at (4, 0); conjugate axis of length $8\sqrt{3}$.

C 19. Find an equation of the hyperbola whose asymptotes are given by $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and with foci at (10, 0) and (−10, 0); use the following steps:
   a. Use the relationships given on page 359 to find two equations involving a and b.
   b. Solve these equations simultaneously by substitution for a^2 and b^2.

20. Find an equation of the hyperbola with asymptotes $y = 3x$ and $y = -3x$, and foci at (0, 6) and (0, −6).

21. Find an equation of the hyperbola with foci at (0, 4) and (0, −4) and passing through the point $(\sqrt{15}, 3)$.

22. Consider the set of points (x, y) whose distance from the point (8, 0) is twice their distance from the line $x = 2$. Show that this set is a hyperbola by writing the relation of the set in one of the standard forms on page 359.

23. Use the methods of the example in the text on page 359 to derive an equation for a hyperbola with foci $(c, 0)$ and $(-c, 0)$ and absolute value of difference of focal radii $2a > 0$ where $a < c$.

24. Repeat Exercise 23 for the hyperbola with foci $(0, c)$ and $(0, -c)$ and absolute value of the difference of focal radii $2a > 0$, where $a < c$.

10-7 Inverse and Other Kinds of Variation

When two pulleys are connected (Figure 13), the one with smaller diameter revolves more rapidly. If d_1 represents the diameter of one pulley and n_1 the number of revolutions per minute (r/min) of the pulley, and if d_2 and n_2 represent the corresponding numbers for the other pulley, then

$$d_1 n_1 = d_2 n_2, \qquad \text{or} \qquad \frac{d_1}{d_2} = \frac{n_2}{n_1}.$$

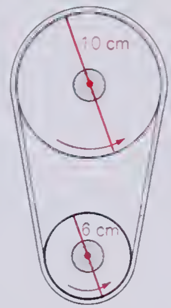


Figure 13

In the pulleys shown in Figure 13, $d_1 = 6$ cm and $d_2 = 10$ cm. If the smaller pulley revolves at 120 r/min, you can compute the revolutions per minute of the other pulley:

$$\frac{6}{10} = \frac{n_2}{120}$$
$$n_2 = 72$$

For this set of pulleys, $dn = 720$. Because this implies that

$$n = \frac{720}{d} \qquad \text{or} \qquad d = \frac{720}{n},$$

you say that n and d **vary inversely** as each other, or are **inversely proportional** to each other.

In general, any function defined by an equation of the form

$$xy = k,$$

where k is a nonzero constant, is called an **inverse variation,** and k is the **constant of variation.** The graph of such a function is of the form shown in Figure 14 if $k > 0$, and of the form shown in Figure 15 if $k < 0$. It can be shown that these graphs are hyperbolas, with foci on the lines with equations $y = x$ and $y = -x$, respectively, and with the coordinate axes as asymptotes. Ordinarily, in practical situations, you have $x > 0$, $y > 0$, and $k > 0$, and the graph is limited to one such as the first-quadrant branch of Figure 14.

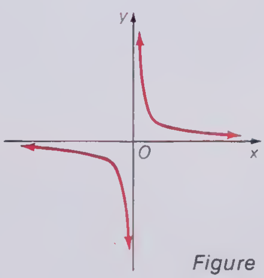


Figure 14

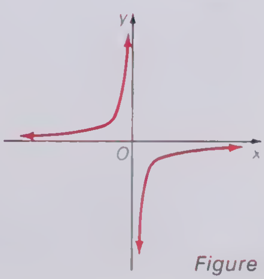


Figure 15

As with direct variation (recall page 89), there is an important relationship among the coordinates of two ordered pairs (x_1, y_1) and (x_2, y_2) of the inverse variation specified by the equation $xy = k$, $k \neq 0$. You have

$$x_1 y_1 = k \qquad \text{and} \qquad x_2 y_2 = k, \qquad \text{and so} \qquad x_1 y_1 = x_2 y_2.$$

Hence, $x_1 \neq 0$, $x_2 \neq 0$, $y_1 \neq 0$, and $y_2 \neq 0$, and you have

$$\frac{x_1}{x_2} = \frac{y_2}{y_1}.$$

We state this result as a theorem.

Theorem. For all real numbers x_1, y_1, x_2, y_2 ($x_1 \neq 0$, $x_2 \neq 0$, $y_1 \neq 0$, and $y_2 \neq 0$), if (x_1, y_1) and (x_2, y_2) are ordered pairs of an inverse variation, then

$$\frac{x_1}{x_2} = \frac{y_2}{y_1}.$$

Everyday examples of inverse variation are:

1. For a fixed distance, the greater the speed, the proportionately less the time needed to cover it (and vice versa): $rt = D$
2. For a stated income, the greater the rate of interest, the proportionately less the amount of principal needed: $pr = I$
3. For a given area, the greater the length of a rectangle, the proportionately less the width: $lw = A$
4. Boyle's Law in physics. If the temperature is kept constant, the greater the pressure on a gas, the proportionately less the volume: $pv = K$

In Section 8-1, the idea of direct variation was extended to include variation defined by $y = ax^2$, that is, "y varies directly as x^2." (The graph of that variation is a parabola.) Similarly, the idea of inverse variation is extended to include variation defined, for example, by

$$y = \frac{k}{x^2},$$

that is, **y varies inversely as x^2.** (The graph of such a variation is not a curve with which we are familiar.) Corresponding extensions involving other powers may also be made.

Still another form of variation is typified by the relationship between the electrical resistance of a wire and the length and diameter of the wire. The electrical resistance R of a wire varies *directly* as the length l of the wire and *inversely* as the square of its diameter d. An equation expressing this fact is

$$R = \frac{kl}{d^2}.$$

You call such a variation a **combined variation.** If z varies directly as x and also directly as y, then the equation relating these variables is of the form $z = kxy$, and you say z varies **jointly** as x and y.

EXAMPLE If 20 m of wire of diameter 1.5 mm has a resistance of 12 Ω, what is the resistance of 20 m of the same type of wire if the diameter is increased to 2 mm?

SOLUTION You have a combined variation $R = \frac{kl}{d^2}$.

Since $R = 12$ when $l = 20$ and $d = 1.5$,

$$12 = \frac{k(20)}{(1.5)^2}, \quad \text{or} \quad k = 1.35.$$

$$\therefore R = \frac{1.35l}{d^2}$$

Replacing l with 20 and d with 2, you have

$$R = \frac{1.35(20)}{(2)^2} = 6.75.$$

∴ the resistance is 6.75 Ω. Answer.

Oral Exercises

To describe the relation of the variable in the left member of the equation to the other variables, state whether the equation is a direct, inverse, combined, or joint variation.

1. $y = \frac{x}{7}$
2. $z = 4xy$
3. $y = \frac{7x}{z}$
4. $y = \frac{4}{x}$
5. $z = \frac{x}{8y}$
6. $y = \frac{1}{8x}$
7. $y = 2.5x$
8. $z = \frac{xy}{10}$

Written Exercises

Give an equation in x and y that defines the given variation and contains the given ordered pair.

A

1. y varies inversely as x; $(3, 6)$
2. y varies inversely as x; $(4, -\frac{1}{12})$
3. y varies inversely as x^2; $(\frac{3}{5}, 5)$
4. y varies inversely as x^3; $(-\frac{3}{2}, 4)$
5. y varies inversely as $\sqrt{x}$; $(9, -\frac{5}{4})$
6. y varies inversely as $\sqrt[3]{x}$; $(0.008, 750)$

For the given variation determine x_2 or y_2.
In Exercises 7–9, y varies inversely as x:

7. $(\frac{1}{2}, 6)$, $(x_2, 2)$
8. $(-3, 5)$, $(2, y_2)$
9. $(\frac{3}{5}, 25)$, $(90, y_2)$

In Exercises 10–12, y varies inversely as x^2:

10. $(3, 4)$, $(x_2, 9)$
11. $(-\frac{3}{2}, 8)$, $(9, y_2)$
12. $(2\sqrt{2}, \frac{1}{3})$, $(x_2, 6)$

13. If z varies jointly as x and y, and $z = 6$ when $x = 4$ and $y = 3$, find z when $x = 5$ and $y = 5$.

14. If z varies jointly as x and y, and $z = \frac{3}{2}$ when $x = 8$ and $y = \frac{1}{3}$, find z when $x = 4$ and $y = 12$.

B 15. If z varies directly as x and inversely as y, and $z = 10$ when $x = 3$ and $y = 4$, find z when $x = 12$ and $y = 25$.

16. If z varies directly as x and inversely as the cube of y, and $z = 4$ when $x = 3$ and $y = \frac{1}{2}$, find z when $x = 2$ and $y = \frac{1}{3}$.

17. In a certain variation z varies jointly as x and y and inversely as w, and $z = 2$ when $x = 3$, $y = 4$, and $w = 8$. Find z when $x = \frac{1}{2}$, $y = 3$, and $w = 5$.

18. In a certain variation, z varies jointly as x and y and inversely as w^2, and $z = 300$ when $x = 60$, $y = 20$, and $w = 4$. Find z when $x = 80$, $y = 30$, and $w = 10$.

19. Sketch the graph of $y = \dfrac{12}{x^2}$.

20. Sketch the graph of $y = \dfrac{-6}{x^2}$.

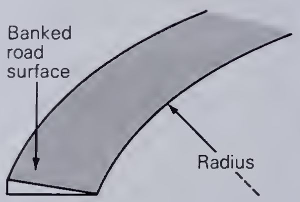
C 21. Use the definition on page 358 to find an equation for the hyperbola with foci at $(2, 2)$ and $(-2, -2)$ and with difference of focal radii 4; draw its graph.

22. Find an equation of the hyperbola that is identical to the graph in Exercise 21, but rotated 45° clockwise. That is, its asymptotes are the lines $y = x$ and $y = -x$.

Problems

A 1. The minimum voltage needed to produce penetrating x rays varies inversely with the wavelength of the x rays. (The wavelength of x rays may be expressed in picometers (pm); 1 pm $= 10^{-12}$ m.) If 62,000 V is needed to produce x rays of wavelength 20 pm, what voltage is needed to produce x rays of wavelength 80 pm?

2. The slope at which a curve in a road should be banked varies inversely with the radius of the curve and directly with the square of the maximum speed of the cars that will use it. If a curve of radius 1250 m carrying cars traveling at a maximum speed of 24.5 m/s has a slope of 0.049, what slope should the road have on a curve of radius 700 m if the maximum speed is 19.6 m/s?



3. The change in the length of a steel bar under fixed load is directly proportional to the length of the bar and inversely proportional to its cross-sectional area. If a bar 40 m long with a cross-sectional area of 70 cm² stretches by 0.028 cm, how much will a bar 100 m long with an area of 490 cm² stretch?

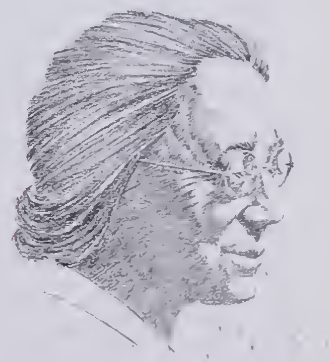
4. The speed of a satellite orbiting the earth is inversely proportional to the square root of its distance from the earth's center. If an artificial satellite 8000 km from the earth's center travels at $\sqrt{5} \times 10^5$ m/s, what is the speed of the moon, which is 400,000 km from the center of the earth?

5. The power produced by an electric circuit varies directly as the square of the voltage and inversely as the resistance. If a voltage of 110 V in a circuit with a resistance of 20 Ω produces 605 W, what voltage applied to a circuit with a resistance of 30 Ω will produce 480 W?

B 6. The velocity of sound in a diatomic gas at 0°C varies inversely as the square root of the density. If sound travels at 3416 m/s in oxygen, which is 16 times as dense as hydrogen, what is the speed of sound in hydrogen?

7. Newton's law of gravitation states that two objects attract each other with a force that varies jointly with the masses of the two objects and inversely with the square of the distance between them. If two 30 kg masses 1 m apart are found to attract each other with a force of 6×10^{-8} N, and a 1 kg mass on the surface of the earth (6×10^6 m from its center of gravity) is attracted by the earth with a force of 9.8 N, compute the mass of the earth.

Emmy Noether
1882–1935



Emmy Noether was a noted German mathematician who had a great influence on the development of modern algebra. From 1922 to 1933 she taught mathematics at the University of Göttingen. As a woman, she was able to secure a professorship only through the enthusiastic support of the university's mathematics department.

Her research centered on noncommutative algebras and their transformations. Noether published thirty-seven papers in all, yet her impact extended far beyond her own work. Her perceptive insight, advice, and encouragement affected the research of her associates and students.

In 1933, when the Nazis assumed power, she was ordered to leave the university, as were many other accomplished professors of Jewish background. She then came to the United States and taught at Bryn Mawr College and at the Institute of Advanced Study in Princeton.

After Emmy Noether's death, Albert Einstein was quoted as saying, "In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began."

Self-Test 2

VOCABULARY

focus (p. 350)	hyperbola (p.358)
directrix (p. 350)	asymptote (p. 358)
parabola (p. 350)	transverse axis (p. 360)
axis of symmetry (p. 351)	conjugate axis (p. 360)
vertex (p. 351)	conic section (p. 360)
ellipse (p. 353)	inverse variation (p. 363)
focal radii (p. 353)	constant of variation (p. 363)
major axis (p. 355)	combined variation (p. 364)
minor axis (p. 355)	

Sketch the graph of each relation.

1. $2x = y^2 - 2y - 1$ 2. $4x^2 - 9y^2 = 36$ *Obj. 1, p. 348*
3. Find an equation of the form $x^2 + y^2 + ax + by + c = 0$ for the circle with center $(3, -1)$ and radius 4.   *Obj. 2, p. 348*
4. Find an equation of the set of points (x, y) the sum of whose distances from $(0, 5)$ and $(0, -5)$ is 26.
5. If z varies directly as x^2 and inversely as y, and $z = \frac{3}{4}$ when $x = 3$ and $y = 2$, find z when $x = 12$ and $y = \frac{1}{2}$.   *Obj. 3, p. 348*
6. The illuminance expressed in lux (lx), of an unshaded electric bulb directly above a flat surface is inversely proportional to the square of the distance to the surface. If the illuminance is 4.5 lx when the bulb is 4 m from the surface, what is the illuminance when the bulb is 3 m from the surface?   *Obj. 4, p. 348*

Check your answers with those at the back of the book.

Solving Quadratic Systems

OBJECTIVES for Sections 10-8 through 10-10:
1. Solve simple quadratic systems graphically.
2. Solve simple quadratic systems by substitution.

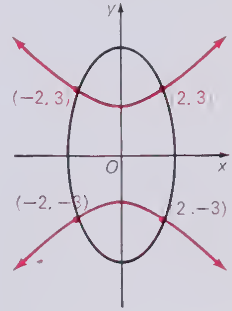
10-8 Graphic Solutions of Systems

Graphical methods can be used to determine or to estimate real-number solutions of systems of equations in two variables in which one or both of the equations are quadratic.

EXAMPLE Find the solution set of the system over $\Re$:

$$4x^2 + y^2 = 25$$
$$x^2 - y^2 = -5$$

SOLUTION Graph both equations on the same coordinate plane and determine points of intersection.



$\{(2, 3), (-2, 3), (-2, -3), (2, -3)\}$. Answer.

Check: $4(2)^2 + (3)^2 \stackrel{?}{=} 25$
$16 + 9 = 25$
$(2)^2 - (3)^2 \stackrel{?}{=} -5$
$4 - 9 = -5$

Clearly, because $(-2)^2 = 2^2$ and $(-3)^2 = 3^2$, all values will check in a similar way.

In solving graphically systems containing one or more quadratic equations, you will discover that:

(1) A system consisting of a linear equation and a quadratic equation may have no real solutions or as many as two.
(2) A system consisting of two quadratic equations may have no real solutions or as many as four.

Oral Exercises

Identify the graphs of the equations.

EXAMPLE $y = 2x^2 + 8$, $y = -7x$ **SOLUTION** A parabola and a line.

1. $y = x^2 - 4$
   $y = 3x$
2. $y = 4x - x^2$ PARABOLA
   $y = 2x - 3$ LINE
3. $x = y^2 - 2y + 1$
   $x + y = 7$
4. $x = 1 - y^2$
   $x = 2y + 2$
5. $9x^2 + 16y^2 = 100$ ELIPSE
   $x^2 + y^2 = 8$ CIRCLE
6. $x^2 + 4y^2 = 25$
   $4x^2 + y^2 = 25$

Written Exercises

1-9 odd

Solve each system over $\Re$ by graphing. Then check the coordinates of each graphical solution by substituting in *both* equations.

A

1–6. Use Oral Exercises 1–6.

7. $x^2 + 4y^2 = 16$
   $4x^2 - y^2 = 64$
8. $y = x^2 - 5$
   $x^2 + y^2 = 25$
9. $x^2 + y^2 = 169$
   $x^2 - 13 = y$

B

10. $x^2 - 4x + y^2 - 2y - 5 = 0$
    $x^2 + y^2 = 25$
11. $x^2 - y^2 = 16$
    $3y = x^2 - 16$
12. $3x^2 - 4y^2 = 12$
    $3x = -2y^2 + 6$

Estimate the solution(s) of each system over $\mathcal{R}$ to the nearest $\frac{1}{2}$ unit by graphing. Check that each solution *approximately* satisfies the two given equations.

C 13. $2x^2 + 2y^2 = 9$
$8x^2 + 4y^2 = 27$

14. $x^2 + 4y^2 = 16$
$x^2 - 8y^2 = 4$

15. $4x = y^2 - 8$
$4x^2 - y^2 = 16$

10-9 Linear-Quadratic Systems: Substitution

If a system of equations involves a linear equation and a quadratic equation, you can solve the system by substitution. In this process the quadratic equation is replaced by one involving a single variable.

EXAMPLE 1 Find the solution set of the system:

$$x^2 + 9y^2 = 37$$
$$x - 2y = -3$$

SOLUTION

1. Transform the *linear equation* to express x in terms of y.

$$x - 2y = -3$$
$$x = 2y - 3$$

2. Replace the given quadratic equation with the equation obtained from it by replacing x with "$\mathbf{2y - 3}$."

$$x^2 + 9y^2 = 37$$
$$(2y - 3)^2 + 9y^2 = 37$$
$$4y^2 - 12y + 9 + 9y^2 = 37$$

3. Solve the new quadratic equation.

$$13y^2 - 12y - 28 = 0$$
$$(13y + 14)(y - 2) = 0$$
$$y = -\tfrac{14}{13},\ y = 2$$

4. Solve two linear systems:

$$x = 2y - 3 \qquad\qquad x = 2y - 3$$
$$y = -\tfrac{14}{13} \qquad\qquad y = 2$$

$$x = 2(-\tfrac{14}{13}) - 3 \qquad x = 2(2) - 3$$
$$x = -\tfrac{67}{13} \qquad\qquad x = 1$$

Thus, the solutions are $(-\frac{67}{13}, -\frac{14}{13})$ and $(1, 2)$.

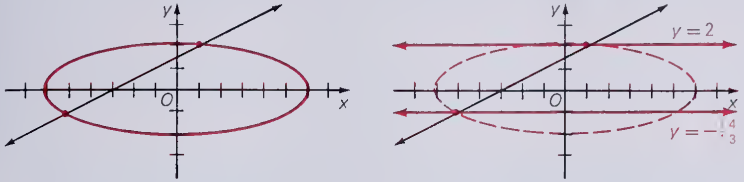
5. Below we check each ordered pair in the first equation. (Checking the second is left to you.)

$$(-\tfrac{67}{13})^2 + 9(-\tfrac{14}{13})^2 \stackrel{?}{=} 37 \qquad (1)^2 + 9(2)^2 \stackrel{?}{=} 37$$
$$\tfrac{4489}{169} + \tfrac{1764}{169} \stackrel{?}{=} 37 \qquad 1 + 36 \stackrel{?}{=} 37$$
$$\tfrac{6253}{169} \stackrel{?}{=} 37 \qquad 37 = 37$$
$$37 = 37$$

$\therefore$ the solution set is $\{(-\frac{67}{13}, -\frac{14}{13}), (1, 2)\}$. Answer.

Figure 16 depicts the graphical situation in the foregoing example. Figure 16a shows the graph of the original system of equations, while

Figure 16b shows the result after Step 3, when the ellipse has been replaced with the two horizontal lines with equations $y = 2$ and $y = -\frac{14}{13}$.



a Figure 16 b

Notice also, in Example 1, that the linear equation was transformed to express x in terms of y rather than y in terms of x in order to make the resulting computation simpler.

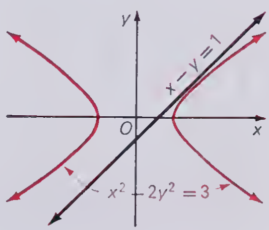
It is important to be aware of the fact that systems of equations involving one or more quadratic equations may have complex as well as real solutions. While graphing may be used to identify such real solutions as exist, substitution will yield complex as well as real solutions.

EXAMPLE 2 Find the solution set of the system:

$$x^2 - 2y^2 = 3$$
$$x - y = 1$$

SOLUTION Solve the linear equation for y in terms of x:

$$y = x - 1$$



Substitute $x - 1$ for y in the quadratic equation and simplify:

$$x^2 - 2(x - 1)^2 = 3$$
$$x^2 - 2x^2 + 4x - 2 = 3$$
$$x^2 - 4x + 5 = 0$$

Solve for x using the quadratic formula:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Then since $y = x - 1$, you have

$$x = 2 + i \qquad \text{or} \qquad x = 2 - i.$$
$$y = 2 + i - 1 \qquad\qquad y = 2 - i - 1$$
$$y = 1 + i \qquad\qquad y = 1 - i$$

$\therefore$ the solution set is $\{(2 + i, 1 + i), (2 - i, 1 - i)\}$. Answer.

Note that, as in Example 2, when a system has no real solutions, the graphs of the equations do not intersect.

Oral Exercises

State whether each ordered pair in the proposed solution set is a solution to the system when it is solved over $\mathcal{C}$.

1. $x^2 + y^2 = -5;$ $\{(3i, 2), (-3i, 2), (3i, -2), (-3i, -2)\}$
   $y = 2$
2. $x^2 - y^2 = -15;$ $\{(4i, i), (-4i, -i), (-4i, i), (4i, -i)\}$
   $x = 4y$
3. $2x^2 - y^2 = 1;$ $\{(2i, 3i), (-2i, 3i), (2i, -3i), (-2i, -3i)\}$
   $3x + 2y = 0$

Written Exercises

Find the solution set of each system over $\mathcal{C}$.

A

1. $y = x^2 + 4$
   $y = 5x$
2. $x^2 - y^2 = 8$
   $x - 3y = 0$
3. $x^2 + y^2 = 6$
   $x + y = -2\sqrt{3}$
4. $x^2 + y^2 = 13$
   $2x + y = 4$
5. $x^2 - y^2 = 16$
   $3x + y = 0$
6. $11 - x^2 = 2y$
   $3x + y = 9$

B

7. $\sqrt{x^2 + y^2} = x + 2$
   $2x = y - 2$
8. $\sqrt{x^2 + y^2} = 1 - x$
   $2x + y = -1$
9. $\dfrac{x}{y} + \dfrac{2y}{x} = \dfrac{18}{xy}$
   $x - 3y = 1$
10. $\dfrac{x}{y} - \dfrac{6y}{x} = \dfrac{5}{xy}$
    $x - 2y = 1$
11. $x^2 + y^2 - 5xy = 15$
    $y - 5 = 2x$
12. $y^2 - x^2 - 3xy = 27$
    $2x + y = -2$
13. $3\sqrt{x^2 + y^2} - \dfrac{5}{3\sqrt{x^2 + y^2}} = 0$
    $x + y = 1$
14. $\dfrac{7}{\sqrt{x^2 - 2y^2}} - \sqrt{x^2 - 2y^2} = 0$
    $2y - x = 1$

C

15. Prove that the solution set of any system

$$y = ax^2 + bx + c$$
$$y = mx + k,$$

with a, b, c, m, and k all real numbers, must consist of one or two ordered pairs all of whose components are real or all of whose components are imaginary.

16. Prove that the solution set of any system

$$ax^2 + by^2 = c$$
$$y = mx + k,$$

with a, b, c, m, and k all real numbers, must consist of one or two ordered pairs all of whose components are real or all of whose components are imaginary.

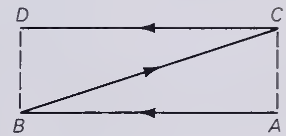
Problems

A 1. Find the dimensions of a rectangle whose area is 48 cm^2 and whose perimeter is 38 cm.

2. The area of a trapezoid is 24 cm^2 and one base has length 5 cm. If the other base is 1 cm shorter than twice the altitude, find the other base and the altitude.

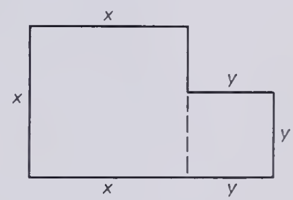
3. Points P and Q on the positive x- and y-axes, respectively, are 17 units apart. Traveling *along* the axes, one would traverse 23 units in going from P to Q. What are the coordinates of P and Q?

4. To travel from A to D, a sailboat tacked along route $ABCD$, traveling a total of 22 km. If $ABDC$ is a rectangle in which the length of $\overline{BD}$ is 8 km, find AB and BC.



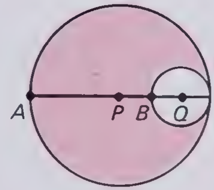
5. A square walk with a uniform width of 1 m is to be built around a park statue. If the area of the walk is to be 20 m^2, what should be its inside and outside dimensions?

6. The foundation of a house is to be in the shape of two adjacent squares, as shown. If the total area of the foundation is to be 500 m^2 and its perimeter is to be 100 m, what should the values of x and y be?



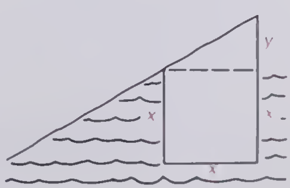
B 7. The difference of the areas of two squares is 45. The length of a side of one square is 3 less than twice the length of a side of the other square. Find the length of a side of each square.

8. In the diagram, the shaded region, between the circle with center P and the circle with center Q, has an area of 8π. If $m(\overline{AB}) = 4$, what are the radii of the two circles?



9. A parabola which has its vertex on the line $y = 2x + 1$ and whose equation is of the form $y = 2(x - h)^2 + k$ passes through the point (1, 7). What are the equations of all such parabolas?

10. A fishing pier is to be built in the shape of a square surmounted by a right triangle as shown in the diagram. If the area of the pier is to be 1050 m^2 and the part of its perimeter along the water is to be 100 m long, what are the dimensions of the pier?



10-10 Quadratic-Quadratic Systems

Substitution often provides a means of solving systems of two quadratic equations in two variables.

EXAMPLE 1 Find the solution set of the system:

$$x^2 + 4y^2 = 17$$
$$3x^2 - y^2 = -1$$

SOLUTION

1. Solve the second equation for y^2 in terms of x.
$$3x^2 - y^2 = -1$$
$$y^2 = 3x^2 + 1$$

2. Replace y^2 in the first equation with "$3x^2 + 1$."
$$x^2 + 4(3x^2 + 1) = 17$$
$$x^2 + 12x^2 + 4 = 17$$
$$13x^2 = 13$$
$$x^2 = 1$$

3. Solve the resulting equation. $x = 1$ **or** $x = -1$

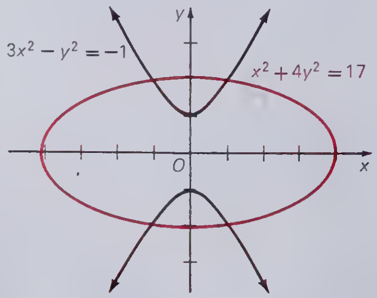
4. Solve two simple systems.

$y^2 = 3x^2 + 1$	$y^2 = 3x^2 + 1$
$x = 1$	$x = -1$
$y^2 = 3(1)^2 + 1$	$y^2 = 3(-1)^2 + 1$
$y^2 = 4$	$y^2 = 4$
$y = 2$ or $y = -2$	$y = 2$ or $y = -2$
$\{(1, 2), (1, -2)\}$	$\{(-1, 2), (-1, -2)\}$

5. Checking the solutions in the original system is left to you.

$\therefore$ the solution set is $\{(1, 2), (1, -2), (-1, 2), (-1, -2)\}$. Answer.

Figure 17 shows that the result of Step 3 is the replacement of the ellipse having equation $x^2 + 4y^2 = 17$ with the two straight lines having equations $x = 1$ and $x = -1$.



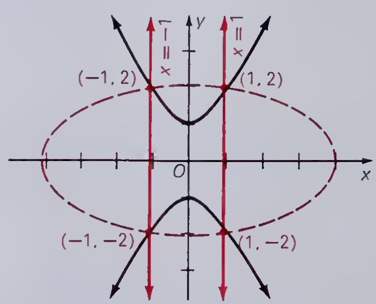


Figure 17

Linear combinations (Section 4-2) of quadratic equations can also be used to find solution sets of systems of such equations.

EXAMPLE 2 Find the solution set of the system:

$$x^2 + y^2 = 10$$
$$9x^2 + y^2 = 18$$

SOLUTION Subtracting the first equation from the second produces the linear combination

$$8x^2 = 8 \qquad \text{or} \qquad x^2 = 1,$$

which can be used together with equation 1 to form the equivalent system:

$$\begin{array}{c} x^2 + y^2 = 10 \\ x = 1 \end{array} \qquad \text{or} \qquad \begin{array}{c} x^2 + y^2 = 10 \\ x = -1 \end{array}$$

Substituting 1 and -1 for x in turn in the first equation then produces values for y. You have:

$1^2 + y^2 = 10$	$(-1)^2 + y^2 = 10$
$y^2 = 9$	$y^2 = 9$
$y = 3$ or $y = -3$	$y = 3$ or $y = -3$
$\{(1, 3), (1, -3)\}$	$\{(-1, 3), (-1, -3)\}$

Checking the solutions in both equations is left to you.

$\therefore$ the solution set is $\{(1, 3), (1, -3), (-1, 3), (-1, -3)\}$. Answer.

Oral Exercises

Express y^2 in terms of x in each of the following equations.

1. $x^2 + y^2 = 17$
2. $4x - 2y^2 = 5$
3. $7y^2 - x^2 = 14$
4. $x^2 + 4y^2 - 8 = 0$
5. $3x^2 + 6y^2 = 2$
6. $-2x^2 - 3y^2 = 18$

Written Exercises

Find the solution set over $\mathcal{C}$.

A

1. $x^2 + 4y^2 = 13$; $x^2 - y^2 = 8$
2. $3x^2 + y^2 = 16$; $x^2 + 2y^2 = 12$
3. $y^2 - 4x^2 = 16$; $2x^2 + y^2 = 16$
4. $3x^2 + 4y^2 = 15$; $4x^2 - y^2 = 1$
5. $x^2 + y^2 = 2$; $4x^2 + 3y^2 = 23$
6. $x^2 - 3y^2 = 6$; $x^2 + 2y^2 = 16$
7. $x^2 - 4y^2 = 9$; $4x^2 + 9y^2 = 36$
8. $y^2 - x^2 = 3$; $x^2 - 9y^2 = 5$
9. $3x^2 + 5y^2 = 12$; $7x^2 - 3y^2 = -5$
10. $3x^2 + 2y^2 = 21$; $2x^2 - 5y^2 = -24$
11. $x^2 + y^2 = 5$; $y = x^2 - 3$
12. $4x^2 + y^2 = 17$; $y = 2x^2 - 1$

B

13. $x^2 + y^2 = 15$; $x = 2y^2 + 6$
14. $y^2 - 6x^2 = 4$; $y = 3x^2 - 2$
15. $x^2 + y^2 = 29$; $xy = 10$

C 16. Use determinants (or linear combinations) to show that if the system

$$\begin{aligned} ax^2 + by^2 &= 1 \qquad (ad - bc \neq 0) \\ cx^2 + dy^2 &= 1 \end{aligned}$$

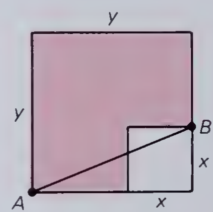
has 4 real solutions, then they lie on the graph of the circle

$$x^2 + y^2 = \frac{a + d - (b + c)}{ad - bc}.$$

Problems

A 1. A rectangular flower bed that measures 5 m along a diagonal has an area of $12\ \text{m}^2$. What are its dimensions?

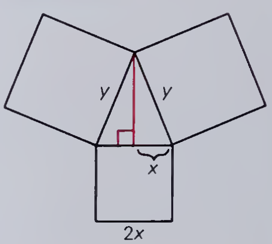
2. Find the dimensions of a rectangular television screen with a diagonal of length 500 cm and an area of $120{,}000\ \text{cm}^2$.



3. Find the coordinates of all the points on the ellipse $2x^2 + y^2 = 20$ that are 4 units from the origin.

4. In the diagram, the shaded region between the square of side y and the square of side x has an area of 12. If $AB = \sqrt{5}$, find x and y.

B 5. Find the dimensions of a right triangle whose area is $30\ \text{cm}^2$ and whose perimeter is 30 cm.



6. A building complex consisting of three square buildings is to be built around a courtyard in the shape of an isosceles triangle of altitude 80 m. If the combined area of the ground floors of the three buildings is to be $34{,}400\ \text{m}^2$, what should the dimensions x and y be?

Self-Test 3

VOCABULARY linear-quadratic system (p. 370)
quadratic-quadratic system (p. 374)

1. Solve the system over $\mathcal{R}$ by graphing: $4x^2 + y^2 = 16$, $2x + y = 4$ — *Obj. 1, p. 368*

Solve each system over $\mathcal{C}$.

2. $x^2 + 5y^2 = 45$, $2y = 1 - x$

3. $9x^2 + 4y^2 = 100$, $y^2 - x^2 = 12$ — *Obj. 2, p. 368*

Check your answers with those at the back of the book.

Chapter Summary

1. The distance from point $P_1(x_1, y_1)$ to point $P_2(x_2, y_2)$ is given by the formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

2. The midpoint (M) of the segment with endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

3. If m_1 and m_2 are the slopes of nonvertical lines L_1 and L_2, respectively, then L_1 and L_2 are perpendicular if and only if

$$m_1 m_2 = -1.$$

4. An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

5. A parabola whose equation is of the form

$$y = a(x - h)^2 + k \qquad \text{or} \qquad x = a(y - k)^2 + h$$

has vertex $V(h, k)$ and axis of symmetry

$$x = h \qquad \text{or} \qquad y = k.$$

6. The graph of a relation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a \neq b)$$

will be an ellipse.

7. The graph of a relation in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

will be a hyperbola.

8. A function specified by an equation of the form $xy = k$, $k \neq 0$, is called an inverse variation. You say that x and y vary inversely as each other or are inversely proportional to each other.

   In a function specified by an equation of the form $z = kxy$, you say that z varies jointly as x and y.

9. The points of intersection of the graphs of the equations of a system represent the real solutions of the system. A linear-quadratic system may have as many as two real solutions; a quadratic-quadratic system may have as many as four real solutions. These systems may also have complex solutions.

Chapter Review

1. Find the distance between $(-4, 1)$ and $(-2, -5)$. 10-1

   a. $10\sqrt{2}$ b. $4\sqrt{10}$ c. $2\sqrt{10}$ d. $2\sqrt{5}$

2. Find the coordinates of the midpoint of the line segment with endpoints $(-a, b)$, (a, b).

   a. $(0, b)$ b. $\left(-\frac{a}{2}, \frac{b}{2}\right)$ c. $(a, 0)$ d. $\left(\frac{a}{2}, -\frac{b}{2}\right)$

3. Find an equation of the line containing $(1, \frac{9}{2})$ and perpendicular to the line through $(3, 7)$ and $(-1, 2)$. 10-2

   a. $8x + 10y = 41$ b. $8x - 10y = 53$ c. $y = -0.8x + 5.3$

4. Find an equation of the line containing $(-2, 3)$ and perpendicular to the graph of $2x - 3y = 8$.

   a. $3x + 2y = 0$ b. $3y - 2x = 13$ c. $2x + 3y = 5$

5. Write an equation of the circle with center at $(1, -4)$ and containing the point $(-3, -6)$. 10-3

   a. $x^2 - 2x + y^2 + 8y = 99$ b. $x^2 - 2x + y^2 + 8y = 3$

   c. $x^2 + 2x + y^2 + 4y = 15$ d. $x^2 + 2x - y^2 - 8y = 9$

6. Find the vertex of the parabola with equation $x = 2y^2 + 6y + 1$ 10-4

   a. $(1, -\frac{3}{2})$ b. $(\frac{29}{2}, \frac{3}{2})$ c. $(\frac{5}{2}, -\frac{7}{2})$ d. $(-\frac{7}{2}, -\frac{3}{2})$

7. Find an equation of the form $y = ax^2 + bx + c$ for the parabola with focus $F(-1, 2)$ and directrix $y = 4$.

   a. $y = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{11}{4}$ b. $y = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{3}{2}$ c. $y = -\frac{1}{4}x^2 - \frac{1}{3}x + 4$

8. Give the coordinates of the foci of the ellipse described by $\frac{x^2}{9} + \frac{y^2}{16} = 1$. 10-5

   a. $(3, 0)$, $(-3, 0)$ b. $(0, \sqrt{7})$, $(0, -\sqrt{7})$ c. $(0, 4)$, $(0, -4)$

9. Find an equation for the ellipse with major axis of length 8; x-intercepts 2 and -2.

   a. $\frac{x^2}{4} + \frac{y^2}{16} = 1$ b. $\frac{x^2}{16} + \frac{y^2}{4} = 1$ c. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

10. Find equations of the asymptotes of the hyperbola described by $16x^2 - 25y^2 = 400$. 10-6

    a. $y = \frac{5}{4}x$, $y = -\frac{5}{4}x$ b. $y = \frac{4}{5}x$, $y = -\frac{4}{5}x$ c. $y = \frac{25}{16}x$, $y = -\frac{25}{16}x$

11. Determine x_1 so that both ordered pairs $(1, \frac{2}{3})$ and $(x_1, 9)$ are members of the same inverse variation. 10-7

    a. $\frac{27}{2}$ b. 6 c. $\frac{1}{6}$ d. $\frac{2}{27}$

12. If a varies jointly as b and c, and $a = 27$ when $b = 18$ and $c = 3$, find a when $b = 6$ and $c = \frac{1}{3}$.

a. 1 b. $\frac{2}{3}$ c. 81 d. 12

13. Determine the number of points of intersection. 10-8

$$y = x^2 + 2$$
$$4y = x^2 + y^2$$

a. 0 b. 1 c. 2 d. 4

14. The sum of two integers is 3. The sum of the squares of the two integers is 369. Find the integers. 10-9

a. -11 and 14 b. -13 and 16 c. -14 and 17 d. 15 and -12

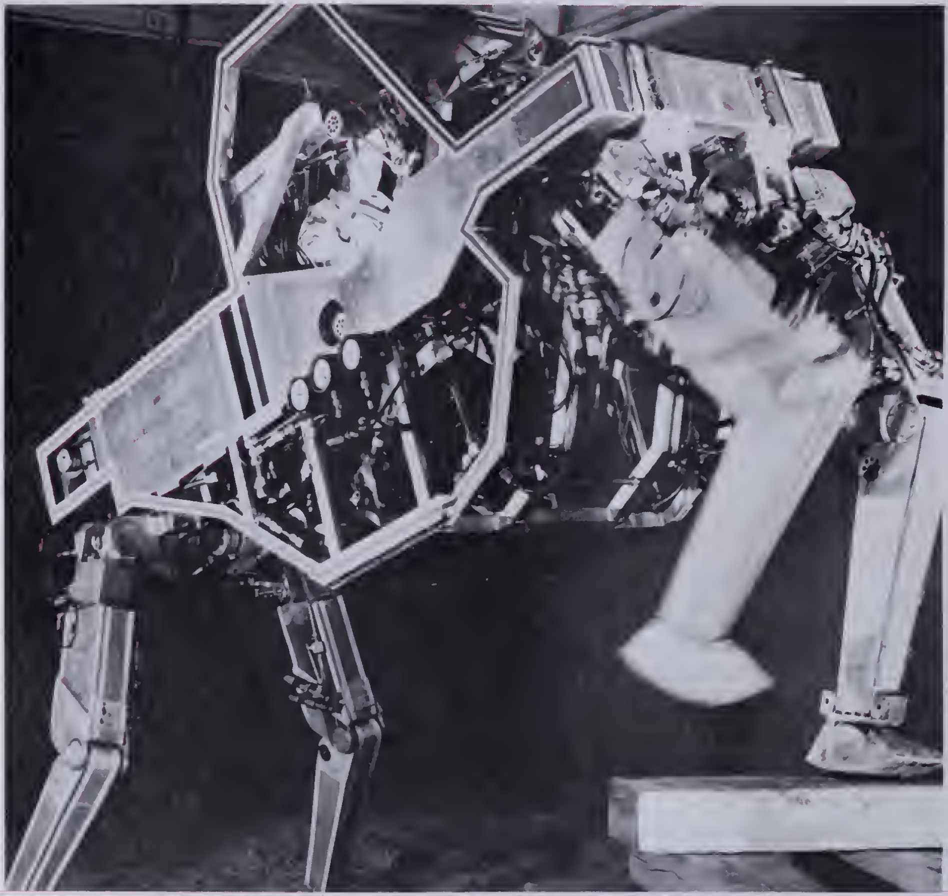
15. Find the solution set over $\mathcal{C}$. 10-10

$$5x^2 + 16y^2 = 26$$
$$-4x^2 + 25y^2 = 17$$

a. $\{(-\sqrt{2}, -1), (\sqrt{2}, 1)\}$ b. $\{(-\sqrt{2}, 1), (\sqrt{2}, -1), (-\sqrt{2}, -1)\}$
c. $\{(-\sqrt{2}, -1), (-\sqrt{2}, 1), (\sqrt{2}, -1), (\sqrt{2}, 1)\}$

Chapter Test

1. Find the perimeter of a triangle with vertices (2, 1), (5, 1), and (4, 4). 10-1
2. Find an equation of the line perpendicular to and containing the midpoint of the segment with endpoints $(-2, 1)$ and $(2, 3)$. 10-2
3. Find the center and the radius of the circle with equation 10-3
$$100x^2 - 100x + 100y^2 + 21 = 0.$$
4. Sketch the graph of $y = 2x^2 - 2x + \frac{7}{2}$. Identify the focus, vertex, and axis of symmetry. 10-4
5. An ellipse has foci at (0, 3) and (0, -3). The sum of its focal radii is 10. Find an equation of the ellipse. 10-5
6. a. Sketch the graph of $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. 10-6
   b. Write the equations for the asymptotes.
   c. State the foci of the hyperbola.
7. If x varies inversely as y^2, and $x = 5$ when $y = 4$, find x when $y = 8$. 10-7
8. Solve over $\mathcal{R}$ by graphing: $x^2 + y^2 = 25$ 10-8
$$3y - 4x = 0$$
9. Solve over $\mathcal{C}$ by substitution: $9x^2 - 5y^2 = 36$ 10-9
$$y = 2x - 3$$
10. Solve over $\mathcal{C}$: $4x^2 + 25y^2 = 360$ 10-10
$$x^2 - y^2 = 61$$



This research prototype quadruped is more than three meters high and has a mass of more than thirteen hundred kilograms. The operator uses hand and leg movements to control the front and rear legs of the quadruped.

11 Exponents and Logarithms

Extending the Laws of Exponents

***OBJECTIVES** for Sections 11-1 and 11-2:*
1. Simplify expressions involving rational and real-number exponents.
2. Solve simple exponential equations.

11-1 Rational Exponents

In Section 6-1 you saw that, by suitable definitions, the laws of exponents can be extended to apply to powers with any integer as an exponent. Thus,

$$4^{-1} \cdot 4^2 = 4^{-1+2} = 4^1 = 4, \qquad \text{and} \qquad 3^{-2} \cdot 3^0 = 3^{-2+0} = 3^{-2} = \tfrac{1}{9}.$$

Can we now use these laws to help define powers so that any rational number can be used as an exponent? Let us look first at a special case, say $3^{\frac{1}{2}}$. For $3^{\frac{1}{2}}$ to have a meaning consistent with the familiar laws of exponents (page 172), it should be true that

$$(3^{\frac{1}{2}})^2 = 3^{(\frac{1}{2} \cdot 2)} = 3^1 = 3.$$

Since $3^{\frac{1}{2}}$ is to denote a number whose square is 3, we define it to be $\sqrt{3}$ (rather than $-\sqrt{3}$) so that the inequality

$$3^0 < 3^{\frac{1}{2}} < 3^1$$

is true.

Thus, $3^{\frac{1}{2}}$ represents the positive square root of the positive number 3. Similar reasoning requires that

$$3^{\frac{5}{2}} = (3^{\frac{1}{2}})^5 = (\sqrt{3})^5 \qquad \text{and} \qquad 3^{-\frac{5}{2}} = (3^{\frac{1}{2}})^{-5} = (\sqrt{3})^{-5}.$$

These observations suggest the following definition.

> If p denotes an *integer*, r a *positive integer*, and b a *positive real number*, then
>
> $$b^{\frac{p}{r}} = (\sqrt[r]{b})^p.$$
>
> When p *as well as* r is a positive integer, we define $0^{\frac{p}{r}} = 0$.

In particular, if $p = 1$,

$$b^{\frac{1}{r}} = \sqrt[r]{b}.$$

The fact (page 277) that $\sqrt[r]{b^p} = (\sqrt[r]{b})^p$ implies that

$$(b^p)^{\frac{1}{r}} = (b^{\frac{1}{r}})^p,$$

and either member of this latter equation is thus equal to $b^{\frac{p}{r}}$.

Using powers with rational exponents, you can write radical expressions in **exponential form**, that is, as powers or products of powers. Then, because the laws of exponents apply to these powers (Exercises 51–54, page 384), you can use the laws to simplify the exponential expressions.

EXAMPLE 1 $2\sqrt[3]{8x^7y^{-3}z} = 2(8)^{\frac{1}{3}}(x^7)^{\frac{1}{3}}(y^{-3})^{\frac{1}{3}}(z)^{\frac{1}{3}} = 2(2)(x^{\frac{6}{3}} \cdot x^{\frac{1}{3}})(y^{-\frac{3}{3}})(z^{\frac{1}{3}})$

$$= 4x^2y^{-1}x^{\frac{1}{3}}z^{\frac{1}{3}} = \frac{4x^2}{y}\sqrt[3]{xz} \quad (y \neq 0).$$

EXAMPLE 2 $\sqrt{x}\,\sqrt[3]{x^2} = x^{\frac{1}{2}} \cdot x^{\frac{2}{3}} = x^{\frac{1}{2}+\frac{2}{3}} = x^{\frac{7}{6}} = x^{\frac{6}{6}} \cdot x^{\frac{1}{6}} = x\sqrt[6]{x}.$

EXAMPLE 3 $\left(\frac{1}{64}\right)^{-\frac{5}{6}} = (64)^{\frac{5}{6}} = [(64)^{\frac{1}{6}}]^5 = 2^5 = 32.$

The laws of exponents can be used to develop another useful fact about radical expressions. Thus the laws of integral exponents permit you to write for any integer s,

$$\begin{aligned}(b^{\frac{p}{r}})^{rs} &= [(b^{\frac{p}{r}})^r]^s \\ &= [b^p]^s \\ &= b^{ps}\end{aligned}$$

Because $b^{ps} = (b^{\frac{p}{r}})^{rs}$, $r > 0$, and $b > 0$, if $s > 0$, you have $rs > 0$ and so you can assert that $b^{\frac{ps}{rs}} = b^{\frac{p}{r}}$, or

$$\sqrt[rs]{b^{ps}} = \sqrt[r]{b^p}$$

For example,

$$\sqrt[12]{81x^8} = \sqrt[3 \cdot 4]{3^{1 \cdot 4}x^{2 \cdot 4}} = \sqrt[3]{3x^2}.$$

Note that in extending powers to include all rational exponents, we have restricted the base b to be a *positive* real number. Without that restriction, we could not always define b^r to be a real number. For example, $(-2)^{\frac{1}{2}}$ could not represent a real number, since there is no real number whose square is negative. In the Exercises below, assume that all variables denote positive real numbers, unless otherwise specified.

Oral Exercises

State each expression in exponential form.

1. $\sqrt[3]{5}$
2. $\sqrt{x^3}$
3. $\sqrt[5]{n^4}$
4. $\dfrac{1}{\sqrt{3}}$
5. $\sqrt[4]{3x^5}$
6. $\sqrt[4]{(3x)^5}$
7. $\sqrt{a + b}$
8. $\sqrt[3]{2(x + y)}$

State each expression in radical form.

9. $5^{\frac{1}{2}}$
10. $7^{\frac{2}{3}}$
11. $2^{\frac{1}{5}}$
12. $3^{-\frac{1}{4}}$
13. $(3x)^{\frac{1}{2}}$
14. $2x^{-\frac{1}{3}}$
15. $(y + 5)^{\frac{3}{8}}$
16. $(u + v^2)^{\frac{3}{2}}$

Written Exercises

Evaluate.

A

1. $8^{\frac{1}{3}}$
2. $27^{\frac{2}{3}}$
3. $25^{-\frac{1}{2}}$
4. $16^{\frac{3}{4}}$
5. $49^{\frac{3}{2}}$
6. $32^{-\frac{2}{5}}$
7. $(\frac{1}{64})^{\frac{2}{3}}$
8. $(\frac{27}{125})^{-\frac{1}{3}}$
9. $(\frac{1}{81})^{-\frac{3}{4}}$
10. $0.008^{\frac{4}{3}}$
11. $(9 + 16)^{\frac{1}{2}}$
12. $(25 + 144)^{-\frac{1}{2}}$

Convert to exponential form and then write in simplified radical form.

13. $\sqrt[4]{x^6}$
14. $\sqrt[6]{2^3y^9}$
15. $\sqrt[4]{4n^2}$
16. $\sqrt[6]{\dfrac{1}{8r^{12}}}$
17. $\sqrt[8]{16}$
18. $\sqrt[10]{\frac{1}{32}}$
19. $\sqrt{\sqrt{64}}$
20. $\sqrt[9]{\frac{8}{27}}$
21. $\sqrt[12]{125}$
22. $\sqrt[3]{\left(\dfrac{8}{125}\right)^2}$
23. $\sqrt[4]{\left(\dfrac{81}{16}\right)^{-2}}$
24. $\sqrt[4]{\sqrt{81}}$

Express in simplest radical form.

25. $\sqrt[6]{16}\sqrt[3]{16}$
26. $\sqrt{2}\sqrt[6]{32}$
27. $\sqrt[8]{9}\sqrt[4]{27}$
28. $\sqrt[12]{5}\sqrt[12]{125}$
29. $\dfrac{\sqrt[8]{16}}{\sqrt[6]{2}}$
30. $\dfrac{\sqrt[4]{125}}{\sqrt[4]{5}}$
31. $\dfrac{\sqrt[6]{9}}{\sqrt[3]{81}}$
32. $\dfrac{\sqrt[3]{16}}{\sqrt[12]{16}}$

Express in simplest radical form.

B 33. $\sqrt{\sqrt[3]{25}}$ 34. $\sqrt{\sqrt{8} \cdot \sqrt[4]{2}}$ 35. $\sqrt{7 \cdot \sqrt[3]{7}}$

36. $\sqrt[3]{\sqrt[4]{3} \cdot \sqrt[4]{9}}$ 37. $\sqrt{\sqrt[6]{5} \cdot \sqrt{5}}$ 38. $\sqrt[3]{\sqrt[4]{8}} \cdot \sqrt{\sqrt{8}}$

Solve over $\mathcal{R}$.

EXAMPLE $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$

SOLUTION Use the fact that $x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2$ to factor the left member of the equation.

$$x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$$
$$(x^{\frac{1}{3}} - 2)(x^{\frac{1}{3}} + 3) = 0$$
$$x^{\frac{1}{3}} = 2 \quad \text{or} \quad x^{\frac{1}{3}} = -3$$

$\therefore$ the solution set is $\{8, -27\}$.

39. $x^{\frac{2}{3}} = 4$ 40. $3y^{\frac{3}{2}} = \frac{1}{9}$ 41. $r^{-\frac{3}{4}} = \frac{1}{8}$

42. $\frac{1}{4}z^{-\frac{5}{6}} = 8$ 43. $(n + 1)^{\frac{2}{3}} = 25$ 44. $4k^{-\frac{2}{5}} = 9$

45. $t - 3t^{\frac{1}{2}} + 2 = 0$ 46. $v^{\frac{2}{3}} - v^{\frac{1}{3}} - 2 = 0$ 47. $u^{\frac{4}{3}} - 10u^{\frac{2}{3}} + 9 = 0$

48. $8x^3 - 9x^{\frac{3}{2}} + 1 = 0$ 49. $4x^{\frac{4}{5}} - 5x^{\frac{2}{5}} + 1 = 0$ 50. $4x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 1 = 0$

Let a and b denote positive real numbers, r and s positive integers, and p and q integers. Prove each statement.

C 51. $b^{\frac{p}{r}} \cdot b^{\frac{q}{s}} = b^{\frac{ps+rq}{rs}}$ 52. $(b^{\frac{p}{r}})^{\frac{q}{s}} = b^{\frac{pq}{rs}}$

(*Hint:* Raise each member of the equation to the rsth power.)

53. $a^{\frac{p}{r}} \cdot b^{\frac{p}{r}} = (ab)^{\frac{p}{r}}$ 54. $\dfrac{a^{\frac{p}{r}}}{b^{\frac{p}{r}}} = \left(\dfrac{a}{b}\right)^{\frac{p}{r}}$

11-2 Real-Number Exponents

In Chapter 1 we defined powers with natural-number exponents, in Chapter 6 we extended this definition to integral powers, and then in Section 11-1 we extended it to powers with rational-number exponents. Thus, we have defined such powers as

$$2^3 = 2 \cdot 2 \cdot 2 = 8, \quad 2^0 = 1, \quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \quad \text{and} \quad 2^{\frac{1}{2}} = \sqrt{2}.$$

Also,

$$2^{\frac{3}{2}} = \sqrt{2^3} \qquad \text{and} \qquad 2^{1.7} = 2^{\frac{17}{10}} = \sqrt[10]{2^{17}}.$$

Can we now define powers such as $2^{\sqrt{3}}$, which have irrational numbers as exponents, to have a meaning consistent with the familiar laws of exponents? The answer to the question is "Yes," as the following discussion shows.

You can construct a table of coordinates of points of the graph of the function $\{(x, y): y = 2^x\}$ for selected rational values of x (these are graphed in Figure 1 at the right):

x	2^x
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1

x	2^x
$\frac{1}{2}$	$1.4\ldots$
1	2
$\frac{3}{2}$	$2.8\ldots$
2	4
$\frac{5}{2}$	$5.6\ldots$

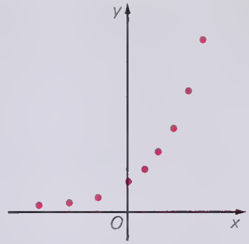


Figure 1

In order for the graph to be represented by a smooth unbroken curve (Figure 2), it must be true that powers such as $2^{\sqrt{3}}$, in which the exponents are irrational, exist. You can see that since

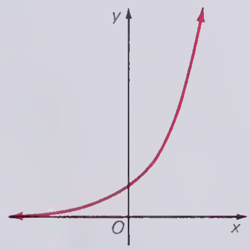
$$1.5 < \sqrt{3} < 2,$$

you have

$$2^{\frac{3}{2}} < 2^{\sqrt{3}} < 2^2,$$

or

$$2.8 < 2^{\sqrt{3}} < 4.$$



$\{(x, y): y = 2^x\}$

Figure 2

The power $2^{\sqrt{3}}$ can be approximated by the successive powers

$$2^1,\ 2^{1.7},\ 2^{1.73},\ 2^{1.732},\ \ldots$$

in which the exponents are rational numbers represented by taking more and more places in the decimal representing $\sqrt{3}$. Since these powers steadily increase but remain less than 2^2, it follows from the Axiom of Completeness (page 242) that they converge to a certain positive real number, called $2^{\sqrt{3}}$. To four decimal places $2^{\sqrt{3}} \approx 3.3220$.

Similar reasoning leads to the definition of b^x where b is any *positive* real number and x any irrational number. Furthermore, it can be proved that the laws of exponents continue to hold for these powers. For example,

$$(3^{\sqrt{2}})^{\sqrt{2}} = 3^{\sqrt{2}\cdot\sqrt{2}} = 3^2 = 9; \qquad 2^{1-\pi} \cdot 2^{\pi} = 2^{(1-\pi)+\pi} = 2^1 = 2.$$

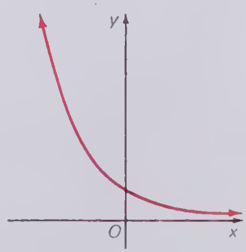
The curve shown in Figure 2 continuously rises with increasing abscissa and is typical of the graph of every function of the form

$$\{(x, y): y = b^x,\ b > 1\}.$$

On the other hand, the graph of

$$\{(x, y): y = b^x,\ 0 < b < 1\}$$

falls with increasing abscissa, as illustrated by the graph of the function $\{(x, y): y = (\frac{1}{2})^x\}$, shown in Figure 3. Notice that this can also be described by $\{(x, y): y = 2^{-x}\}$.



$\{(x, y): y = (\frac{1}{2})^x = 2^{-x}\}$

Figure 3

In either Figure 2 or Figure 3, any vertical line and any horizontal line above the x-axis intersects the graph in just one point. In general, you have:

> For $b > 0$, $b \neq 1$, $b^{x_1} = b^{x_2}$ if and only if $x_1 = x_2$.

EXAMPLE Solve $8^{2r+1} = (\frac{1}{2})^{r-4}$ over $\mathcal{R}$.

SOLUTION

1. First express each member as a power of the same base, 2. $\quad 8^{2r+1} = (\frac{1}{2})^{r-4}$; $\quad (2^3)^{2r+1} = (2^{-1})^{r-4}$
2. Simplify each member. $\quad 2^{6r+3} = 2^{-r+4}$
3. **Equate exponents** and solve for r. $\quad \mathbf{6r + 3 = -r + 4}$
4. The check is left for you. $\quad 7r = 1$

$\therefore$ the solution set is $\{\frac{1}{7}\}$. Answer. $\quad r = \frac{1}{7}$

Oral Exercises

State whether the inequality is true or false.

1. $4^{1.4} < 4^{\sqrt{2}}$
2. $8^{\pi} > 8^{4.1}$
3. $7^{\sqrt{5}} < 7^{\frac{5}{2}}$
4. $9^{\sqrt{6}} > 9^{2.9}$
5. $2^{3.4} > 4^{\sqrt{3}}$
6. $5^{2\sqrt{2}} > 25^{1.4}$
7. $8^{2\sqrt{2}} < 64^{1.5}$
8. $2^{3\sqrt{5}} < 8^{2.1}$

Written Exercises

Use the facts that $2^{\sqrt{2}} \approx 2.6651$ and $2^{\sqrt{3}} \approx 3.3220$ to express each of the following in the form 2^x.

A

1. $(2.6651)(3.3220)$
2. $\dfrac{3.3220}{2.6651}$
3. $(2.6651)^3$
4. $\dfrac{1}{3.3220}$
5. $\sqrt{2.6651}$
6. $2(3.3220)$
7. $(2.6651)(3.3220)^2$
8. $\dfrac{(2.6651)^3}{8}$
9. $\dfrac{4}{\sqrt[3]{3.3220}}$

Solve over $\mathcal{R}$.

10. $5^3 = 5^{2x-1}$
11. $9^x = 3^{x+4}$
12. $2^{3x-1} = 4^{x+2}$
13. $5^{x-1} = 125^{2x+3}$
14. $8^{2x-2} = 4^{2-x}$
15. $(\frac{1}{3})^{x-3} = 3^{x-1}$
16. $7^{2x+4} = (\frac{1}{49})^{x-3}$
17. $125^{x+3} = (\frac{1}{25})^{3x-6}$
18. $(\frac{1}{32})^{x+3} = (\frac{1}{8})^{x-3}$

Graph each pair of functions on one set of axes. Label each graph.

B

19. $f(x) = 3^x$
    $g(x) = (\frac{1}{3})^x$
20. $f(x) = \frac{1}{2}(2^x)$
    $g(x) = \frac{1}{2}(3^x)$
21. $f(x) = 4(\frac{1}{2})^x$
    $g(x) = 4(\frac{1}{3})^x$

C **22.** Find positive irrational numbers a and b such that $2^a \cdot 2^b$ is a rational number.

23. Solve the following system graphically, approximating the solution to the nearest half unit.

$$\begin{aligned} y &= 2^x \\ x + y &= 7 \end{aligned}$$

Self-Test 1

VOCABULARY exponential form (p. 382) equating exponents (p. 386)

Simplify. (Assume $y > 0$.)

1. $\sqrt[6]{16^3}$ **2.** $\sqrt[4]{9y^2}$ **3.** $\sqrt[6]{25}\,\sqrt[3]{25}$ **4.** $\dfrac{\sqrt[8]{32}}{\sqrt[8]{2}}$ *Obj. 1, p. 381*

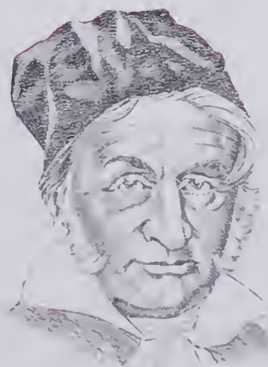
Solve over $\mathcal{R}$.

5. $3^{x-2} = 9^{x+3}$ **6.** $8^{x-1} = 2^{x+3}$ *Obj. 2, p. 381*

7. $5^{x-5} = 125^{3-2x}$ **8.** $(\frac{1}{4})^{x+4} = 8^{2x+4}$

Check your answers with those at the back of the book.

Karl Friedrich Gauss
1777–1855



It was Karl Friedrich Gauss, born six years before Euler's death, who paved the way for many of the scientific discoveries of this century. Born in Germany, the son of a bricklayer, at age three he corrected his father's figuring of the weekly payroll. At ten he discovered the formula for the sum of an arithmetic progression. For his doctoral dissertation, he gave the first correct proof of the Fundamental Theorem of Algebra (see page 332).

Gauss's genius became even more evident in subsequent years. His discoveries dealt not only with pure mathematics but also with applied mathematics. His work with the geometry of curved surfaces provided the foundation for Einstein's work in relativity, and, amazingly, such fields as electronics and aerodynamics rely heavily on the results of Gauss's work with functions of a complex variable.

From Exponents to Logarithms

***OBJECTIVES** for Sections 11-3 and 11-4:*
1. Identify inverses of functions.
2. Convert sentences from exponential to logarithmic form and vice versa.
3. Identify integral logarithms to various bases.

11-3 The Inverse of a Relation

In Section 3-2, you saw that any set of ordered pairs is a relation. If, now, in a relation R, the components of each of the pairs are interchanged, the result is another set of ordered pairs, and, hence, another relation. We denote this latter relation by R^{-1} (read "R inverse" or "the inverse of R"), and say that R and R^{-1} are **inverses** of each other. For example, if

$$R = \{(3, 4), (4, 5), (5, 6)\},$$

then

$$R^{-1} = \{(4, 3), (5, 4), (6, 5)\}.$$

Clearly, the domain and range of R^{-1} are the range and domain of R, respectively. Given an equation of the form $y = R(x)$ defining a relation R, you can obtain an equation defining its inverse simply by interchanging the variables. Thus, $x = R(y)$ defines the inverse of R. For example, the inverse of the relation defined by

$$y = 3x - 2$$

is defined by

$$x = 3y - 2,$$

or, when solved for y in terms of x,

$$y = \tfrac{1}{3}(x + 2).$$

Because for every ordered pair (a, b) in a relation R, the ordered pair (b, a) is in R^{-1}, the graphs of R and R^{-1} are related in an interesting and useful way, as we shall now show.

Figure 4 illustrates the fact that when the same scale is used on both axes, (a, b) and (b, a) are always located symmetrically with respect to the graph of $y = x$. That is, the graph of $y = x$ is the perpendicular bisector of the segment with endpoints (a, b) and (b, a). Therefore:

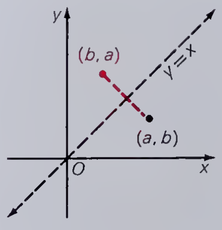


Figure 4

> The graphs of a relation R and its inverse R^{-1} are always the reflections of each other in the line with equation $y = x$.

Figure 5 shows the graphs of

$$R = \{(x, y): y = 3x - 2\}$$

and

$$R^{-1} = \{(x, y): y = \tfrac{1}{3}(x + 2)\},$$

together with the line having equation $y = x$.

Since every function is a relation, every function has an inverse, but the inverse of a function is not always a function. Figure 6 shows the graph of

$$R = \{(x, y): y = x^2 + 1\}$$

and that of its inverse

$$R^{-1} = \{(x, y): x = y^2 + 1\}.$$

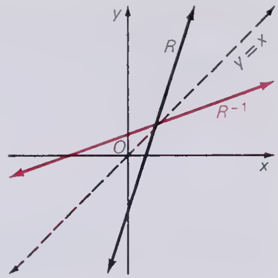


Figure 5

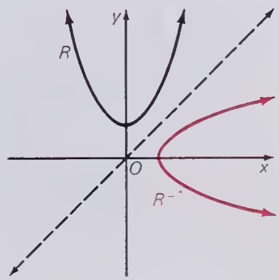


Figure 6

Clearly, R^{-1} is not a function, because for many values of x, there are two values of y. If a function f is to have an inverse f^{-1} that is a function, then not only must each element in the domain of f be paired with exactly one element in the range, but also each element in the range must be paired with exactly one element in the domain. Such a function is called a **one-to-one function.** If f is a one-to-one function that maps x_1 onto y_1, f^{-1} will map y_1 onto x_1. It must be true that for every x in the domain of f,

$$f^{-1}[f(x)] = x,$$

and for every x in the domain of f^{-1},

$$f[f^{-1}(x)] = x.$$

For the example shown in Figure 5, you have

$$f(x) = 3x - 2$$

and

$$f^{-1}(x) = \tfrac{1}{3}(x + 2),$$

and you can verify that

$$f^{-1}[f(x)] = \tfrac{1}{3}[(3x - 2) + 2] = x$$

and

$$f[f^{-1}(x)] = 3[\tfrac{1}{3}(x + 2)] - 2 = x.$$

Oral Exercises

State the inverse of the function.

1. $f(x) = 2x$
2. $f(x) = \dfrac{x}{8}$
3. $f(x) = x - 5$
4. $f(x) = -x$

Written Exercises

For each of the following, find $f^{-1}(x)$ and show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for each x in the domains of f and f^{-1}, respectively.

A
1. $y = 3x - 3$
2. $y = 2x + 6$
3. $y = -\frac{1}{2}x + 5$
4. $y = \sqrt{x}, x \geq 0$
5. $y = x^3$
6. $y = \sqrt{x + 1}, x \geq -1$

In Exercises 7–13, write the inverse relation of the given relation; graph both relations; and tell whether or not the inverse is a function.

7. $y = 2x + 3$
8. $y = \dfrac{4}{x}$
9. $y = x^2 + 2$
10. $y = x^3 + 1$
11. $y = \sqrt{x - 1}, x \geq 1$
12. $y = \sqrt{x^2 + 4}$

B
13. $y = x^2 - 2x$ (*Hint:* When solving the inverse relation for y, use the quadratic formula, treating x as a constant.)
14. Show that the line $y = x$ is the perpendicular bisector of the line segment joining (a, b) and (b, a).
15. Let $f(x) = 2^x$. Find $f^{-1}(1)$, $f^{-1}(2)$, $f^{-1}(4)$, $f^{-1}(8)$, and $f^{-1}(\frac{1}{2})$ and sketch the graphs of f and f^{-1} on one set of axes.

C
16. Draw the graph of $F = \{(x, y)\text{: } 9x^2 + 16y^2 = 144 \text{ and } y > 0\}$. Is it a one-to-one function?

11-4 The Logarithmic Function

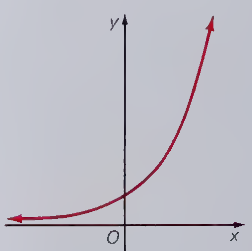
A function defined by an equation of the form

$$y = b^x \qquad (b > 0, b \neq 1)$$

is called an **exponential function with base b.** Its domain is $\mathcal{R}$ while its range is $\{y\text{: } y > 0\}$.

For example, the graph of

$$R = \{(x, y)\text{: } y = 2^x\}$$

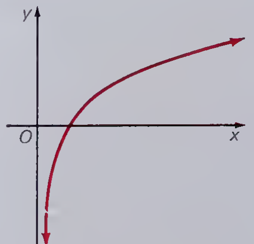


$\{(x, y): y = 2^x\}$

Figure 7

is pictured in Figure 7. Do you see that as x increases, y increases, and so this function is one-to-one? It therefore has an inverse that is a function, namely,

$$R^{-1} = \{(x, y)\text{: } x = 2^y\}.$$



$\{(x, y): x = 2^y\}$

Figure 8

Because the domain and range of R^{-1} are the range and domain, respectively, of R, R^{-1} has $\{x\text{: } x > 0\}$ as domain and $\mathcal{R}$ as range. Its graph is shown in Figure 8.

In the equation

$$x = 2^y,$$

the exponent y is called "the **logarithm** of x to the base 2."

This is written

$$y = \log_2 x.$$

More generally, the function defined by an equation of the form

$$x = b^y \qquad \text{or} \qquad y = \log_b x \qquad (b > 0, b \neq 1, x > 0)$$

is called the **logarithmic function** with base b. Since such a function is the inverse of an exponential function with base b, the graphs of the two are reflections of each other in the graph of $y = x$, as shown in Figure 9.

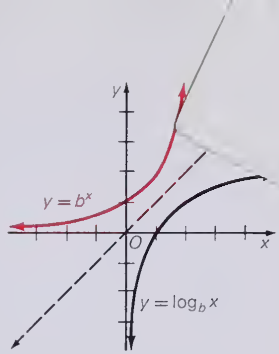


Figure 9

It should be emphasized that

$$\boxed{y = \log_b x \text{ if and only if } x = b^y.}$$

That is, these two equations are equivalent. By definition, since $b^0 = 1$, $\log_b 1 = 0$ for all permissible numbers b.

Either the logarithmic form or the exponential form should be used as is most convenient. Thus, you can say that the following are true:

$2^5 = 32$	is equivalent to	$\log_2 32 = 5.$
$4^{-2} = \frac{1}{16}$	is equivalent to	$\log_4 \frac{1}{16} = -2.$
$\log_{10} 1000 = 3$	is equivalent to	$10^3 = 1000.$
$\log_{100} 10 = \frac{1}{2}$	is equivalent to	$100^{\frac{1}{2}} = 10.$

The close relationship between exponential and logarithmic functions produces two additional useful facts about the latter functions. You saw on page 386 that for $b > 0$, $b \neq 1$, $b^{y_1} = b^{y_2}$ if and only if $y_1 = y_2$. It follows that:

For $b > 0$, $b \neq 1$,

$$\log_b x_1 = \log_b x_2 \text{ if and only if } x_1 = x_2 \ (x_1, x_2 > 0).$$

Next, since exponential and logarithmic functions are inverses, the fact that, for inverses, $f^{-1}[f(x)] = x$ and $f[f^{-1}(x)] = x$ implies that:

$$b^{\log_b x} = x \qquad (b > 0, b \neq 1, x > 0),$$
$$\log_b b^x = x \qquad (b > 0, b \neq 1).$$

Oral Exercises

Express each of the following in logarithmic form.

EXAMPLE 1 $2^3 = 8$ **SOLUTION** $\log_2 8 = 3$

1. $10^4 = 10{,}000$
2. $5^2 = 25$
3. $4^{-3} = \frac{1}{64}$
4. $7^0 = 1$
5. $32^{\frac{1}{5}} = 2$
6. $(\frac{1}{2})^{-4} = 16$
7. $10^{-2} = 0.01$
8. $(\sqrt{3})^4 = 9$

Express each of the following in exponential form.

EXAMPLE 2 $\log_4 16 = 2$ **SOLUTION** $4^2 = 16$

9. $\log_3 81 = 4$
10. $\log_5 \frac{1}{25} = -2$
11. $\log_{10} 0.001 = -3$
12. $\log_{\frac{1}{2}} \frac{1}{8} = 3$
13. $\log_b 1 = 0$ $1^0=b$
14. $\log_3 \sqrt[4]{3} = \frac{1}{4}$

Written Exercises

Find each logarithm.

EXAMPLE 1 $\log_5 125$

SOLUTION Let $\log_5 125 = x$; then $5^x = 125$.
$\therefore\ 5^x = 5^3$, and $x = 3$. Answer.

A

1. $\log_2 16$
2. $\log_{10} 0.1$
3. $\log_3 \frac{1}{27}$
4. $\log_8 1$
5. $\log_{25} 5$
6. $\log_{\frac{1}{2}} 32$
7. $\log_{\frac{1}{5}} \frac{1}{125}$
8. $\log_{12} 12$
9. $\log_{\sqrt{2}} 8$
10. $\log_9 27$
11. $\log_8 4$
12. $\log_{\sqrt{3}} 9\sqrt{3}$

Solve for x by converting each statement to exponential form and solving.

EXAMPLE 2 $\log_x \sqrt{5} = \frac{3}{2}$

SOLUTION If $\log_x \sqrt{5} = \frac{3}{2}$, then $x^{\frac{3}{2}} = \sqrt{5} = 5^{\frac{1}{2}}$.
Raise both members to the $\frac{2}{3}$ power:
$(x^{\frac{3}{2}})^{\frac{2}{3}} = (5^{\frac{1}{2}})^{\frac{2}{3}}$
$x = 5^{\frac{1}{3}}$ $\quad \therefore\ x = \sqrt[3]{5}$. Answer.

13. $\log_x 64 = 3$
14. $\log_6 x = 3$
15. $\log_x 7 = \frac{1}{2}$
16. $\log_{100} x = -\frac{3}{2}$
17. $\log_x 27 = \frac{3}{4}$
18. $\log_{\sqrt{2}} \frac{1}{4} = x$
19. $\log_x 3 = -2$
20. $\log_{1000} 100 = x$
21. $\log_x \sqrt{5} = \frac{1}{4}$
22. $\log_x 8 = 6$
23. $\log_x \frac{1}{25} = -\frac{2}{3}$
24. $\log_{\frac{1}{8}} 4 = x$

Evaluate.

EXAMPLE 3 $\log_2 (\sqrt{2})^6$

SOLUTION Since $\log_b b^x = x$,

$$\log_2 (\sqrt{2})^6 = \log_2 (2^{\frac{1}{2}})^6$$
$$= \log_2 2^3$$
$$= 3 \quad \text{Answer.}$$

B **25.** $\log_6 6^{10}$ **26.** $\log_2 4^7$ **27.** $\log_{100} 10^8$ **28.** $\log_4 16^9$

Solve for x.

EXAMPLE 4 $\log_{2x} 125 = 3$

SOLUTION Since $\log_{2x} 125 = 3$, $(2x)^3 = 125$

$$8x^3 = 125$$
$$x^3 = \frac{125}{8}$$
$$x = \frac{5}{2} \quad \text{Answer.}$$

29. $\log_{4x} 9 = 2$ **30.** $\log_3 (2x + 1) = 4$ **31.** $\log_2 32 = 2x - 1$

32. $\log_3 (x^2 + 2) = 3$ **33.** $\log_2 16 = 3x - 1$ **34.** $\log_{x^2} 64 = \frac{3}{2}$

C **35.** $\log_2 (\log_2 16) = \log_6 x$ **36.** $\log_2 (\log_9 3) = \log_x 7$

37. $\log_{10} [\log_3 (\log_2 8)] = x$ **38.** $\log_4 (\log_5 25) = \log_3 x$

Self-Test 2

VOCABULARY inverse relations (p. 388) exponential function (p. 390)
one-to-one functions (p. 389) logarithmic function (p. 391)

1. Find $f^{-1}(x)$ if $f(x) = x^3 + 1$. *Obj. 1, p. 388*
2. Is the inverse of the function $\{(x, y): x^2 = y\}$ a function? Give a reason for your answer.
3. Write in logarithmic form: $4^3 = 64$ *Obj. 2, p. 388*
4. Write in exponential form: $\log_4 8 = \frac{3}{2}$

Solve for x.

5. $\log_{10} x = 5$ 6. $\log_x 64 = 2$ *Obj. 3, p. 388*

Check your answers with those at the back of the book.

Using Logarithms

***OBJECTIVES** for Sections 11-5 through 11-10:*
1. *Find the logarithm of a given number and the antilogarithm of a given logarithm.*
2. *Determine the precision and accuracy of a measurement.*
3. *Use logarithms to make calculations.*
4. *Use logarithms to solve exponential equations.*

11-5 Logarithms and Computation

People have always been looking for shorter, easier methods of computation. One method, developed originally to aid astronomers in their complicated and tedious calculations, uses logarithms. Because logarithms are exponents, you can find products (or quotients) of positive numbers by first adding (or subtracting) their logarithms and then finding the number that has the result as its logarithm. This process is justified by the following theorem which gives the **product** and **quotient properties** of logarithms.

> If x_1 and x_2 are positive real numbers, then
>
> **1.** $\log_b x_1 x_2 = \log_b x_1 + \log_b x_2$. **2.** $\log_b \frac{x_1}{x_2} = \log_b x_1 - \log_b x_2$.

To prove the first property, you start with the fact (page 391) that

$$x_1 = b^{\log_b x_1} \quad \text{and} \quad x_2 = b^{\log_b x_2}.$$

Then

$$x_1 x_2 = b^{\log_b x_1} \cdot b^{\log_b x_2}$$

or

$$x_1 x_2 = b^{\log_b x_1 + \log_b x_2}$$

or

$$\log_b x_1 x_2 = \log_b x_1 + \log_b x_2.$$

Property 2 can be established in a similar way (Exercise 27, page 397).

EXAMPLE 1 Express $\log_b \frac{xy}{z}$ without using multiplication or division.

SOLUTION By the quotient property of logarithms: $\log_b \frac{xy}{z} = \log_b xy - \log_b z$.

Then, by the product property:

$$\log_b xy - \log_b z = \log_b x + \log_b y - \log_b z. \quad \text{Answer.}$$

It is easy to find logarithms of integral powers of a base. For example, since $2^3 = 8$, $\log_2 8 = 3$. Logarithms of nonintegral powers of a base can be computed by methods of advanced mathematics to any desired number of decimal places. Some of these values are shown in the following tables:

$b = 2$
$\log_2 1 = 0$
$\log_2 2 = 1$
$\log_2 3 \approx 1.58496$
$\log_2 4 = 2$
$\log_2 5 \approx 2.32193$
$\log_2 6 \approx 2.58496$
$\log_2 7 \approx 2.80736$
$\log_2 8 = 3$
$\log_2 9 \approx 3.16993$
$\log_2 10 \approx 3.32193$

$b = 10$
$\log_{10} 1 = 0$
$\log_{10} 2 \approx 0.301030$
$\log_{10} 3 \approx 0.477121$
$\log_{10} 4 \approx 0.602060$
$\log_{10} 5 \approx 0.698970$
$\log_{10} 6 \approx 0.778151$
$\log_{10} 7 \approx 0.845098$
$\log_{10} 8 \approx 0.903090$
$\log_{10} 9 \approx 0.954242$
$\log_{10} 10 = 1$

Notice in both tables the following illustrations of the product property of logarithms:

$$\log_b (2 \times 3) = \log_b 2 + \log_b 3 = \log_b 6$$
$$\log_b (2 \times 4) = \log_b 2 + \log_b 4 = \log_b 8$$
$$\log_b (2 \times 5) = \log_b 2 + \log_b 5 = \log_b 10$$

Because our numeration system has the base 10, it is convenient to use logarithms to the base 10, which are called **common logarithms.** In this usage, it is customary to omit writing the base 10, and we agree that $\log x$ will mean $\log_{10} x$.

Some other logarithms can be computed easily. For example, since $30 = 3 \times 10$:

$$\begin{aligned} \log 30 &= \log 3 + \log 10 \\ &= \log 3 + 1 \end{aligned}$$

In general, each time you multiply a number by 10, you add 1 to its common logarithm. Thus:

$$\begin{aligned} \log 3 &\approx 0.477121 \\ \log 30 &\approx 1.477121 \\ \log 300 &\approx 2.477121 \\ \log 3000 &\approx 3.477121 \end{aligned}$$
and so on

In order to use logarithms in computation, however, it is necessary to be able to find the approximate logarithm for any positive real number. You will learn how to do that in succeeding sections.

Now you can use the product and quotient properties of logarithms to solve certain logarithmic equations.

EXAMPLE 2 Solve $\log_7(x+1) + \log_7(x-5) = 1$ over $\mathcal{R}$.

SOLUTION

1. Use the product property.
$$\log_7(x+1) + \log_7(x-5) = 1$$
$$\log_7(x+1)(x-5) = 1$$

2. Write the equation in exponential form.
$$(x+1)(x-5) = 7^1$$

3. Solve for x.
$$x^2 - 4x - 5 = 7$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$x = 6 \text{ or } x = -2$$

4. Check.

$x = 6$
$$\log_7(6+1) + \log_7(6-5) \stackrel{?}{=} 1$$
$$\log_7 7 + \log_7 1 \stackrel{?}{=} 1$$
$$1 + 0 \stackrel{?}{=} 1$$
$$1 = 1$$

$x = -2$
$$\log_7(-2+1) + \log_7(-2-5) \stackrel{?}{=} 1$$
$$\log_7(-1) + \log_7(-7) \stackrel{?}{=} 1$$
No, because logarithms of negative numbers are not defined.

$\therefore$ the solution set is $\{6\}$. Answer.

Oral Exercises

Express as the logarithm to base 10 of a single number.

1. $\log 3 + \log 2$
2. $\log 5 + \log 10$
3. $\log 12 - \log 3$
4. $\log 9 - \log 6$
5. $(\log 7) + 2$
6. $(\log 30) - 1$

Written Exercises

Use the table of logarithms to base 10 on page 395 to find a numerical value for each of the following

A

1. $\log_{10} 12$
2. $\log_{10} 25$
3. $\log_{10} 400$
4. $\log_{10} \frac{5}{2}$
5. $\log_{10} \frac{14}{3}$
6. $\log_{10} 84$
7. $\log_{10} 15{,}000$
8. $\log_{10} 0.21$

Express as the logarithm to base 2 of a single number; then use the table on page 395 to evaluate.

9. $\log_2 39 - \log_2 13$
10. $\log_2 55 - \log_2 11$
11. $\log_2 \frac{1}{12} + \log_2 72$
12. $\log_2 \frac{14}{3} + \log_2 \frac{3}{2}$
13. $\log_2 \frac{14}{3} + \log_2 \frac{12}{7}$
14. $\log_2 17 - \log_2 1.7$

Use the product and quotient properties of logarithms to solve the given equation.

15. $\log 6 + \log x = \log 3$
16. $\log y - \log 5 = \log 7$
17. $\log z + \log 5 = 3$
18. $\log 4t^3 - \log t = 2$
19. $\log_2 x^3 - \log_2 27 = 3$
20. $\log_5 8x + \log_5 x^2 = 3$

Use the product and quotient properties of logarithms to solve the given equation.

B 21. $\log_{10} x + \log_{10} (x - 3) = 1$

22. $\log_6 (z + 2) - \log_6 (z - 3) = 1$

23. $\log_2 (x - 5) + \log_2 (x - 1) = 5$

24. $\log_5 2x^2 - \log_5 (x + 5) = 1$

25. $\log_8 (x + 6) + \log_8 (x - 6) = 2$

26. $\log_3 (x^2 - 1) - \log_3 (5x + 5) = 0$

C 27. Prove that for positive numbers x_1 and x_2,

$$\log_b \frac{x_1}{x_2} = \log_b x_1 - \log_b x_2.$$

28. Prove that for any positive number x, $\log_3 x = 2 \log_9 x$. (*Hint:* Let $\log_9 x = L$, and write in exponential form.)

11-6 Using a Table of Common Logarithms

Tables of logarithms have been computed to various numbers of decimal places, but we shall use a table (Table 5 in the Appendix) that gives the first four decimal places of common logarithms.

There is a traditional arrangement of logarithmic tables, which you have to learn to read. The first few lines of Table 5 are:

x	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732

Notice that no decimal points are given. You read the digits from the table, but you have to put the decimal points in the correct places yourself.

For example, to find an approximate value for log 1.24, you find 12 in our column under x. You then move along row 12 to the column headed 4, where you find 0934. This means that

$$\log 1.24 \approx 0.0934.$$

From this, you see that, since $12.4 = 1.24 \times 10$, you have by the product property of logarithms (page 394), $\log 12.4 = \log 1.24 + \log 10$, or

$$\log 12.4 \approx 1.0934.$$

Here are four more examples:

$$
\begin{aligned}
124 &= 1.24 \times \mathbf{10^2}; & \log 124 &\approx \mathbf{2}.0934\\
12400 &= 1.24 \times \mathbf{10^4}; & \log 12400 &\approx \mathbf{4}.0934\\
0.0124 &= 1.24 \times \mathbf{10^{-2}}; & \log 0.0124 &\approx \mathbf{-2} + 0.0934\\
0.000124 &= 1.24 \times \mathbf{10^{-4}}; & \log 0.000124 &\approx \mathbf{-4} + 0.0934
\end{aligned}
$$

Notice that the logarithm of a number is the sum of two parts:

(1) a part between 0 and 1, called the **mantissa,** which is nonnegative and is found from Table 5, and
(2) an integral part, called the **characteristic,** which is the exponent of 10 when the number is expressed in standard (or scientific) notation.

To maintain this pattern, we usually do not simplify sums involving negative characteristics. That is, we do not change $-2 + 0.0934$ to obtain $\log 0.0124 \approx -1.9066$. Instead, we write "$-2 = 8.0000 - 10$," giving us

$$\log 0.0124 \approx \mathbf{8}.0934 - \mathbf{10}.$$

Similarly,

$$\log 0.000124 \approx \mathbf{6}.0934 - \mathbf{10}.$$

Usually the difference with -10 is used, although occasionally it is helpful to use some other difference, for example, $18.0000 - 20$ instead of $8.0000 - 10$.

In general, one may say that Table 5 gives approximate values of the *logarithms* of numbers from 1.000 to 9.999, or of the *mantissas* of other numbers.

EXAMPLE 1 Find log 1470.

SOLUTION We first write $1470 = 1.47 \times \mathbf{10^3}$. So the characteristic is 3. Next, we examine the left-hand column for 14 and the top row for 7, and find that the mantissa is 0.1673.

$\therefore\ \log 1470 \approx 3 + 0.1673 = 3.1673$. Answer.

You can reverse the foregoing procedure to find an approximation for a number with a given logarithm. If $\log x = a$, then x is called the **antilogarithm** of a, abbreviated "**antilog *a*.**"

EXAMPLE 2 Find antilog $9.1523 - 10$.

SOLUTION First find the mantissa, 0.1523, in the body of the table. Then read the first two digits, 14, directly across in the left-hand column. Going directly up from 0.1523, you find the third digit, 2, in the top row. Since the characteristic is -1, you have

$$\text{antilog } 9.1523 - 10 \approx 1.42 \times 10^{-1} = 0.142. \quad \text{Answer.}$$

Oral Exercises

Express the given number in standard notation. (See page 269.)

1. 27
2. 814
3. 0.635
4. 9.7
5. 0.0038
6. 5,210,000
7. 0.0000491
8. 0.0605

State the digits of the given antilogarithm from the table on page 397.

9. antilog 2.1553
10. antilog 4.0128
11. antilog 0.1004
12. antilog 9.1399 − 10
13. antilog 5.0607
14. antilog 7.1703 − 10

Written Exercises

Find each logarithm. Use Table 5.

A

1–8. Use the numbers in Oral Exercises 1–8 above.

9. log 82.3
10. log 10,400,000
11. log 0.007
12. log 0.0395
13. log 190
14. log 0.000528

Find each antilogarithm. Use Table 5.

15–20. Use the antilogarithms in Oral Exercises 9–14 above.

21. antilog 1.8463
22. antilog 8.9031 − 10
23. antilog 6.8041 − 10
24. antilog 6.5340
25. antilog 4.7284
26. antilog 9.8500 − 10

Use Table 5 to express each log as a single number.

B

27. log 25 + log 624
28. log 88 + log 0.325
29. log 7150 − log 5.5
30. log 0.077 − log 28

Solve for x.

C

31. $\log 3 + \log x = 1.7324$
32. $\log x - \log 0.6 = 2.7033$
33. $\log x^2 + \log 7 = 8.4014 - 10$
34. $\log 7.29 - \log x^2 = 9.9085 - 10$
35. $\dfrac{(x-2)\log x}{x^2} > 0$
36. $\dfrac{\log(x-1)}{x+3} \leq 0$

11-7 Interpolation

Table 5 gives direct readings for the logarithms of numbers with numerals having *at most three* significant digits (page 269). A method of estimating logarithms for numbers with four significant digits is given in the following example.

EXAMPLE 1 Find log 1.374.

SOLUTION You can find entries for log 1.370 and log 1.380 but not for log 1.374. You do know, however, that

$$\log 1.370 < \log 1.374 < \log 1.380.$$

Thus,

$$0.1367 < \log 1.374 < 0.1399.$$

If the graph of this logarithmic function were a straight line ($\overleftrightarrow{PR}$ in Figure 10), you could find log 1.374 by using a proportion. The graph, however, is close enough to a straight line over much of its range for us to make the assumption that

the ordinate h of point C'

on the line is an acceptable approximation of

log 1.374,

the ordinate of point C on the curve.

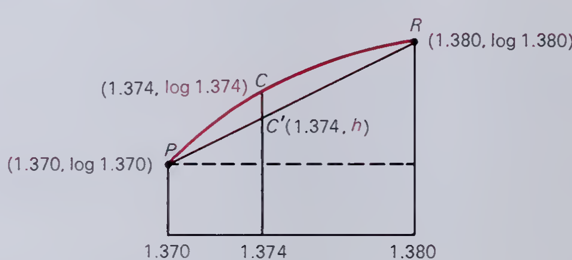


Figure 10

Because P, C', and R are collinear,

$$\underbrace{\text{the slope of segment } PC'} = \underbrace{\text{the slope of segment } PR}$$

$$\frac{h - \log 1.370}{1.374 - 1.370} = \frac{\log 1.380 - \log 1.370}{1.380 - 1.370}$$

$$\frac{h - \log 1.370}{0.004} = \frac{\log 1.380 - \log 1.370}{0.010}$$

$$\therefore\ h \approx \log 1.370 + \tfrac{4}{10}(\log 1.380 - \log 1.370).$$

This means that we assume that because

1.374 is $\frac{4}{10}$ of the way from 1.370 to 1.380,

log 1.374 is $\frac{4}{10}$ of the way from log 1.370 to log 1.380.

Letting $d = \log 1.374 - \log 1.370 = \frac{4}{10}(\log 1.380 - \log 1.370)$, you can arrange the work as follows:

		x	$\log x$		
0.010		1.380	0.1399		0.0032
	0.004	1.374	log 1.374	d	
		1.370	0.1367		

$d \approx \frac{4}{10} \times 0.0032 \approx 0.0013$ (rounded to four places because mantissas in Table 5 are reliable only to four places).

$\therefore \log 1.374 \approx 0.1367 + 0.0013$, or $\log 1.374 \approx 0.1380$. Answer.

The process just described is *linear interpolation* (page 335). It enables you to use a four-place table of logarithms to approximate the logarithm of a number whose decimal numeral is known to four significant digits, with the result usually correct to four decimal places. To approximate the logarithm of a number whose numeral is known to more than four digits, you first round the numeral for the given number to four digits (page 269), and then find the logarithm of the result.

You can also interpolate in reverse to find antilog x if $\log x$ is known and its mantissa is not an entry in the table.

EXAMPLE 2 Find antilog 2.3176.

SOLUTION Ignore the characteristic for now, and locate 0.3176 between entries in Table 5. You find that 0.3160 and 0.3181 are table entries, with antilogarithms 2.07 and 2.08 respectively. Then arrange these facts as shown below:

		x	$\log x$		
0.010		2.080	0.3181		0.0021
	d'	antilog 0.3176	0.3176	0.0016	
		2.070	0.3160		

$$\frac{d'}{0.010} = \frac{0.0016}{0.0021} = \frac{16}{21} \quad \text{or} \quad d' = 0.010 \cdot \left(\frac{16}{21}\right) \approx 0.008.$$

$\therefore$ antilog $0.3176 \approx 2.070 + 0.008 = 2.078$.

Next, note that the characteristic of the given logarithm is 2. Hence,

antilog $2.3176 \approx 207.8$. Answer.

Notice that d' was rounded to one significant digit because reverse interpolation in a four-place table yields at most four significant digits for the antilogarithm.

Oral Exercises

State the inequality you would use as the first step in interpolating to find the logarithm of the number.

EXAMPLE 2.735 **SOLUTION** $\log 2.730 < \log 2.735 < \log 2.740$

1. 1.453
2. 7546
3. 536.5
4. 0.3464

Written Exercises

Find the logarithms of the numbers.

A 1–4. Use the antilogarithms in Oral Exercises 1–4.

5. 0.05929
6. 88.92
7. 65,270
8. 0.006298
9. 6002
10. 1.256
11. 120,700
12. 0.01972

Find the antilogarithms of the numbers.

13. 1.1620
14. 3.6769
15. 8.9651 − 10
16. 9.8230 − 10
17. 2.5072
18. 0.5485
19. 7.8847 − 10
20. 4.7714
21. 9.7508 − 10
22. 1.3239
23. 8.1964 − 10
24. 3.4244

B 25. Find log 1.145 by interpolating between log 1.14 and log 1.15 (the usual way), and then find it again by interpolating between log 1.11 and log 1.18. Which answer is greater?

C 26. Are logarithms calculated by interpolation usually too great or too small? Why?

27. Are antilogarithms calculated by interpolation usually too great or too small? Why?

11-8 Computing Products and Quotients

In computing with four-place logarithms, results can be given to at most four significant digits. If the numbers in the computation are approximations, the accuracy of the result may be further restricted.

Measurements always produce approximations. A measurement is made by comparing some quantity with a measuring device which has a numerical scale. The measurement may be read as a number of the smallest units on the scale.

The **precision** of a measurement is defined to be the *unit* used in making it. For example, a botanist reporting the height of a sapling as 1.9 m is measuring its height precise to the nearest 0.1 m. The maximum possible error is half the unit of precision. In this example it is 0.05 m. Thus, the true height of the sapling, h, satisfies the following inequality:

$$1.85 \leq h < 1.95.$$

Another aspect of measurement is its **accuracy.** The accuracy of a measurement is the relative error. It is usually expressed as the percent the precision is of the actual measurement. The accuracy of the botanist's measurement may be found as follows:

$$\frac{0.1}{1.9} \approx 0.053 = 5.3\%$$

The accuracy varies with the size of the particular measurement. For example, another sapling measured was 0.5 m high. Although the precision of the measurement is still 0.1 m, its accuracy is 20%.

In sum, the precision of a measurement is determined by how the measurement is taken. The accuracy is the relative error, which depends on the precision and the size of the particular measurement.

It is easy to tell the number of significant digits if a measurement is given in standard (scientific) notation (Section 8-5). For example, instead of 200 meters, you would write

$$2 \times 10^2, \qquad 2.0 \times 10^2, \qquad \text{or} \qquad 2.00 \times 10^2,$$

according as the measurement is precise to the nearest hundred meters, the nearest ten meters, or the nearest meter.

For computations in general, we use the following working rules:

To Round the Numerals for Results:

1. Give products, quotients, and powers to the same number of *significant digits* as appear in the *least accurate* approximation involved.
2. Give sums and differences to the same number of decimal places as appear in the approximation with the *least number* of decimal places (the least *precise* measurement).

Since there may be some doubt about the fourth digit obtained by using four-place logarithms, we shall usually round the result to three significant digits.

To guard against errors in computation, you should adopt two practices as a matter of routine.

First, make an estimate of the result. This guards against such gross errors as misplacing a decimal point.

Second, plan and arrange your work systematically and neatly. This makes it easier to check for errors. A practice to follow is always to arrange your work in vertical columns, aligning equality signs vertically and keeping decimal points directly under one another. You can indicate the operations (+) or (−) and put in the characteristics in advance. Also, label the steps of your computation, so that if you have to check back, you will know what you are checking.

EXAMPLE 1 Compute $\dfrac{438 \times 0.410}{2.32}$.

SOLUTION Let $N = \dfrac{438 \times 0.410}{2.32}$.

1. Estimate N.

$$N \approx \frac{400 \times 0.4}{2} = \frac{4 \times 10^2 \times 4 \times 10^{-1}}{2} = \frac{160}{2} = 80.$$

2. Write an equation showing the plan of work.

$$\log N = \log 438 + \log 0.41 - \log 2.32$$

3. Find $\log N$.

$$\begin{aligned}
\log 438 &\approx 2.6415 \\
\log 0.410 &\approx \underline{9.6128 - 10} \quad (+) \\
& \ 12.2543 - 10 \\
\log 2.32 &\approx \underline{0.3655 } \quad (-) \\
\log N &\approx 11.8888 - 10
\end{aligned}$$

4. Find N. $N \approx \text{antilog } 11.8888 - 10 = \text{antilog } 1.8888 \approx 77.42$.

Note that the value arrived at is in reasonable accord with your estimate.
$\therefore$ to three significant digits, the result is 77.4. Answer.

Because logarithms are defined only for positive numbers, if negative numbers are involved in a calculation, you simply first determine by inspection whether the result is a positive or a negative number, and then perform the calculation using absolute values.

EXAMPLE 2 Compute $\dfrac{7.08}{-15.9}$.

SOLUTION Let $N = \dfrac{7.08}{-15.9}$ and note that N is negative.

1. Estimate N. $\dfrac{7.08}{-15.9} \approx -\dfrac{7}{16} \approx -0.5$

2. Use $\log |N| = \log 7.08 - \log 15.9$.

3. $$\begin{aligned}
\log 7.08 &\approx 0.8500 \\
\log 15.9 &\approx 1.2014 \quad (-)
\end{aligned}$$

Since $1.2014 > 0.8500$, to avoid a negative mantissa we replace 0.8500 with $10.8500 - 10$.

$$\begin{aligned}
\log 7.08 &\approx 10.8500 - 10 \\
\log 15.9 &\approx \underline{1.2014 } \quad (-) \\
& \ \ 9.6486 - 10 \\
\therefore \log |N| &\approx \ \ 9.6486 - 10
\end{aligned}$$

4. $|N| \approx \text{antilog } 9.6486 - 10 \approx 0.4452$ and $N \approx -0.4452$.

Note that the result is in reasonable accord with the estimate.

$\therefore$ to three significant digits, $\frac{7.08}{-15.9} \approx -0.445$. Answer.

Oral Exercises

State the logarithmic equation you would use to compute each of the following.

1. $(7.3)(180)$
2. $(45.1)(3090)$
3. $(-226)(0.985)$
4. $(0.0720)(69.4)$
5. $(-5700)(-0.0031)$
6. $\frac{8200}{6.9}$
7. $\frac{0.437}{-16.2}$
8. $\frac{-78}{-0.392}$
9. $\frac{0.0266}{0.913}$

Written Exercises

Compute each of the following using logarithms.

A **1–9.** The product or quotient in Oral Exercises 1–9.

10. $(5.23)(348)(0.266)$

11. $(9790)(0.428)(0.0118)$

12. $\frac{(49.1)(154)}{723}$

13. $\frac{(-662)(0.251)}{29.0}$

14. $\frac{-429}{(56.3)(-3.72)}$

15. $\frac{(11.6))(203)}{(174)(0.926)}$

16. $(221.5)(8.156)$

17. $\frac{4852}{43.27}$

18. $\frac{0.2637}{0.006925}$

B **19.** Given that $\log 2 \approx 0.3010$ and $\log 3 \approx 0.4771$, state an equation you can use to find approximations of the logarithms of each of the following numbers without using a table. Then find the approximations.

a. 4 b. 8 c. 12 d. $\frac{1}{2}$ e. 5 f. 15

20. Show that for any positive number y, $\log y^2 = 2 \log y$, and hence $\log \sqrt{y} = \frac{1}{2} \log y$ for any real number y.

21. Using only the information given in Exercise 19 and the facts that (1) $7^2 \approx 48$ and (2) $11^2 \approx 120$, find approximations of log 7 and log 11 without using tables. (*Hint:* Use the results of Exercise 20.)

C **22.** Express the following equations in terms of the logarithms of the variables.

a. Direct variation: $y = kx$

b. Joint variation: $z = kxy$

c. Inverse variation: $y = \frac{k}{x}$

d. Combined variation: $z = \frac{kx}{y}$

11-9 Computing Powers and Roots

To discover another very useful property of logarithms, recall that

$$x = b^{\log_b x}$$

so that

$$x^n = (b^{\log_b x})^n = b^{n \log_b x}$$

Writing this latter exponential equation in logarithmic form produces the **power property** of logarithms.

> If $x > 0$ and $n \in \mathcal{R}$, then $\log_b x^n = n \log_b x$.

To illustrate this property, look at these tables:

$b = 2$
$\log_2 1 = 0$
$\log_2 2 = 1$
$\log_2 3 \approx 1.58496$
$\log_2 4 = 2$
$\log_2 5 \approx 2.32193$
$\log_2 6 \approx 2.58496$
$\log_2 7 \approx 2.80736$
$\log_2 8 = 3$
$\log_2 9 \approx 3.16993$
$\log_2 10 \approx 3.32193$

$b = 10$
$\log_{10} 1 = 0$
$\log_{10} 2 \approx 0.301030$
$\log_{10} 3 \approx 0.477121$
$\log_{10} 4 \approx 0.602060$
$\log_{10} 5 \approx 0.698970$
$\log_{10} 6 \approx 0.778151$
$\log_{10} 7 \approx 0.845098$
$\log_{10} 8 \approx 0.903090$
$\log_{10} 9 \approx 0.954242$
$\log_{10} 10 = 1$

Notice that in either table:

$$\log_b 2^2 = 2 \log_b 2 = \log_b 4$$
$$\log_b 2^3 = 3 \log_b 2 = \log_b 8$$
$$\log_b 3^2 = 2 \log_b 3 = \log_b 9$$

This property of logarithms is a very powerful aid in estimating values that are otherwise hard to obtain.

EXAMPLE 1 Find $\sqrt[12]{2}$.

SOLUTION Let $N = \sqrt[12]{2} = 2^{\frac{1}{12}}$.

1. Use: $\log N = \frac{1}{12} \log 2$
2. Compute: $\log 2 \approx 0.3010$
   $12 \quad (\div)$
   $\log N \approx 0.0251$
3. $N = \text{antilog } 0.0251 \approx 1.060$

$\therefore$ to three significant digits, $\sqrt[12]{2} \approx 1.06$. Answer.

To find a root of a number between 0 and 1, you must adjust the negative portion of the logarithm to fit the division, as in the following example.

EXAMPLE 2 Find $\sqrt[3]{0.83}$.

SOLUTION Let $N = \sqrt[3]{0.83}$

1. Estimate N. $\sqrt[3]{0.83} \approx \sqrt[3]{0.729} = 0.9$
2. Use the following equation:

$$\log N = \tfrac{1}{3}\log 0.83 = \tfrac{1}{3}(9.9191 - 10)$$

Here 10 is not a multiple of 3, so use

$$\tfrac{1}{3}(8.9191 - 9) \text{ or } \tfrac{1}{3}(29.9191 - 30)$$

(or some other equivalent form). Thus,

$$\log N = 2.9730 - 3, \text{ or } 9.9730 - 10.$$

In either case, the mantissa is 0.9730 and the characteristic is -1.

3. $N = \text{antilog}\,(2.9730 - 3) = 0.9398$. This result is in reasonable agreement with the estimate.

$\therefore$ to three significant digits, $\sqrt[3]{0.83} \approx 0.940$. Answer.

EXAMPLE 3 Compute $\sqrt[3]{(0.831)^2}$.

SOLUTION Let $N = \sqrt[3]{(0.831)^2}$.

1. Estimate N. $\sqrt[3]{(0.8)^2} = \sqrt[3]{0.64} = \sqrt[3]{640 \times 10^{-3}}$

$$\approx \sqrt[3]{9^3 \times 10^{-3}} = 9 \times 10^{-1} = 0.9$$

2. Use the following equation:

$$\log N = \log\,(0.831)^{\frac{2}{3}} = \tfrac{2}{3}\log\,(0.831)$$

3. Compute: $\log\,(0.831) \approx$

$$\begin{array}{rl} 8.9196 - 9 & \\ 2 & (\times) \\ \hline 17.8392 - 18 & \\ 3 & (\div) \\ \hline 5.9464 - 6 & \end{array}$$

$\therefore \log N \approx 5.9464 - 6$

4. $N \approx \text{antilog}\,(5.9464 - 6) \approx 0.8838$. This result is in reasonable agreement with the estimate.

$\therefore$ to three significant digits, $\sqrt[3]{(0.831)^2} \approx 0.884$. Answer.

EXAMPLE 4 If \$1000 is deposited in a savings account paying 6% interest compounded quarterly, what will the account amount to in 10 yr if no deposits or withdrawals are made? Use the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where P represents the principal invested at $100r$ percent annual interest (r is expressed as a decimal), n represents the number of times the interest is compounded during a year, and A represents the amount accumulated after t yr.

SOLUTION

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$P = 1000,\ r = 0.06,\ n = 4, \text{ and } t = 10$$

$$\therefore A = 1000\left(1 + \frac{0.06}{4}\right)^{4(10)}$$

$$A = 1000(1.015)^{40}$$

$$\log A = \log 1000 + 40 \log 1.015$$

$$\begin{array}{rrl} \log 1.015 \approx & 0.0065 & \\ & \underline{40} & (\times) \\ & .2600 & \\ \log 1000 = & \underline{3.0000} & (+) \\ & 3.2600 & \end{array}$$

$\therefore \log N \approx 3.2600$ and $N \approx \text{antilog } 3.2600 \approx 1820$.
$\therefore$ there will be approximately \$1800 after 10 yr. Answer.

Note: A more accurate result can be obtained if log 1.015 can be found to more places, since any "error" is multiplied by 40. To four significant digits, $\log 1.015 \approx 0.006466$. Using this value, you can find that the result is approximately \$1814.

Oral Exercises

State the logarithm of the number in terms of the base.

1. $(3.76)^4$ **2.** $(2.16)^5$ **3.** $(48.6)^{\frac{1}{3}}$ **4.** $(74.1)^{\frac{3}{2}}$

State the logarithm of the indicated root.

5. $\sqrt{661}$ **6.** $\sqrt[5]{8580}$ **7.** $\sqrt[3]{65,000}$ **8.** $\sqrt{0.335}$

Written Exercises

Compute by logarithms. Round results to three significant digits.

A **1–8.** Use Oral Exercises 1–8.

Compute by logarithms. Round results to three significant digits.

9. $\sqrt[6]{0.0955}$
10. $\sqrt[4]{0.801}$
11. $\sqrt[10]{70.3}$
12. $\sqrt[12]{17{,}800}$
13. $(27.9)^3(0.163)$
14. $\dfrac{1240}{(6.29)^4}$
15. $\dfrac{\sqrt[3]{23{,}300}}{(84.5)^2}$
16. $\dfrac{(0.648)^3}{\sqrt{0.079}}$
17. $\sqrt[10]{687}\,\sqrt[6]{0.00342}$
18. $\sqrt[4]{\dfrac{916}{34.5}}$
19. $\sqrt{(13.7)(0.44)^3}$
20. $\sqrt[3]{\dfrac{8.12}{57.4}}$
21. $\dfrac{\sqrt[3]{48}(29.1)^4}{6160}$
22. $\sqrt[5]{\dfrac{14{,}000}{(0.531)^3(725)}}$
23. $\sqrt[7]{\dfrac{(8.82)^5(0.0054)}{767}}$
24. $\sqrt[3]{\dfrac{\sqrt{895}(1.03)^5}{(17.8)^2}}$

In Exercises 25–30, assume $\log x = 1.8$ and $\log y = 0.4$. Solve for z without using a table of logarithms. (*Hint:* Take logs of both sides.)

B 25. $z = \sqrt{x^2y}$
26. $z = \dfrac{\sqrt[3]{x^4}}{y}$
27. $z = (\sqrt[3]{x}\,\sqrt{y})^5$
28. $z = \dfrac{x^4\sqrt{y}}{y}$
29. $z = \dfrac{\sqrt[3]{x}}{y^4}$
30. $z = \sqrt{\dfrac{x^4}{y^3}}$

Solve for x without using a table of logarithms.

EXAMPLE $\log x = 2(\log 45 - \log 5) + \frac{1}{2}\log 64$

SOLUTION $x = (45 \div 5)^2 \times 64^{\frac{1}{2}} = 81 \times 8 = 648$

31. $\log x = \frac{1}{2}(\log 63 - \log 7) + 2\log 5$
32. $\log x = \frac{1}{3}(\log 45 - 2\log 3 + \log 25)$
33. $\log x = \frac{1}{2}(\frac{1}{2}\log 16 + \log 12) - \frac{1}{3}(\log 25 + \log 40)$
34. $\log x = 2(\log 15 - 2\log\sqrt{5}) - \frac{3}{2}(\log\sqrt[3]{2} + \log 3 + \log\sqrt[3]{6})$

C 35. Show that $\log\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2\log(x + \sqrt{x^2 - 1})$.

36. Show that $\dfrac{\log(x + h) - \log x}{h} = \log\left(1 + \dfrac{h}{x}\right)^{\frac{1}{h}}$.

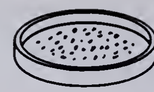
37. Show that $\log\dfrac{y}{a + \sqrt{a^2 + y^2}} = \log\dfrac{\sqrt{a^2 + y^2} - a}{y}$.

Problems

A 1. Find the amount after 20 yr in an account in which \$5000 is invested at a rate of 10%, compounded semiannually.

2. Find the amount after 9 yr in an account in which \$3000 is invested at 8%, compounded quarterly.

3. The number of bacteria N in a culture is given by the equation $N = 450{,}000(2)^{0.25t}$, where 450,000 bacteria were initially present and t is the time elapsed expressed in minutes. How many bacteria will there be in the culture after 0.8 min?



4. The amount N of carbon-14 remaining after t yr of radioactive decay is given by the equation $N = N_0(0.97)^{0.004t}$, where N_0 is the initial amount. Of 2 g initially present, how much will remain after 7500 yr?

5. The population P of a certain town increases according to the formula $P = 5000(1.1)^{0.2t}$, where t represents number of years after 1980. If this trend continues, what will the population be in 2028?

B 6. The period of revolution T of an artificial earth satellite in hours is given by

$$T = \frac{2\pi}{3600}\sqrt{\frac{r^3}{GM}},$$

where r is the distance to the satellite from the center of the earth. If $G = 6.67 \times 10^{-11}$, $M = 5.97 \times 10^{24}$ kg, the mass of the earth, and $r = 6.59 \times 10^6$ m, find the period of the satellite.

programming in BASIC

In the BASIC language, you can use exponents written in fractional or decimal form. For example:

```
10 PRINT 2↑(1/2);2↑.5;2↑1.567
20 END
RUN

 1.41421  1.41421  2.96288
END
```

Notice that a fractional exponent must be enclosed in parentheses because of the order of operations described on page 97.

Exercises

1. Use the computer to find $2^{\frac{1}{12}}$, $0.83^{\frac{1}{3}}$, and $0.831^{\frac{2}{3}}$. Compare the results with those of Examples 1–3 (pages 406–407).
2. $\log_{10} 2 = .301030$ means 10↑.301030 = 2. Test this by having the computer find 10↑.301030.
3. Test 2↑3.32193.
4. Check the results of Exercises 2 and 3 with the tables on page 406.
5. Check the value of $2^{\sqrt{3}}$ given on page 385.
6. Use a computer to check your work in the exercises on pages 408–409.

11-10 Solving Equations Using Logarithms

You can use logarithms to find approximate solutions for some exponential equations over $\mathcal{R}$.

EXAMPLE 1 Solve $x^{\frac{3}{7}} = 4.63$ over $\mathcal{R}$ and express the solution to three significant digits.

SOLUTION

1. Raise both sides to the $\frac{7}{3}$ power. $\qquad x^{\frac{3}{7}} = 4.63$, $\quad x = (4.63)^{\frac{7}{3}}$
2. Use one of the properties on page 391 to equate **common logarithms**. $\qquad \log x = \log (4.63)^{\frac{7}{3}}$
3. Apply the power property of logarithms to simplify the right member and solve for $\log x$. $\qquad \log x = \frac{7}{3}\log 4.63$, $\quad \log x \approx \frac{7}{3}(0.6656)$, $\quad \log x \approx 1.5531$
4. Find x. $\qquad x \approx \text{antilog } 1.5531 \approx 35.73$

$\therefore$ the solution set is $\{35.7\}$. Answer.

EXAMPLE 2 A culture of bacteria contains N bacteria after t h according to

$$N = N_0(2.72)^{0.04t}$$

where N_0 is the number present originally. How long will it take for 10,000 bacteria to multiply to 30,000?

SOLUTION Substitute 30,000 for N and 10,000 for N_0.

$$30{,}000 = 10{,}000(2.72)^{0.04t}$$

Divide each member by 10,000.

$$3 = (2.72)^{0.04t}$$

Equate common logarithms.

$$\log 3 = \log (2.72)^{0.04t}$$

Apply the power property of logarithms and solve for t.

$$\log 3 = 0.04t \log 2.72$$

$$t = \frac{\log 3}{(0.04)(\log 2.72)}$$

$$t \approx \frac{0.4771}{(0.04)(0.4346)}$$

(Solution continued on page 412.)

$\log t \approx \log 0.4771 - \log 0.04 - \log 0.4346$

$$\begin{array}{rrl} \log 0.4771 \approx & 9.6786 - 10 & \\ \log 0.04 \approx & \underline{8.6021 - 10} & (-) \\ & 1.0765 & \\ \text{Change to} & 11.0765 - 10 & \\ \log 0.4346 \approx & \underline{9.6381 - 10} & (-) \\ & 1.4384 & \end{array}$$

$\log t \approx 1.4384$

$t \approx \text{antilog } 1.4384 \approx 27.4$

$\therefore$ it would take approximately 27.4 hours for the number of bacteria to multiply from 10,000 to 30,000. Answer.

EXAMPLE 3 Express $\log_5 11$ in terms of common logarithms.

SOLUTION Let $N = \log_5 11$.

1. Write in exponential form. $\quad 5^N = 11$
2. Equate common logarithms. $\quad \log 5^N = \log 11$
3. Apply the power property of logarithms and solve for N. $\quad N \log 5 = \log 11$

$$N = \frac{\log 11}{\log\ 5}$$

$\therefore \log_5 11 = \dfrac{\log 11}{\log\ 5}$. Answer.

The result of Example 3 suggests the following:

Relationship Between the Logarithms of a Number n to Two Different Bases a and b

$$\log_b n = \frac{\log_a n}{\log_a b} \quad (n > 0, a > 0, a \neq 1, b > 0, b \neq 1)$$

Can you justify the steps in the following proof of this relationship?

$$\begin{aligned} \text{Let } x &= \log_b n. \\ b^x &= n \\ \log_a b^x &= \log_a n \\ x \log_a b &= \log_a n \\ x &= \frac{\log_a n}{\log_a b} \\ \text{or } \log_b n &= \frac{\log_a n}{\log_a b} \end{aligned}$$

In particular, if $n = a$ in the preceding equation, you obtain, since $\log_a a = 1$:

$$\boxed{\log_b a = \frac{1}{\log_a b} \quad (a > 0, a \neq 1, b > 0, b \neq 1)}$$

For example, using the values in the tables on page 395 and a little arithmetic, you can write

$$\log_2 6 = \frac{\log_{10} 6}{\log_{10} 2} = \frac{0.778151}{0.301030} \approx 2.58496.$$

Another very important base for logarithms is the irrational number e ($e \approx 2.7182818$) which arises naturally in many practical situations and is used a great deal in more advanced mathematics. To find $\log_e x$ given a table of common logarithms, you simply use

$$\log_e x = \frac{\log_{10} x}{\log_{10} e}.$$

Since $\log_{10} e \approx 0.4343$, you have

$$\log_e x = \frac{\log_{10} x}{0.4343} = 2.303 \log_{10} x.$$

Oral Exercises

State the equation you would use as the first step in solving for x.

1. $3^x = 560$
2. $5.4^x = 133$
3. $25^x = 3.29$
4. $6.78^x = 83.2$
5. $x^{3.5} = 1950$
6. $x^{1.8} = 7.62$
7. $x^{1.2} = 6.9$
8. $x^{\frac{5}{3}} = 22.7$

Written Exercises

Solve over $\mathcal{R}$. Use Table 5 as needed and express each solution to three significant figures.

A

1–8. Solve Oral Exercises 1–8.

9. $7(4^x) = 30$
10. $106(3.1)^x = 180$
11. $9(2^{-x}) = 4$
12. $3(9.12)^{2x} = 5.77$
13. $8x^{\frac{3}{2}} = 86.3$
14. $12\sqrt[3]{x^4} = 4.98$
15. $6^{3.2x} = 97.1$
16. $(0.28)^{\frac{3}{4}}(x + 1) = 50$

Approximate to three significant digits.

17. $\log_{3.02} 8.63$
18. $\log_5 8.5$
19. $\log_{2.6} 9.1$
20. $\log_e 5$

Solve over $\mathcal{R}$. Express each solution to three significant digits.

B 21. $10^{x-2} = 355$ 22. $10^{4x+1} = 879$ 23. $5^{2x-1} = 8.12$

24. $3^{x-2} = 2^x$ 25. $7^{2x} = 5.2^{x-1}$ 26. $6^{x+1} = 4.5^{3x-1}$

Use the relationship on page 412 to prove each of the following. Do not use a table of logarithms.

C 27. $\log_9 2 = \dfrac{\log_3 2}{2}$ 28. $\log_e 25 = \dfrac{2 \log 5}{\log e}$

29. $\log_b x = 2 \log_{b^2} x$ 30. $\log_b x = n \log_{b^n} x$

31. $\log_{ab} x = \dfrac{\log_a x}{1 + \log_a b}$ 32. $\log_{\frac{a}{b}} x = \dfrac{\log_a x}{1 - \log_a b}$

33. $(\log_a x)(\log_b y) = (\log_b x)(\log_a y)$ (*Hint:* Consider $\log_x y$.)

Problems

A 1. How long will it take for \$1000 invested at 8%, compounded semiannually, to grow to \$3000?

2. What amount of money must be invested at 8% compounded quarterly in order for the account to contain \$7000 after 13 yr?

3. A certain radioactive element decays according to the formula

$$N = N_0(\tfrac{1}{2})^{0.25t},$$

where t is the elapsed time in minutes, N_0 the initial amount, and N the final amount. How long will it take for 100 g of the element to decay to 21 g?

4. The amount N in a bank account that pays 5% interest compounded instantaneously can be given by the formula $N = N_0e^{0.05t}$, where t is in years and N_0 is the amount originally deposited. According to this formula, how long will it take \$2000 to grow to \$6000? (Use $e \approx 2.72$.)

B 5. A savings bond pays \$4 after 7.5 yr for every \$3 originally invested. To what rate of interest, compounded quarterly, is this equivalent?

programming in BASIC

The BASIC language has a built-in function LOG(X). The base of this function, however, is *e* (see page 413). To find logarithms to another base, *b*, use the formula on page 412:

$$\log_b n = \text{LOG(N) / LOG(B)} \quad (b > 0, b \neq 1, n > 0).$$

Exercises

1. Use the computer to check Example 1, page 411. (That is, compute 4.63↑(7/3).)
2. Use the computer to check Example 2, page 411, by finding

   LOG(3) / (0.04*LOG(2.72)).

3. Prove that

$$\frac{\log_a m}{\log_a n} = \frac{\log_b m}{\log_b n}$$

$(m > 0, n > 0, n \neq 1, a > 0, a \neq 1, b > 0, b \neq 1)$.

4. Write a program that will print out common logarithms of the positive integers 20 through 40.
5. The value of e can be found to any desired number of decimal places by using the series

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

   to as many terms as needed. Write a program to find the value of e, stopping when you reach a term that is less than 0.000005. Print each term and the accumulated sum.

Self-Test 3

VOCABULARY common logarithm (p. 395), mantissa (p. 398), characteristic (p. 398), antilogarithm (p. 398), precision (p. 402), accuracy (p. 403)

Use Table 5 to find the following:

1. $\log 28.36$ **2.** antilog $8.6841 - 10$ *Obj. 1, p. 394*

3. For the measurement 2.5 m, give (a) the precision and (b) the accuracy. *Obj. 2, p. 394*

Use logarithms to compute each of the following.

4. $(0.0058)(27.6)$ **5.** $\dfrac{\sqrt[3]{253}}{0.614}$ *Obj. 3, p. 394*

Solve over $\mathcal{R}$, giving answer to three significant digits.

6. $4(5^x) = 38$ **7.** $2x^{\frac{1}{4}} = 74.5$ *Obj. 4, p. 394*

Check your answers with those at the back of the book.

Careers

in Microbiology

Biology is the study of living things—their structure, evolutionary development, behavior, and life processes. Microbiologists restrict their study to microscopic and submicroscopic organisms, such as bacteria, viruses, protozoa, algae, and fungi.

The field of microbiology has a wide range of applications. In medicine, the study of bacteria and viruses has led to the control of many diseases and to the discovery of antibiotics. Since microorganisms play a vital part in the decomposition of matter in the soil, agriculture has benefited from greater knowledge about their function. The food industry is concerned with organisms that cause spoilage. Knowledge of bacterial growth is also important in sanitation and water pollution control.

Microbiologists prepare cell cultures (above) and use an electron microscope (below).

EXAMPLE Under certain conditions, bacterial growth can be represented by $N \approx N_0(2.72)^{kt}$, where N_0 is the initial number of bacteria, N is the number after t hours, and k is a constant depending on the type of bacteria and the conditions (see Example 2, section 10-11). A biologist has a culture of bacteria, whose growth is described by $N = N_0(2.72)^{0.03t}$. When there are about 10^5 of these bacteria, a mutation occurs in one of them. The mutants can make better use of some of the chemicals being supplied to the culture, and therefore their growth is described by $M = M_0(2.72)^{0.07t}$. How long after the mutation occurs (assuming there are no other mutations) will the number of mutants, M, equal the number of original type bacteria, N?

SOLUTION We want to find t such that $M = N$. That is,

$$M_0(2.72)^{0.07t} = N_0(2.72)^{0.03t}$$

Since when the mutation occurred, there were 10^5 of the original type of bacteria, and 1 mutant, we must solve

$$1(2.72)^{0.07t} = 10^5(2.72)^{0.03t}$$

Taking the log of each member

$$\begin{aligned} 0.07t \log(2.72) &= \log 10^5 + 0.03t \log(2.72) \\ 0.04t \log(2.72) &= 5 \\ 0.04t\,(0.4346) &= 5 \\ t &\approx 288 \text{ h or } 12 \text{ d.} \end{aligned}$$

ON THE CALCULATOR

Equations in which the variable appears as an exponent are usually solved using logarithms. Some of these are solved as shown below.

EXAMPLE Solve $4^x = 28$ for x.

SOLUTION Taking the logarithm of both sides, we get

$$\begin{aligned} \log 4^x &= \log 28 \\ x \log 4 &= \log 28 \\ x &= \frac{\log 28}{\log 4} \end{aligned}$$

To solve this equation on your calculator, use these steps:

28 [log] [÷] 4 [log] [=] 2.4036775. Answer.

Exercises

Solve for x.

1. $5^x = 100$
2. $7^x = 2035$
3. $12^x = 0.5$
4. $2^{3x} = 4096$
5. $3^{2x} = 59{,}049$
6. $4^{5x} = 1.4142136$

Chapter Summary

1. Radical expressions may be written equivalently in *exponential form:* $(\sqrt[r]{b})^p = b^{\frac{p}{r}}$, provided $\sqrt[r]{b}$ and p are real numbers.
2. The laws of exponents apply to real-number exponents for positive bases.
3. Relations in which reversing the components of the ordered pairs of each produces the other are called *inverses* of each other. The inverse of a one-to-one function is a function. If f^{-1} is the inverse of a one-to-one function f, then $f^{-1}[f(x)] = x$ for each x in the domain of f, and $f[f^{-1}(x)] = x$ for each x in the domain of f^{-1}.
4. The inverse of the *exponential function* $\{(x, y)\colon y = b^x, b > 0, b \neq 1\}$ is the *logarithmic function* $\{(x, y)\colon y = \log_b x, b > 0, b \neq 1, x > 0\}$. If $x \in \Re$, $x > 0$, $b > 0$, $b \neq 1$, then $b^{\log_b x} = x$; if $x \in \Re$, $b > 0$, $b \neq 1$, then $\log_b b^x = x$.
5. The *precision* of a measurement is defined to be the unit of measure. The *accuracy* of a measurement is defined to be the *relative error*, that is, the *maximum possible error* divided by the measurement itself.
6. The *characteristic* of the common logarithm of a number may be found by inspection of the number in scientific notation; the *mantissa* is determined from the table.

7. In using *linear interpolation,* you assume that small portions of the graph of $y = \log x$ are straight line segments.
8. The laws of exponents are the basis for the laws of logarithms:

$$\log (x_1x_2) = \log x_1 + \log x_2 \qquad \log \frac{x_1}{x_2} = \log x_1 - \log x_2$$

$$\log x_1^{\ n} = n \log x_1$$

9. To find the logarithm of a number to another base, you use the relationship:

$$\log_b n = \frac{\log_a n}{\log_a b}.$$

Chapter Review

1. Write $\sqrt[3]{x + y^2}$ in exponential form. 11-1

   a. $x^{\frac{1}{3}} + y^{\frac{2}{3}}$ b. $(x + y^2)^{\frac{1}{3}}$ c. $x + y^{\frac{2}{3}}$ d. $x^{\frac{1}{3}} + y^{\frac{1}{3}}$

2. Write $\frac{2}{\sqrt[6]{8}}$ in simple radical form.

   a. $\sqrt{2}$ b. $\frac{1}{\sqrt{2}}$ c. $\sqrt[3]{6}$ d. $\frac{1}{2}$

3. Solve $4^{x+2} = 2^{3x-1}$ over $\mathcal{R}$. 11-2

   a. $\{3\}$ b. $\{\frac{3}{2}\}$ c. $\{5\}$ d. $\{4\}$

4. If $F = \{(x, y): y = 2x + 1\}$, write the equation describing F^{-1}. 11-3

   a. $\{(x, y): x = \frac{1}{2}(y - 1)\}$ b. $\{(x, y): x = 2y + 1\}$
   c. $\{(x, y): y = \frac{1}{2}x - 1\}$ d. $\{(x, y): x = \frac{1}{2}(x + 1)\}$

5. Find $\log_3 81$. 11-4

   a. 9 b. 27 c. 6 d. 4

6. Solve $\log_x 64 = 3$ over $\mathcal{R}$.

   a. 4 b. 16 c. 8 d. 24

In Review Items 7–8, use the product and quotient properties of logarithms to solve the given equations over $\mathcal{R}$.

7. $\log x + \log 4 = 2$ 11-5

   a. 5 b. 25 c. $\frac{1}{2}$ d. 400

8. $\log_5 y - \log_5 (y - 4) = \log_5 2$

   a. $\frac{1}{2}$ b. 3 c. 5 d. 8

9. State the characteristic of $\log 0.0081$. 11-6

   a. -3 b. -2 c. 3 d. -4

In Review Items 10–16, use Table 5 when necessary.

10. Find log 40000.
 a. 3.6021 b. 5.6021 c. 6.6021 − 10 d. 4.6021

11. Use interpolation to find log 1437. *11-7*
 a. 3.1581 b. 3.1575 c. 3.1579 d. 3.1564

12. Use interpolation to find antilog 1.5085.
 a. 32.25 b. 3.228 c. 3.225 d. 32.28

13. Compute $-\frac{3.88}{15.7}$ using logarithms. *11-8*
 a. −0.211 b. −0.223 c. −0.247 d. 2.47

14. Compute $\frac{\sqrt[3]{1440}}{10.5}$ using logarithms. *11-9*
 a. 1.075 b. 1.143 c. 1.315 d. 1.126

15. Solve $x^{\frac{2}{3}} = 10.5$ over $\mathcal{R}$. *11-10*
 a. 34 b. 56 c. 15.75 d. 17

16. Solve $7^x = 210.3$ over $\mathcal{R}$.
 a. 2.75 b. 3.93 c. 2.84 d. 1.57

Chapter Test

1. Evaluate $27^{-\frac{4}{3}}$. *11-1*
2. Solve $2x^{-\frac{1}{3}} = 3$ over $\mathcal{R}$.
3. Solve $9^{2x+1} = 81^{3x-2}$ over $\mathcal{R}$. *11-2*
4. If $F(x) = \{(x, y)\colon y = 2x^3 - 3\}$, find $F^{-1}(x)$. *11-3*
5. Solve $\log_x 125 = 3$ over $\mathcal{R}$. *11-4*
6. Evaluate $\log_3 3^5$.
7. Use Table 5 to evaluate $\log_{10} 144$. *11-5*
8. Use the product and quotient properties of a logarithm to solve $\log 4 + \log x - \log 2 = 2$.

In Test Items 9–14, use Table 5 when necessary.

9. Find log 0.0013. 10. Find antilog 4.8470. *11-6*
11. Use interpolation to find log 3759. *11-7*
12. Compute $\frac{(2.03)(136)}{(0.0854)}$ using logarithms. *11-8*
13. Compute $\sqrt[4]{1280}$ using logarithms. *11-9*
14. Solve $2.72^{0.25x} = 25$ over $\mathcal{R}$. *11-10*

This experimental windmill will generate enough electricity to power about thirty homes.

12 Permutations, Combinations, and Probability

Permutations

***OBJECTIVES** for Sections 12-1 through 12-3:*
1. *Apply fundamental counting principles.*
2. *Find the number of permutations of the elements of a set.*
3. *Find the number of permutations of the elements of an* r*-element subset of an* n*-element set.*
4. *Find the number of permutations of elements that are not all different.*

12-1 Two Fundamental Counting Principles

Can you find the number of elements in the union of two finite sets if you know the number of elements in each of the sets? Consider the following examples:

(*1*) $A = \{1, 2, 3, 4\}$, $B = \{5, 6\}$, $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(*2*) $A = \{1, 2, 3, 4\}$, $C = \{2, 4, 6\}$, $A \cup C = \{1, 2, 3, 4, 6\}$

In (*1*), the number of elements in the union is just the sum of the numbers of elements in the given sets: $6 = 4 + 2$. In (*2*), however, the union has only 5 elements, instead of $4 + 3 = 7$, since the elements 2 and 4 belong to both A and C. In set notation, $\{2, 4\} = A \cap C$.

Thus to find the number of elements in the union of two sets, you must also know the number of elements in their intersection.

> If a finite set A contains r elements, a finite set B contains s elements, and their intersection ($A \cap B$) contains t elements, then the union of A and B ($A \cup B$) contains $r + s - t$ elements.

If $A \cap B = \emptyset$, as in (*1*) on page 421, then $t = 0$ and $A \cup B = r + s$.

EXAMPLE 1 The Mathematics Club at East High School has 34 members, the Spanish Club has 28 members, and 8 students are members of both organizations. If all members attended, what would be the attendance at a joint meeting of the two clubs?

SOLUTION You want to determine the number N of members in the union of two sets whose intersection contains 8 members. You have:

$$N = 34 + 28 - 8 = 54$$

$\therefore$ the attendance at the joint meeting would be 54. Answer.

To discover a second counting principle, consider this problem: If A is the set of *two* integers $\{1, 2\}$ and B is the set of *three* integers $\{4, 5, 6\}$, how many different ordered pairs (a, b) are there with $a \in A$ and $b \in B$?

For each of the *two* ways that you can choose the first entry, a, there are *three* ways you can choose the second entry, b. Thus the set of all such ordered pairs (a, b) is

$$\{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)\},$$

which contains $2 \cdot 3 = 6$ elements. This set of ordered pairs is called the **Cartesian product** of A and B and is denoted by $A \times B$.

The result in this example can be generalized as follows:

> If a finite set A contains r elements and a finite set B contains s elements, then there are rs different ordered pairs (a, b) with $a \in A$ and $b \in B$ (that is, $A \times B$ contains rs elements).

This principle can be extended to any number of sets and applied in many counting situations.

EXAMPLE 2 How many four-digit numerals for even numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6?

SOLUTION To help you think through such problems, it is useful to employ a diagram such as this: $\boxed{}\boxed{}\boxed{}\boxed{}$ or this: ____ ____ ____ ____.

For the thousands digit, you can use any of the six digits 1, 2, 3, 4, 5, or 6, but not 0. Therefore, you write 6 in the first space: $\boxed{6}\boxed{}\boxed{}\boxed{}$

For the hundreds and tens digits you can use any one of the given seven digits, so write 7 in each of these places: $\boxed{6}\boxed{7}\boxed{7}\boxed{}$

In the units place, you can use any one of the four digits 0, 2, 4, or 6, but not 1, 3, or 5. Therefore, write 4 in the units place: $\boxed{6}\boxed{7}\boxed{7}\boxed{4}$

The second counting principle tells you that there are $6 \times 7 \times 7 \times 4$, or 1176, ways of forming the required even numerals. Answer.

Oral Exercises

If A and B are as given, state the number of elements in $A \cap B$, $A \cup B$, and $A \times B$.

1. $A = \{4, 6, 8\}$, $B = \{5, 7, 9\}$
2. $A = \{1, 3, 5\}$, $B = \{4, 5, 6\}$
3. $A = \{3\}$, $B = \{1, 2, 3, 4, 5\}$
4. $A = \{1\}$, $B = \{2\}$
5. $A = B = \{1, 2, 3, 4, 5\}$
6. $A = \{1, 2, 3\}$, $B = \emptyset$

In Exercises 7–8, the numbers of elements in three of the sets A, B, $A \cup B$, $A \cap B$, and $A \times B$ are given. State the numbers of elements in the other two.

7. $A{:}3,\ B{:}5,\ A \cup B{:}7,\ A \cap B{:}?,\ A \times B{:}?$
8. $A{:}6,\ B{:}?,\ A \cup B{:}8,\ A \cap B{:}5,\ A \times B{:}?$

Written Exercises

A

1. How many ordered pairs of letters are there that use only the letters A, B, C, D, and E?
2. How many three-digit numerals can be formed that do not contain a 7?
3. How many sequences of 4 letters can be formed from the letters J, K, L, M, N, and O if no letter may be used more than once?
4. How many different sequences of heads and/or tails are possible if a coin is flipped 7 times? (If you let H stand for heads and T for tails, one such sequence could be $THHTHTH$.)
5. Out of a faculty of 85 at Central High School, 47 teach at least one eleventh grade class and 56 teach at least one twelfth grade class. How many on the faculty teach both eleventh and twelfth grade classes?

6. In a survey of 375 dog and cat owners there were 215 dog owners and 193 cat owners. How many in the survey own a dog and no cat?

7. There are 8 different routes between cities A and B, and 7 different routes between cities B and C. How many different routes are there from city A to city C by way of city B?

8. How many different employee identification symbols are possible if each symbol consists of two letters of the alphabet followed by three digits?

B 9. In Exercise 8, how many symbols are possible if the two letters must be different and all three of the digits cannot be zero?

10. How many seven-digit phone numbers are possible if 0 and 1 cannot be used as the first digit and the first three digits cannot be 555, 637, or 936?

11. How many numerals for positive even integers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, and 5?

12. How many numerals for numbers between 450 and 700 can be formed using only the digits 2, 3, 4, 5, 6, and 7?

13. How many of the numbers in Exercise 12 will be odd?

C 14. How many multiples of 3 less than 1000 can be formed from the digits 1, 4, 5, 7, and 9? (*Hint:* The sum of the digits of any multiple of 3 is also a multiple of 3.)

12-2 Linear and Circular Permutations

You can list the members of the set $\{a, b, c\}$ in six different orders:

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba.$$

Each ordering, or arrangement, of the letters is called a (*linear*) *permutation* of the set $\{a, b, c\}$. A **permutation** is any arrangement of the elements of a set in a definite order.

Notice that the first letter listed can be any member of $\{a, b, c\}$. This means there are *three* choices for first place, so we write **3** in the first space of a diagram: | 3 | | |. *After a letter has been selected for first place,* the choice for second place is made from the set of *two* letters remaining. Therefore, we write **2** in the second space: | 3 | 2 | |. *After letters have been assigned to both the first and the second places,* there is only *one* choice for third place; so we write **1** in the third space: | 3 | 2 | 1 |. Thus, the number of permutations of the elements of $\{a, b, c\}$ is

$$3 \times 2 \times 1.$$

The product $3 \times 2 \times 1$ can be written in brief **factorial notation** as $3!$ (read "three factorial" or "factorial three"). Thus, factorial five is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120,$$

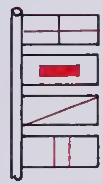
and in general,

$$n! = n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1, \text{ where } n \text{ is any natural number.}$$

The preceding example illustrates the following fact:

> The number of permutations of the members of a set containing n different elements is $n!$.

EXAMPLE 1 How many different signals can be made using the four flags pictured at the right if all the flags must be used in each signal?



SOLUTION You want to determine the number of permutations of 4 things:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$\therefore$ 24 signals can be made using the four flags. **Answer.**

Now suppose you are asked to find the number of permutations of five letters taken three at a time. In the diagram $\boxed{}\boxed{}\boxed{}$, the first space could be filled in *five* ways, the second in *four* ways, and the last in *three*. Thus $\boxed{5}\boxed{4}\boxed{3}$ would represent the situation. From the fundamental counting principle there are $5 \cdot 4 \cdot 3$, or 60, ways in which the letters could be arranged.

In a set, the number of permutations of n different elements taken r at a time is denoted by ${}_nP_r$. Other representations are $P(n, r)$ and P_r^n. To obtain a formula for ${}_nP_r$, notice that the diagram representing the situation contains r spaces to be filled as shown:

$$\boxed{n}\boxed{n-1}\boxed{n-2}\boxed{\cdots}\boxed{n-(r-1)}$$

Thus, you have the following result:

> The number of permutations of r members of a set containing n different elements is
>
> $${}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

Note that if $r = n$, then ${}_nP_n = n!$.

EXAMPLE 2 How many different three-letter sequences can you form from the letters of the alphabet if no two letters in a sequence are the same?

SOLUTION You want to determine the number of permutations of 3 elements from a set of 26 elements:

$$_{26}P_3 = 26 \cdot 25 \cdot 24 = 15{,}600$$

$\therefore$ you can form 15,600 different three-letter sequences. Answer.

There is a special type of permutation, called a **circular permutation,** which is an arrangement of objects in a circular pattern. A common example is the seating of people around a circular table. In such an arrangement there is no first place, so that if each person shifts position by one place counterclockwise (or clockwise) the relative positions are not changed. In fact, if there are n people at the table, each person can shift position n times and return to his or her original position without disturbing the arrangement. Therefore, if you use the formula for a linear permutation to find the number of possible arrangements, you will have counted each different arrangement n times. Thus, there are

$$\frac{n!}{n} = \frac{n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{n} = (n-1)!$$

distinguishable permutations.

The diagram below shows the $(3-1)!$, or 2, circular permutations of the 3-element set $\{a, b, c\}$ and the corresponding $3!$, or 6, linear permutations.

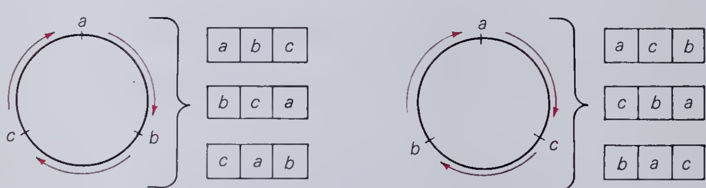


Figure 1

EXAMPLE 3 In how many ways can six persons be seated around a circular table?

SOLUTION Since this is a circular permutation of 6 things, there are $(6-1)!$, or 120, possible different seating arrangements.

You may think of this in a slightly different manner. Since a rotation of any permutation does not produce a new permutation, one of the positions can be considered fixed, and $\boxed{1}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ describes the situation. We see again that there are 120 different arrangements. Answer.

The analysis of problems involving circular permutations of objects which do not have a definite top or bottom, such as bracelets or key

rings, is somewhat different. In these cases it seems reasonable to consider that flipping an arrangement over does not change the arrangement. Thus, since flipping over the first arrangement in Figure 1 in Example 2 yields the second arrangement, and vice versa, there is only $\frac{(3-1)!}{2}$, or 1, permutation of 3 objects about a key ring or bracelet. In general, for n objects, there are $\frac{(n-1)!}{2}$ such permutations, provided $n > 2$.

EXAMPLE 4 In how many ways can 5 keys be arranged on a key ring?

SOLUTION You have

$$\frac{(n-1)!}{2} = \frac{4!}{2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2} = 12. \quad \text{Answer.}$$

Oral Exercises

State as a number.

1. ${}_5P_3$
2. ${}_3P_2$
3. ${}_7P_2$
4. ${}_8P_1$
5. ${}_9P_9$
6. ${}_{12}P_2$

Written Exercises

A

1. In how many different ways can the 8 notes of a C scale be arranged to form a melody with no note repeated if considerations of rhythm are ignored?
2. Answer Exercise 1 with the additional restriction that the melody must begin on the lower C and end on the higher C.
3. Given 9 starting batters on a baseball team, how many batting orders are possible?
4. How many arrangements are there of the letters in the word PICTURE?
5. In how many ways can 4 persons be seated in a row of 9 chairs?
6. In how many ways can the 5 positions on a basketball team be assigned among 10 players?
7. In how many ways can a president, a vice-president, a secretary, and a treasurer be chosen from the 12 members of a club?
8. In how many ways can 6 distinct dishes be arranged around a revolving platter at a buffet dinner?
9. In how many ways can 10 numbers be arranged around a circular dartboard?
10. In how many different ways can a host couple and 6 guests be seated at a round table?

11. In how many ways can 7 keys be arranged on a key ring?

12. In how many different ways can 10 different colored beads be arranged on a bracelet?

B 13. In how many ways can 6 students be seated in a row if 2 particular students must be seated next to each other?

14. In how many ways can 6 students be seated in a row if 2 particular students must not be seated next to each other?

15. How many arrangements of the letters in the word VERTICAL begin with three vowels?

16. How many arrangements of the letters in the word GRACIOUS both begin and end with a vowel?

Show that each of the following is true when $n = 7$ and $r = 4$.

17. $n({}_{n-1}P_{r-1}) = {}_nP_r$

18. $(n - r){}_nP_r = {}_nP_{r+1}$

19. ${}_nP_{n-r} = \dfrac{n!}{r!}$

20. $({}_nP_r)({}_{n-r}P_{n-r}) = {}_nP_n$

Prove each of the following for all positive integers n, r, and s, where $n > r + s$.

C 21–24. Prove the statements in Exercises 17–20 above.

25. ${}_nP_4 - {}_nP_3 = (n - 4)({}_nP_3)$

26. ${}_nP_r - {}_nP_{r-1} = (n - r){}_nP_{r-1}$

27. $({}_nP_r)({}_{n-r}P_s) = {}_nP_{r+s}$

12-3 Permutations with Repeated Elements

Finding the number of distinguishable permutations of a set of elements that are not all different involves an extension of the method used in Section 12-2. For example, to find the number of distinguishable permutations of the letters in the word

reverse,

we must consider the fact that the letter *e* occurs **three** times and the letter *r* **twice** in the word, and that simply interchanging any of the three *e*'s or the two *r*'s with each other does not produce a distinguishable permutation.

To analyze the situation in detail, let us label the *r*'s and *e*'s with subscripts:

$$r_1 \quad e_1 \quad v \quad e_2 \quad r_2 \quad s \quad e_3$$

There are, of course, ${}_7P_7 = 7!$ permutations of these 7 letters. If we let P denote the number of *distinguishable* permutations, then for each of

these P permutations, there are $3!$ permutations of the e's (e_1, e_2, and e_3) and $2!$ permutations of the r's (r_1 and r_2). It follows that

$$2! \cdot 3! P = {}_7P_7 = 7!, \text{ so that}$$

$$P = \frac{7!}{2! \cdot 3!} = 420.$$

Using similar reasoning, we can assert that, in general:

> The number of distinguishable permutations of n elements taken n at a time, with n_1 elements alike, n_2 of another kind alike, and so on, is
>
> $$\frac{n!}{n_1! n_2! \cdots}.$$

Oral Exercises

Find the number of distinguishable permutations of the letters in the given word.

1. all **2.** book **3.** foot **4.** mitt **5.** sills

Written Exercises

Find the number of distinguishable arrangements of the letters in the given word.

A **1.** series **2.** sequence **3.** ellipse **4.** parabolas

5. coefficient **6.** nonsense **7.** microbiology **8.** multiplicity

How many different positive integers can be formed using all the digits of the given numeral?

9. 314159 **10.** 1414214 **11.** 1732050808 **12.** 2718281828

B **13.** How many distinguishable circular permutations of the letters in the word ROOTS can be formed?

14. How many distinguishable circular permutations of the letters in the word SERIES can be formed?

C **15.** Find the number of distinguishable five-letter sequences that can be formed from the letters in the word CIRCLE. Use the following steps:

a. Find the number of distinguishable five-letter sequences containing exactly one C.

b. Find the number of distinguishable five-letter sequences containing two C's.

c. Add the results of steps a and b.

Self-Test 1

VOCABULARY Cartesian product (p. 422) | permutation (p. 424) | factorial notation (p. 425) | circular permutation (p. 426)

1. How many numerals for three-digit odd integers can be formed using only the digits 3, 4, 5, 6, and 7? *Obj. 1, p. 421*
2. In how many ways can 6 names be arranged on a ballot if each name must be used exactly once? *Obj. 2, p. 421*
3. In how many ways can 4 countries on a map be colored if 7 colors are available and each country must be a different color? *Obj. 3, p. 421*
4. How many distinguishable arrangements are there of the letters in the word ESSENCE? *Obj. 4, p. 421*

Check your answers with those at the back of the book.

Combinations

OBJECTIVES for Sections 12-4 and 12-5:
1. *Find the number of combinations of* n *elements taken* r *at a time.*
2. *Find the number of ways in which specified subsets can be selected from two or more given sets.*

12-4 Counting Subsets

Can you list the three-element subsets of the set T, where $T = \{a, b, c, d\}$? To obtain any of these subsets, all you have to do is remove one of the members of the original set. Thus the three-element subsets are:

$$\{a, b, c\} \qquad \{a, b, d\} \qquad \{a, c, d\} \qquad \{b, c, d\}.$$

Hence, denoting the number of **three**-element subsets of a **four**-element set by ${}_4C_3$, you have ${}_4C_3 = 4$.

You can classify the *permutations* of T's elements *taken three at a time* according to the three-element subset involved in each permutation. For example, the **3!**, or 6, arrangements

$$\begin{array}{ccc} abd & bad & dab \\ adb & bda & dba \end{array}$$

are the permutations of the subset $\{a, b, d\}$. Similarly, each of the other

three-element subsets yields **3!** other permutations of the letters a, b, c, and d taken three at a time. Thus, you have

$$ {}_4C_3 \times \mathbf{3!} = {}_4P_3 \qquad \text{or} \qquad {}_4C_3 = \frac{{}_4P_3}{\mathbf{3!}}. $$

This formula is consistent with our observation that ${}_4C_3 = 4$, since

$$ \frac{{}_4P_3}{3!} = \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4. $$

Moreover, the formula suggests the general relationship between ${}_nC_r$, the number of r-element subsets of a set with n elements, and ${}_nP_r$, the number of permutations of the n elements taken r at a time for $0 < r < n$:

$$ {}_nC_r = \frac{{}_nP_r}{\boldsymbol{r!}}. $$

Since ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$, you have:

> The number of r-element subsets of a set containing n elements is
>
> $$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. $$

Note that the numerator and the denominator of the expression on the right are both products of r factors.

An r-element subset of a set with n elements is often called a **combination** of n elements taken r at a time. Thus, ${}_nC_r$, also denoted by $\binom{n}{r}$, $C(n, r)$, or C_r^n, is the number of combinations of n elements taken r at a time.

EXAMPLE How many 5-card hands can be dealt from a standard 52-card bridge deck?

SOLUTION You are asked for the number of 5-card subsets of a 52-card set. Letting $r = 5$ and $n = 52$, you can begin by noting that the denominator of

$$ {}_{52}C_5 = \frac{\qquad\qquad\qquad}{5!} $$

contains **5** factors. Therefore, the required numerator contains **5** descending factors starting with 52. Thus

$$ {}_{52}C_5 = \frac{\mathbf{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 2{,}598{,}960. $$

$\therefore$ there are 2,598,960 possible 5-card hands in a 52-card deck. Answer.

If you multiply the numerator and denominator of the expression for ${}_nC_r$ by $(n-r)!$, you obtain the equivalent expression:

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!(n-r)!}$$

or

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

The symbol ${}_nC_0$ denotes the number of subsets with no elements in a set having n elements. Since there is just one such subset, namely the empty set $\emptyset$, ${}_nC_0 = 1$. Thus, you have:

$${}_nC_0 = \frac{n!}{0!n!} = \frac{1}{0!}, \qquad \text{or} \qquad 1 = \frac{1}{0!}$$

Therefore, for the formula to hold when r is a whole number, we must *define* 0! to be 1. Now, you should verify that this definition also makes the formula for ${}_nC_r$ valid in the case $r = n$.

You can discover a useful fact about ${}_nC_r$ by noticing that whenever r elements are selected from a set of n elements, $n-r$ elements are left behind. Therefore, the combinations of r elements selected, and the combinations of $n-r$ elements left, are paired one-to-one and are consequently the same in number; that is,

$${}_nC_r = {}_nC_{n-r}.$$

You can use this fact to simplify computations; for example,

$${}_{50}C_{48} = {}_{50}C_2 = \frac{50\cdot 49}{1\cdot 2} = 1225.$$

Oral Exercises

Express as a number.

1. ${}_4C_1$
2. ${}_5C_2$
3. ${}_{12}C_2$
4. ${}_6C_4$
5. ${}_{14}C_{14}$
6. ${}_{15}C_1$

Written Exercises

A

1. How many ways can a selection of 4 records be chosen from a record club offering 11 records?
2. In how many ways can a committee consisting of 12 people choose a steering subcommittee of 5?
3. How many roads are needed to connect 10 cities if there is to be exactly one road between any two of the cities?

4. At a chess tournament with 12 players, each player played exactly one game with all but one of the other players. How many games of chess were played at the tournament?

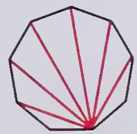
5. How many ways can 4 blue socks be chosen from a bag that contains 10 blue socks and 8 red socks?

6. Of nine points, no 3 are collinear. How many triangles are there that have 3 of these points as vertices?

7. How many ways are there of scoring exactly 70% on a 10-question true-false test?

B 8. In a club consisting of 20 members, how many ways are there of choosing a committee of 6 if the club president must be on the committee and another member of the club is not able to serve on the committee?

9. How many diagonals are there in a regular nine-sided polygon? (Note that a diagonal cannot connect two *adjacent* vertices.)



10. In how many ways can 10 people be divided into two basketball teams of 5 players each?

11. How many committees of 6 can be chosen from a management group of 10 people if the president and first vice-president are not to serve on the same committee?

12. Answer Exercise 6 above if 5 of the 9 points are collinear, but no other 3 are.

13. How many different sums of money can be made using 1 penny, 1 nickel, 1 dime, 1 quarter, and 1 half-dollar?

14. Find n if **a.** ${}_nC_2 = {}_{100}C_{98}$ **b.** ${}_nC_4 = {}_nC_3$

Use the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$ to prove:

C 15. ${}_nC_r = {}_nC_{n-r}$ 16. ${}_nC_r = \dfrac{n}{r}({}_{n-1}C_{r-1})$ 17. $(n-r)({}_nC_r) = n({}_{n-1}C_r)$

18. Show that the total number of subsets of a set with n elements is 2^n. (*Hint:* Each member of the set either is or is not selected in forming a subset.)

12-5 Combinations and Products

You can use the second fundamental principle of counting (page 422) to help determine the number of ways specified subsets can be selected from two or more given sets.

EXAMPLE In how many ways can 3 white and 2 red marbles be chosen from an urn containing 12 white and 8 red marbles?

SOLUTION For the white marbles, you compute

$$_{12}C_3 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220.$$

For the red marbles, you compute

$$_8C_2 = \frac{8 \cdot 7}{1 \cdot 2} = 28.$$

Then, by the second counting principle, the number of ways both selections can be made is just

$$28 \times 220 = 6160. \quad \text{Answer.}$$

Oral Exercises

An urn contains 10 red marbles and 8 blue marbles. State an expression for the number of ways of drawing each of the following combinations.

EXAMPLE 2 red, 3 blue **SOLUTION** $_{10}C_2 \times {_8C_3}$

1. 5 red, 4 blue
2. 7 red, 8 blue
3. 6 red, 3 blue
4. 10 red, no blue
5. 1 red, 8 blue
6. 10 red, 8 blue

Written Exercises

A **1–6.** Find the number of ways the combinations in Oral Exercises 1–6 may be drawn.

7. A family dinner special at a certain restaurant allows a choice of any 3 out of 5 dishes from Group A and 4 out of 6 dishes from Group B. How many different choices are possible?

8. Parts A, B, and C of a test contain 5, 6, and 7 questions respectively, and a student must choose 2, 3, and 4 questions respectively from the three parts. In how many ways can this be done?

Exercises 9–17 refer to a standard bridge card deck containing 52 cards with 13 cards in each of four suits (clubs, diamonds, hearts, and spades) and 4 cards in each denomination (ace, two, three, etc.). In each case, tell how many ways there are of choosing the given cards.

9. 2 aces and 3 kings

10. 2 hearts and 3 spades

11. 3 face cards (jacks, queens, or kings) and 2 tens

12. 2 red cards and 2 clubs
13. 1 ace, 1 two, 1 three, 1 four, and 1 five
14. 3 jacks and 2 cards neither of which is a jack
15. 2 kings, 2 queens, and a card that is neither a king nor a queen

B 16. 2 cards of one denomination and 2 cards of another denomination (*Hint:* First find the number of ways of choosing the 2 denominations.)

17. 2 cards of one denomination, 2 cards of another denomination, and a card that is of any third denomination

18. How many five-letter sequences containing exactly 3 vowels and 2 consonants can be made from the letters in the word ANTIDOTES? (*Hint:* Find the number of possible *combinations* of letters first, then the number of permutations of each combination.)

C 19. Find the number of distinguishable five-letter sequences that can be made from the letters in the word VERTICES. (*Hint:* Separate into two cases: (1) sequences containing both E's; (2) all other sequences.)

20. How many 5-letter arrangements can be formed using the letters in the word CREATION if each arrangement has 3 vowels and 2 consonants and if no letter is repeated?

21. There are 12 empty seats in a row of a theater. In how many ways can 3 people be seated in the row? In how many ways can they be seated next to each other in the row?

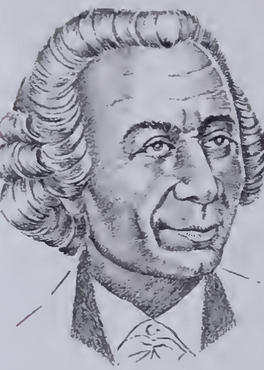
Self-Test 2

VOCABULARY combination (p. 431)

1. In how many ways can 4 titles be chosen from a summer reading list of 10 books? *Obj. 1, p. 430*
2. In how many ways can a committee of 6 be chosen from a group of 11 people? *Obj. 2, p. 430*
3. In how many ways can you select 2 blue marbles and 3 red marbles from an urn containing 6 blue marbles and 6 red marbles?
4. Parts A and B of a test contain 6 and 8 questions respectively. In how many ways can a student select 4 questions from part A and six questions from part B?

Check your answers with those at the back of the book.

Leonhard Euler
1701–1783



Leonhard Euler is probably the greatest Swiss scientist in history, and certainly one of the most prolific mathematicians of all times. Euler, who spent most of his lifetime at St. Petersburg Academy in Russia and the Berlin Academy, was one of several great mathematicians who could work almost anywhere under any conditions. He was able to perform long calculations in his head, so that even after he was totally blind, his mathematical output did not stop. His writings included works on calculus, topology, navigation, celestial mechanics, and algebra. In trigonometry Euler extended the sine, cosine, and tangent functions to angles other than those occurring in right triangles.

In addition, many of the mathematical symbols used today were developed by Euler. He adopted the symbols

$$i \text{ to represent } \sqrt{-1},$$

$$\Sigma \text{ to indicate summation,}$$

and

$$f(x) \text{ to denote a function of } x.$$

Binomial Expansions

OBJECTIVES for Sections 12-6 and 12-7:
1. ***Expand a binomial.***
2. ***Use Pascal's triangle to expand a binomial.***
3. ***Find a given term in a binomial expansion.***

12-6 The Binomial Theorem

When you expand natural-number powers of binomials, you discover an interesting pattern:

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a - b)^1 = a - b$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

These examples suggest:

1. The number of terms in the expansion of $(a \pm b)^n$ is $n + 1$.
2. If the binomial is a sum, all terms in the expansion are added; if the binomial is a difference, the terms are alternately added and subtracted, the even-numbered terms being subtracted.
3. The coefficient of the first term is 1.
4. The coefficient of any other term is the product of the coefficient of the preceding term and the exponent of a in the preceding term divided by the number of the preceding term.
5. The exponent of a in any term after the first is one less than the exponent of a in the preceding term.
6. The exponent of b in any term after the first is one greater than the exponent of b in the preceding term.
7. The sum of the exponents of a and b in each term is n.

EXAMPLE 1 Expand $(3n - 2)^5$.

SOLUTION

$$\frac{5 \cdot 1}{1} = 5 \qquad\qquad \frac{2 \cdot 10}{4} = 5$$

$$1 \cdot (3n)^5 - 5(3n)^4(2) + 10(3n)^3(2)^2 - 10(3n)^2(2)^3 + 5(3n)(2)^4 - 2^5$$

Term Number 1 2 3 4 5 6

The arrows show how the numerical coefficients of the second and fifth terms are computed. (Explain the others.) In simplified form,

$(3n - 2)^5 = 243n^5 - 810n^4 + 1080n^3 - 720n^2 + 240n - 32.$ Answer.

The pattern displayed by the expansion of binomials suggests the *Binomial Theorem,* which states that for any positive integer n the expansion of $(a + b)^n$ is:

$$a^n + \frac{n}{1}a^{n-1}b^1 + \frac{n}{1} \cdot \frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}a^{n-3}b^3 + \cdots$$
$$+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-(r-2)}{r-1}a^{n-(r-1)}b^{r-1} + \cdots + b^n.$$

The rth term is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}, \qquad r > 1.$$

Observe that in the rth term, the exponent of b is $r - 1$, the exponent of a is $n - r + 1$, the denominator of the coefficient is $(r - 1)!$, and the numerator of the coefficient is $n(n - 1) \ldots (n - r + 2)$, which consists of $r - 1$ consecutive integers decreasing from n. Thus, the numerator and denominator of the coefficient contain the same number of factors.

EXAMPLE 2 Find the fifth term in the expansion of $(2x + y)^{10}$.

SOLUTION You have $n = 10$ and $r = 5$.

1. Find the exponent of b (in this case, y). It is $5 - 1$, or **4**. $\quad y^4$
2. The exponent of a (in this case, $2x$) is then $10 - 4$, or **6**. $\quad (2x)^6y^4$
3. The denominator of the coefficient is then $(5 - 1)!$, or $4 \cdot 3 \cdot 2 \cdot 1$. $\quad \dfrac{\quad}{4 \cdot 3 \cdot 2 \cdot 1}(2x)^6y^4$
4. The numerator of the coefficient contains 4 factors starting with 10. $\quad \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}(2x)^6y^4$

Simplifying, you have, $210(2)^6x^6y^4$ or $13{,}440x^6y^4$. Answer.

Oral Exercises

Find the requested parts of the binomial expansion of $(4x - 3)^4$.

$$1 \cdot (4x)^4 - ?(4x)^?3 + ?(4x)^2 3^? - 4(4x)3^3 + ?$$

1. The numerical coefficient of the second term.
2. The exponent of $4x$ in the second term.
3. The numerical coefficient of the third term.
4. The exponent of 3 in the third term.
5. The fifth term.

Written Exercises

Expand each binomial and express the result in simplified form.

A

1. $(a + 3)^4$
2. $(x - 2)^5$
3. $(3 - r)^5$
4. $(2x + y)^6$
5. $(2c + \frac{1}{2})^5$
6. $(\frac{1}{3}x - 1)^4$
7. $\left(2 - \frac{b}{2}\right)^6$
8. $(p^2 + q)^7$
9. $(x^2 - 1)^8$
10. $(y^2 + z^2)^7$
11. $(a^2 + 2b)^6$
12. $(c^3 + d^2)^8$

Find and simplify the specified term.

B

13. $(a + b)^{10}$; fourth
14. $(x^2 - 1)^{12}$; sixth
15. $(p + 3q)^9$; third
16. $(n^3 - 2)^{11}$; eighth
17. $\left(2c - \frac{d}{2}\right)^{12}$; seventh
18. $(1 - u^2)^{15}$; fifth

Assume that the Binomial Theorem as stated on page 437 holds when n is a positive *rational number* (and the expansion is an infinite series). Give the first three terms of each expansion.

C

19. $(x^2 + y)^{\frac{1}{2}}$
20. $(4 + k)^{\frac{1}{2}}$
21. $(9 - r)^{\frac{3}{2}}$
22. $(8a^3 + 1)^{\frac{4}{3}}$

12-7 Pascal's Triangle

You can look at the expansion of a positive integral power of a binomial from the point of view of combinations of terms selected from each of the binomial factors. Consider the following expansion:

$$(a + b)^3 = (a + b)(a + b)(a + b) = aaa + baa + aab + aba + bba + bab + abb + bbb$$

You obtain each product shown in the expansion by multiplying three variables, one from each of the binomial factors of $(a + b)^3$. The term baa, for example, is the result of choosing b from the first binomial factor, a from the second, and a from the third. If you combine similar terms in the expansion to obtain

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

then 3, the coefficient of a^2b, is the number of ways of selecting one b from the three factors, that is ${}_3C_1$. Similarly, because you obtain a^3 by choosing no b from the three factors, the coefficient of a^3 is 1, or ${}_3C_0$. In fact, you can rewrite the expansion as follows:

$$(a + b)^3 = {}_3C_0a^3 + {}_3C_1a^2b + {}_3C_2ab^2 + {}_3C_3b^3$$

The reasoning used in determining the coefficients in the expansion of $(a + b)^3$ can be extended to determining the coefficients in the expansion of $(a + b)^n$. Thus,

$$(a + b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_nb^n,$$

where the rth term is ${}_nC_{r-1}a^{n-(r-1)}b^{r-1}$.

If you write the expansions of $(a + b)^n$ for successive values of n in the form of a triangle, you have

$$\begin{array}{ll}
(a+b)^0 = & \qquad\qquad\qquad 1 \\
(a+b)^1 = & \qquad\qquad\quad a + b \\
(a+b)^2 = & \qquad\quad a^2 + 2ab + b^2 \\
(a+b)^3 = & \quad a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 = & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{array}$$

Now, looking only at the coefficients, you see the triangle:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Notice that each term other than 1 is the sum of the term to the right and the term to the left of it in the row directly above. Thus, the next row is:

$$\begin{array}{ccccccccccc}
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
 & & + & & + & & + & & + & & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This array is known as **Pascal's triangle,** named after the French mathematician and philosopher Blaise Pascal.

EXAMPLE 1 Use Pascal's triangle to expand $(2m - 1)^6$.

SOLUTION You use the seventh row of this triangle:

Sixth Row:		1	5	10	10	5	1
Seventh Row:	**1**	**6**	**15**	**20**	**15**	**6**	**1**

$$\therefore (2m - 1)^6 = 1(2m)^6 - 6(2m)^5 + 15(2m)^4 - 20(2m)^3 + 15(2m)^2 - 6(2m) + 1$$
$$= 64m^6 - 192m^5 + 240m^4 - 160m^3 + 60m^2 - 12m + 1. \quad \text{Answer.}$$

EXAMPLE 2 Find the seventh term in the expansion of $(t + 2)^{10}$.

SOLUTION The rth term is given by ${}_nC_{r-1}(t)^{n-r+1}(2)^{r-1}$.

The seventh term is ${}_{10}C_6 t^{10-7+1}(2)^{7-1}$.

Since ${}_{10}C_6 = {}_{10}C_4 = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$, the seventh term is

$$210t^4(2)^6 = 210(64)t^4 = 13{,}440t^4. \quad \text{Answer.}$$

Oral Exercises

Use the seventh row of Pascal's triangle as given in Example 1 above to find the eighth row of the triangle.

Written Exercises

Use Pascal's triangle to expand each binomial. Express answers in simplified form.

A

1. $(c + 1)^5$
2. $(r - 2)^6$
3. $(2x + y)^7$
4. $\left(\dfrac{a}{3} + b\right)^4$
5. $(1 - y^2)^6$
6. $(\frac{1}{2}a + 2)^5$
7. $(r^2 - t^2)^8$
8. $(2k + n^2)^6$

For each expansion, express in simplified form the term that contains the given expression.

B

9. $(1 + m)^9$; m^3
10. $(x + \frac{1}{2})^{10}$; x^7
11. $\left(\dfrac{p}{2} + q\right)^8$; q^5
12. $(a + \frac{1}{3}b)^{12}$; a^8
13. $(1 - 2x)^{10}$; $(2x)^5$
14. $(r^2 - t^3)^{15}$; $(t^3)^4$

C

15. Prove: ${}_nC_3 = {}_{n-1}C_2 + {}_{n-1}C_3$, for $n \geq 4$
16. Prove: ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$, for $n \geq r + 1$
17. Add the entries in the first few rows of Pascal's triangle and make a conjecture about the sum of the entries in the nth row as a function of n. Test the conjecture for $n = 8$.

Self-Test 3

VOCABULARY Binomial Theorem (p. 437) Pascal's triangle (p. 440)

1. Expand $(a^2 - 2)^5$. *Obj. 1, p. 436*

2. Use Pascal's triangle to expand $\left(x - \dfrac{y}{2}\right)^6$. *Obj. 2, p. 436*

3. Find the seventh term in the expansion of $(2p + q)^{10}$. *Obj. 3, p. 436*

Check your answers with those at the back of the book.

Probability

OBJECTIVES for Sections 12-8 through 12-11:
1. ***List a sample space for an experiment, and identify an event.***
2. ***Find the probability of an event and of its complement.***
3. ***Find the probability of mutually exclusive events.***
4. ***Find the probability of the occurrence of a second event given the occurrence of a first event.***

12-8 Sample Spaces and Events

Suppose you conduct an experiment by tossing three coins—a dime, a nickel, and a quarter. If h represents heads and t tails, any possible outcome of the experiment is an element of

$$\{(h, h, h), (h, h, t), (h, t, h), (h, t, t), (t, h, h), (t, h, t), (t, t, h), (t, t, t)\},$$

where the components of the ordered triples represent in order the result of tossing the dime, the nickel, and the quarter. This set is called a *sample space* or *universe* of the experiment. A **sample space** is a set S of elements that correspond one-to-one with the outcomes of an experiment. Each of the elements corresponding to an outcome is called a **sample point.**

EXAMPLE 1 A die is rolled and the number of spots on its top face when it comes to rest is observed. List a sample space for the experiment.

SOLUTION $\{1, 2, 3, 4, 5, 6\}$.

Now suppose you are interested in whether the number of spots observed in Example 1 is an even number. You can call the result of an even number an *event*, and, in this case, the event is the occurrence of any of the outcomes 2, 4, or 6. The set $\{2, 4, 6\}$ can be seen to be a subset of the sample space. An **event** is any subset of a sample space.

EXAMPLE 2 Two dice are cast. List a sample space for this experiment and then list the event that the sum of the spots showing is 8.

SOLUTION We can use ordered pairs to list the outcomes. Thus, we have the set of thirty-six sample points shown at the right.

$$\begin{matrix} \{(1, 1), & (1, 2), & (1, 3), & (1, 4), & (1, 5), & (1, 6) \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), & (2, 5), & (2, 6) \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), & (3, 5), & (3, 6) \\ (4, 1), & (4, 2), & (4, 3), & (4, 4), & (4, 5), & (4, 6) \\ (5, 1), & (5, 2), & (5, 3), & (5, 4), & (5, 5), & (5, 6) \\ (6, 1), & (6, 2), & (6, 3), & (6, 4), & (6, 5), & (6, 6)\} \end{matrix}$$

The event that the sum of the spots showing is 8 is then $\{(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)\}$.

You should keep in mind that the sample space always consists of the possible outcomes of the experiment, not the events in which you may be interested. Thus, in Example 2 it would be incorrect to consider the sample space to be the set consisting of the possible sums of the spots showing on the dice.

The list of outcomes in the sample space in Example 2 suggests that it would be useful to discuss such sample spaces in terms of Cartesian products. Thus, the sample space for the first die is

$$A = \{1, 2, 3, 4, 5, 6\}$$

and that for the second die is the same. Then the sample space for the experiment is just $A \times A$. A *lattice* which portrays this sample space is shown in Figure 2. Experiments such as tossing coins or dice, in which the outcome is purely a matter of chance, are said to have *random* outcomes.

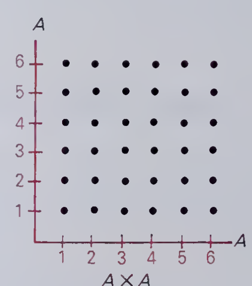


Figure 2

EXAMPLE 3 A letter is selected at random from those in the word HEART.

a. List the sample space.

b. List the event that the letter selected is a vowel.

SOLUTION **a.** {H, E, A, R, T}. **b.** {E, A}.

Oral Exercises

For the experiment in which a letter is selected at random from those in the word PENCIL, state the following:

1. the sample space
2. the event that the letter selected is a vowel
3. the event that the letter selected is a consonant
4. the event that the letter selected is the fifth letter of the alphabet

Written Exercises

In Exercises 1–8: (a) list a sample space for the given experiment, and (b) list the given event.

A

1. A number is chosen from the integers 1 through 11. Event: The number is a multiple of 3.
2. A sock is drawn from a bag containing 4 white socks and 3 black socks. Event: The sock is black.
3. A coin is flipped twice. Event: Heads appears at least once.
4. A tag is drawn from a box containing three tags numbered 1, 2, and 3. The tag is returned to the box, and another tag is drawn. Event: Both tags have the same number.
5. Same experiment as in Exercise 4 except that the first tag is not returned before the second is drawn. Event: The sum of the two numbers is even.
6. Two cards are drawn randomly from the four aces of a standard bridge deck. Event: The cards are both red.
7. Two *different* letters are chosen randomly from the letters A, B, C, and E. Event: At least one of the two letters is a vowel.
8. Same experiment as in Exercise 7. Event: The two letters are not both vowels.

In Exercises 9–16, the experiment is drawing the given number of cards from a standard 52-card bridge deck. Tell *how many* elements are in the sample space and *how many* elements are in the given event. Recall that the sample space consists of all the possible outcomes of the experiment.

B

9. A single card is drawn; it is a heart.
10. Of two cards drawn, both are kings.
11. Of two cards drawn, both are red.
12. Of two cards, both are face cards.
13. Of two cards drawn, both are clubs.
14. Of three cards drawn, all are spades.

C

15. Of five cards drawn, three are diamonds and two are spades.
16. Five cards are drawn, and all four suits are represented among them.

12-9 Mathematical Probability

Consider the following experiment: From a bag containing 5 blue and 12 white marbles, one marble is drawn.

If the experiment is designed so that each marble is just as likely to be drawn as any other, we say that the experiment has 17 *equally likely* outcomes. The event that the marble drawn is white consists of 12 outcomes. Therefore, if you replace the marble drawn and repeat the experiment many times, it seems reasonable to expect that about $\frac{12}{17}$ of the time you will find that you have drawn a white marble. This ratio, $\frac{12}{17}$, is called the *probability* that the outcome of any single trial of the experiment will be the drawing of a white marble. This example suggests the following definition:

> Let S be a sample space of an experiment in which there are n possible outcomes, each equally likely. If an event A is a subset of S such that A contains h elements, then the probability of an event A, denoted by $P(A)$, is given by $P(A) = \dfrac{h}{n}$.

EXAMPLE 1 If two cards are drawn at random from a standard 52-card bridge deck, what is the probability that both cards are hearts?

SOLUTION Since there are 13 hearts in the deck, there are ${}_{13}C_2$ ways in which two of them can be drawn. There are 52 cards altogether, so there are ${}_{52}C_2$ possible ways in which two of them can be drawn. If A represents drawing 2 hearts, then

$$P(A) = \frac{{}_{13}C_2}{{}_{52}C_2} = \frac{\dfrac{13 \cdot 12}{1 \cdot 2}}{\dfrac{52 \cdot 51}{1 \cdot 2}} = \frac{78}{1326} = \frac{1}{17}. \quad \text{Answer.}$$

In the foregoing example, the answer $\frac{1}{17}$ does not tell you anything *certain* about what is going to happen. It does not, for example, tell you that you will get exactly one pair of hearts out of 17 draws. You might get one such draw, or you might get none, or you might even get 17. However, if you perform the experiment a very large number of times, the ratio of the number of times you draw 2 hearts to the total number of draws will probably come close to $\frac{1}{17}$.

An event A in the sample space S is called *certain* if $A = S$; it is called *impossible* if $A = \emptyset$. Since $P(S) = \dfrac{n}{n} = 1$, while $P(\emptyset) = \dfrac{0}{n} = 0$, the probability is 1 for a certain event and 0 for an impossible one. Do you see that the probability of an event which is neither certain nor impossible is a number between 0 and 1?

By the symbol $\bar{A}$ (read "the **complement** of A"), we mean the set of the elements of S that are *not* members of A. If A has h members, then $\bar{A}$ contains $n - h$ elements. Therefore, $P(\bar{A})$ is the probability that A does *not* occur, and

$$P(\bar{A}) = \frac{n-h}{n} = 1 - \frac{h}{n} = 1 - P(A).$$

The **odds** that the event A will occur are given by

$$\frac{P(A)}{P(\bar{A})}, \quad \text{or} \quad \frac{h}{n-h}, \quad \text{or} \quad h \text{ to } n - h.$$

Thus, in the original experiment the odds are *12 to 5 in favor of* drawing a white marble or *5 to 12 against* drawing a white marble.

EXAMPLE 2 Two marbles are drawn at random from an urn containing 14 red and 12 blue marbles.

a. What is the probability that at least one marble is blue?

b. What are the odds that at least one marble is blue?

SOLUTION a. The probability that at least one marble is blue, is just the probability that *not both* marbles are *red*. Let A represent the event that both *are* red. Then

$$P(A) = \frac{{}_{14}C_2}{{}_{26}C_2} = \frac{14 \cdot 13}{26 \cdot 25} = \frac{7}{25},$$

and the probability that at least one marble is blue is just

$$P(\bar{A}) = 1 - \tfrac{7}{25} = \tfrac{18}{25}.$$

b. The odds that at least one marble is blue are $\dfrac{18}{25 - 18}$, or 18 to 7.

Oral Exercises

A single die is tossed. What is the probability that the number of spots showing is:

1. six?
2. even?
3. odd?
4. less than 3?

5–8. Give the odds for each of the events in Exercises 1–4 above.

Written Exercises

A 1. One letter is selected at random from the first 10 letters of the alphabet. What is the probability that the letter is:

a. a vowel?
b. a consonant?
c. before E in the alphabet?
d. in the word SIDEWALK?

2. In Exercise 1, what are the odds for each event?

3. Two dice are thrown. Refer to the sample points in Example 2, page 442, to decide the probability of each of the following events.
   a. The sum of the numbers showing is 7.
   b. Both dice show the same number.
   c. The dice show different numbers.
   d. The sum of the numbers showing is 4 or 6.
4. In Exercise 3, what are the odds for each event?
5. Two socks are drawn at random from a drawer containing 3 red, 5 green, and 4 blue socks. What is the probability of each of the following events?
   a. Both are red.
   b. Both are green.
   c. Both are blue.
   d. Neither is red.
   e. Neither is green.
   f. Neither is blue.
6. Are the events of parts a and d in Exercise 5 above complements of each other? Give a reason for your answer.
7. A coin is tossed 4 times. What is the probability of getting:
   a. all heads?
   b. exactly 1 tail?
   c. exactly 2 heads?
   d. exactly 3 tails?
   e. at least 1 head?
   (*Hint:* Consider the complement.)
8. On a 10-question true-false test:
   a. How many possible ways are there of answering all the questions?
   b. If the questions are answered at random, what is the probability of answering exactly 7 questions correctly?
   c. If the questions are answered at random, what is the probability of answering exactly 8 questions correctly?

B

9. A three-digit numeral with no repeated digits is made from the digits 1 through 7. What is the probability that the number indicated is
   a. odd?
   b. a multiple of 5?
   c. between 300 and 500?
   d. between 300 and 650?
10. Four cards are drawn at random from the 13 hearts in a standard deck. What is the probability that the selection contains:
   a. at least two face cards (jack, queen, or king)?
   b. no face cards?
   c. both the queen and the king?
   d. the queen and not the king?
11. A committee of six is to be chosen by lot from a group of 10 people. If Herb and Nora are among those being considered, what is the probability of each of these events?
   a. Both Herb and Nora will be on the committee.
   b. Neither Herb nor Nora will be on the committee.
   c. Nora but not Herb will be on the committee.
   d. At least one of the two will be on the committee. (*Hint:* See part b.)

12. The letters of the word QUIET are rearranged at random. What is the probability of each of the following events?
a. Q will be followed directly by U.
b. Q and U will be together in either order.

13. Two different letters are chosen at random from the word FACETIOUS. What is the probability of each of the following events?
a. Both are vowels. b. Both are consonants.

14. A committee of 6 is to be chosen at random from 6 juniors and 5 seniors. What is the probability that the committee will contain exactly 3 seniors and 3 juniors?

15. Four boys and three girls are seated at random at 7 desks in a row. What is the probability that the boys and girls are in alternate seats?

C 16. In Exercise 15 what is the probability that the three girls are in 3 adjacent seats?

12-10 Mutually Exclusive Events

The diagram* in Figure 3 shows the sample space S for the experiment of drawing a number from $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The diagram also shows two events A and B in S. A corresponds to the drawing of a number less than 4, so that $A = \{1, 2, 3\}$. B corresponds to the drawing of an even number; that is, $B = \{2, 4, 6, 8\}$. Therefore,

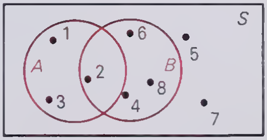


Figure 3

$$P(A) = \tfrac{3}{8} \quad \text{and} \quad P(B) = \tfrac{4}{8} = \tfrac{1}{2}.$$

What is the probability that either A or B (or both) will occur? This amounts to asking for $P(A \cup B)$. Since $A \cup B = \{1, 2, 3, 4, 6, 8\}$, $P(A \cup B) = \frac{6}{8} = \frac{3}{4}$. To see the relationship between $P(A \cup B)$, $P(A)$, and $P(B)$, notice that the intersection of A and B $(A \cap B)$ is $\{2\}$. Also, $P(\{2\}) = \frac{1}{8}$ and

$$P(A \cup B) = \frac{6}{8} = \frac{3 + 4 - 1}{8} = \frac{3}{8} + \frac{4}{8} - \frac{1}{8}.$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

You can prove that this relationship holds for any two events A and B in a sample space. If the events have no outcome in common, that is, $A \cap B = \emptyset$, we say that the events are **mutually exclusive.** For mutually exclusive events, $P(A \cap B) = P(\emptyset) = 0$.

*Such diagrams are used to picture set relationships and are called *Venn diagrams* in honor of the English mathematician John Venn (1834–1923).

Thus:

> If A and B are mutually exclusive events,
> $$P(A \cup B) = P(A) + P(B).$$

EXAMPLE Three marbles are drawn at random from an urn containing 4 white, 3 red, and 5 blue marbles. What is the probability that at least one of them is red?

SOLUTION The desired probability is the sum of the probabilities that *exactly* one marble is red (A), that *exactly* two marbles are red (B), or that *exactly* three marbles are red (C).

$$P(A) = \frac{{}_3C_1 \times {}_9C_2}{{}_{12}C_3} = \frac{3 \times 36}{220} = \frac{108}{220}$$

$$P(B) = \frac{{}_3C_2 \times {}_9C_1}{{}_{12}C_3} = \frac{3 \times 9}{220} = \frac{27}{220}$$

$$P(C) = \frac{{}_3C_3}{{}_{12}C_3} = \frac{1}{220}$$

$\therefore$ the required probability is $\frac{108}{220} + \frac{27}{220} + \frac{1}{220} = \frac{136}{220} = \frac{34}{55}$. **Answer.**

Oral Exercises

A single die is tossed. State whether the events are mutually exclusive.

1. The number of spots showing is less than 3; the number of spots showing is odd.
2. The number of spots showing is odd; the number of spots showing is even.
3. Six spots are showing; the number of spots showing is even.
4. The number of spots showing is greater than 4; the number of spots showing is less than or equal to 3.

Written Exercises

A 1. Six coins are flipped simultaneously. What is the probability there are:

a. at least 5 heads?
b. at least 4 heads?
c. at least 3 tails?
d. no heads?
e. at least 4 heads or at least 3 tails?

2. Two marbles are drawn at random from a bag containing 3 red, 5 blue, and 6 green marbles. What is the probability of drawing:
   a. at least one red marble?
   b. at least one green marble?
   c. two marbles of the same color?
   d. two marbles of different colors?
3. What is the probability of getting 8 or more questions correct on a 10-question true-false test if the questions are answered at random?
4. Two cards are drawn from a standard 52-card deck. What is the probability that the cards are:
   a. of the same suit?
   b. a pair?
   c. both spades or both jacks?
5. Three letters are chosen at random from the word RECORD. What is the probability that the selection will contain:
   a. E or O but not both?
   b. E or O or both?
6. Three dice are thrown. What is the probability of the following events?
   a. All three dice show the same number.
   b. Exactly 2 of the dice show the same number.

B

7. Two letters are chosen at random from the word GEESE and two are chosen at random from the word PLEASE. What is the probabity that the selection contains 4 E's or no E's?
8. Five letters are chosen at random from the first 10 letters of the alphabet. Find the probability that the selection contains A or B but not both.

C

9. Four beads are drawn at random from a box containing 4 black, 4 white, and 2 red beads. What is the probability that the selection will contain exactly 2 black beads or exactly 2 white beads or both? (Note that the two events of drawing exactly 2 white beads and of drawing exactly 2 black beads are not mutually exclusive.)
10. In Exercise 9, what is the probability of drawing *exactly* 2 beads of the *same color?*

12-11 Independent and Dependent Events

Suppose two balls are drawn at random from a bag containing 4 red and 3 black balls. What is the probability that both balls drawn are red?

Let A be the event of drawing a red ball the first time, and B, the event of drawing a red ball the second time. The sample space S of the experiment depends on whether or not the first ball drawn is returned to the bag before the second ball is chosen. The question amounts to asking for $P(A \cap B)$.

Case I. The first ball is replaced before the second is drawn.

The sample space S consists of all ordered pairs (x, y) where both x and y denote elements of a set of 7 outcomes (4 red, 3 black).

Figure 4 shows the sample space containing 7×7, or 49, sample points. We use r_i and b_i to designate the drawing of red and black balls, respectively. The colored dashed rectangle outlines all possible experiment outcomes in which the first ball is red, and the colored solid rectangle outlines all possible outcomes in which the second ball is red. Since $A \cap B$ consists of the ordered pairs of the form (red, red), there are $4 \times 4 = 16$ elements in $A \cap B$. Therefore,

$$P(A \cap B) = \tfrac{16}{49} = \tfrac{4}{7} \cdot \tfrac{4}{7}.$$

Note:

$$P(A \cap B) = P(A) \cdot P(B).$$

The outcomes in $A \cap B$ are in the colored shaded region in Figure 4.

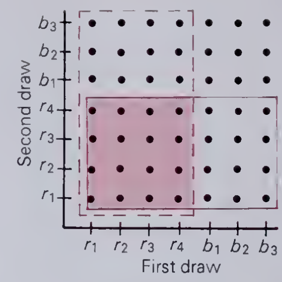


Figure 4

Case II. The first ball is not replaced before the second is drawn.

Any one of the 7 balls may be selected on the first draw, but since this ball will not be replaced, there remain only 6 balls for the second draw. All ordered pairs with equal components must be deleted from the sample space. Note in Figure 5 that one diagonal will not be in the new sample space. Therefore, there are 7×6, or 42, ordered pairs possible. The number of these that are of the form (red, red) is 4×3, or 12, because any of the 4 red balls can be the first, but there are only 3 choices for the second red ball. Thus,

$$P(A \cap B) = \tfrac{12}{42} = \tfrac{2}{7}.$$

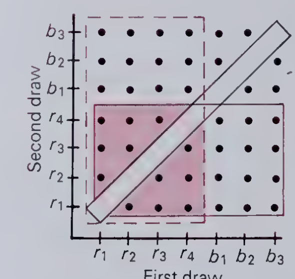


Figure 5

Analyzing this result as

$$P(A \cap B) = \tfrac{12}{42} = \tfrac{4}{7} \cdot \tfrac{3}{6},$$

you can see that

$$\tfrac{4}{7} = P(A).$$

We can interpret the second factor, $\tfrac{3}{6}$, as the *probability that the second ball drawn is red under the condition that the first ball drawn was red;* we shall denote this probability by $P(B \mid A)$.

This example suggests a general law for a **conditional probability.**

Let $P(A)$ denote the probability of an event A, and $P(B \mid A)$ denote the conditional probability of an event B given that event A has occurred. If $P(A \cap B)$ is the probability that A and B occur, then:

$$P(A \cap B) = P(A) \cdot P(B \mid A).$$

This relation is symmetric. Thus:

$$P(A \cap B) = P(B) \cdot P(A \mid B).$$

We say that events A and B are *independent* when the probability of one does not depend on the occurrence of the other. For example, in Case I where the balls are drawn *with replacement,* the events A and B are independent because the outcome on the first draw does not affect the outcome on the second draw. Thus, we define two events A and B as **independent events** if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

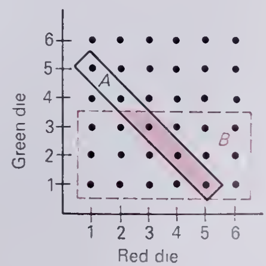
For independent events, $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$. Two events that are not independent are said to be **dependent.**

EXAMPLE 1 A red die and a green die are thrown. What is the probability that the sum of the numbers shown is 6 and that the green die shows a number 3 or less? Are these events independent?

SOLUTION Graph the sample space. Let A be the event that the sum of the numbers is 6. This is shown by the solid rectangle.

Let B be the event that the green die shows a number 3 or less. This is shown by the dashed colored rectangle.

Then $A \cap B$ is shown by the shaded region. You have



$P(A) = \frac{5}{36}$, $P(B) = \frac{18}{36} = \frac{1}{2}$, and

$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$. Answer

Since $\frac{1}{12} \neq \frac{5}{36} \cdot \frac{1}{2}$, $P(A \cap B) \neq P(A) \cdot P(B)$. Hence, A and B are dependent events. Answer

In many cases, rather than use the relationship

$$P(A \cap B) = P(A) \cdot P(B)$$

to determine the independence of events A and B, you use the fact that A and B are obviously independent events and employ the relationship to find $P(A \cap B)$.

EXAMPLE 2 A die is thrown and a coin is tossed. What is the probability that the die shows a 2 or a 3 and the coin shows a head?

SOLUTION Let A be the event that the die shows a 2 or a 3, and B be the event that the coin shows a head. These events are clearly independent. Hence, knowing that $P(A) = \frac{2}{6} = \frac{1}{3}$, and $P(B) = \frac{1}{2}$, you have

$$P(A \cap B) = \tfrac{1}{3} \cdot \tfrac{1}{2} = \tfrac{1}{6}. \quad \text{Answer.}$$

Oral Exercises

In each exercise, use the probabilities given to tell whether A and B are dependent or independent events.

1. $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$, $P(A \cap B) = \frac{1}{20}$
2. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{6}$
3. $P(A) = 0.2$, $P(B) = 0.4$, $P(A \cap B) = 0.15$
4. $P(A) = 0.25$, $P(B) = 0.2$, $P(A \cap B) = 0.05$
5. $P(A) = 0.1$, $P(B) = 0.2$, $P(A \cap B) = 0.1$
6. $P(A) = 0.25$, $P(B) = 0.25$, $P(A \cap B) = 0.25$

Written Exercises

A

1. Given that $P(C) = 0.3$ and $P(D \mid C) = 0.15$, find $P(C \cap D)$.
2. Given that $P(K) = \frac{1}{3}$ and $P(J \cap K) = \frac{1}{6}$, find $P(J \mid K)$.
3. A coin is tossed twice and a die is rolled once. Find the probability of each event.
   a. Two heads are tossed and a five shows on the die.
   b. Exactly one head is tossed and a one or a two shows on the die.
   c. At least one head is tossed, and a two does not show on the die.
   d. At most one head is tossed and a number less than 5 shows on the die.
4. Two marbles are drawn at random from a bag containing 3 green, 4 orange, and 5 red marbles. What is the probability of drawing:
   a. one green and one red?
   b. one orange and one green?
   c. one red and one non-orange?
   d. one orange but no red?

B

5. Two cards are drawn at random from a 52-card deck and then replaced; two more are then drawn. What is the probability of each event?
   a. The first two are spades and the second two are tens.
   b. The first two are face cards and the second two are aces.
   c. All four are hearts.
   d. Two of the cards are hearts and two of them are spades.

6. Two *different* letters are chosen at random from the word NUMERAL and three numbers are chosen at random from the numbers 1, 2, 3, 4, 5, and 6 (*with* repetition allowed). What is the probability of getting:
   a. A and N and three of the same digit?
   b. two vowels and three odd digits?
   c. two consonants and no 3's among the digits?
7. A single die is rolled twice. If A is the event that the sum of the numbers showing is 6, and B is the event of at least one 5 showing, calculate $P(A)$, $P(B)$, $P(A \mid B)$, and $P(B \mid A)$, and show that $P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$.
8. Two cards are drawn at random from a 52-card deck. Calculate the probabilities of the following events to determine which is more likely. A: Both cards are of the same color. B: One card is red and the other is black.
9. The student council consists of 7 girls and 5 boys, of whom 3 girls and 3 boys are seniors. If a 4-person dance committee is to be chosen by lot, what is the probability that the committee will consist of:
   a. 2 boys and 2 girls?
   b. 3 girls and 1 boy?
   c. 2 senior boys and 2 senior girls?
   d. all seniors, of whom at least 2 are girls?
10. A coin is tossed 10 times and 2 dice are tossed. What is the probability of the event that at least 8 heads are tossed and the sum of the numbers on the dice is not 3?
11. Consider two mutually exclusive events A and B such that $P(A) \neq 0$ and $P(B) \neq 0$.
   a. Find $P(A) \cdot P(B \mid A)$.
   b. Tell whether A and B are independent.

In Exercises 12–16, a cube with two adjacent faces painted green, two adjacent faces painted yellow, and two adjacent faces painted red is tossed and lands on one of its faces. The exercises refer to the following events:

A: The cube lands on a green face.
B: The top face is red.
C: Two of the vertical faces are green.
D: Exactly one of the vertical faces is red.

C 12. Find the probabilities of events A, B, C, and D.
13. Are events A and B independent? Explain your answer.
14. Are events B and C independent? Explain your answer.
15. Are events B and D independent? Explain your answer.
16. Which pair(s) of events are mutually exclusive?

Careers

in Actuarial Science

Actuaries base many of their decisions on interpretations of statistical data which they receive.

Actuaries collect and analyze statistical data relating to insurance. Actuaries must apply their knowledge of mathematics and statistics, as well as principles of business and finance. Some of the typical problems actuaries solve are: determining mortality, accident, sickness, disability, and retirement rates; finding the probability of fires, natural disasters, and unemployment; designing insurance and pension plans; calculating premiums; and determining the amount of money an insurance company needs to cover payment of its benefits.

Most actuaries are employed by private insurance companies. They may also serve as consultants for large corporations, where they advise on insurance and pension plans for the employees. The government also employs actuaries who may be involved in the regulation of insurance companies or in social security programs.

EXAMPLE In calculating the premium for a one-year term life insurance policy, an actuarial worker has the following information. Out of a given number of people alive at a given age at the beginning of a year, a certain number will be likely to die during that year. Under the terms of the policy, the beneficiary is to receive \$1000 in the event of the death of the policyholder. The problem is to determine the amount of the premium to be charged in order to cover payments to the beneficiaries.

SOLUTION The premium for persons of a given age is calculated by the following formula:

$$\text{Premium} = \frac{\begin{pmatrix}\text{amt. pd. to}\\ \text{ea. beneficiary}\end{pmatrix} \cdot \frac{100}{103} \cdot \begin{pmatrix}\text{number likely to}\\ \text{die during year}\end{pmatrix}}{\text{number living at beginning of year}}$$

The factor $\frac{100}{103}$ takes into account the interest the collected premiums will earn before any benefits are paid. For age 35, probability tables show that of 9374 people alive at the beginning of a year, 24 will die during the year.

$$\text{Premium} = \frac{1000 \cdot \frac{100}{103} \cdot 24}{9374}$$

$$\approx \$2.49 \qquad \text{(for \$1000 life insurance for 1 year, at age 35)}$$

Self-Test 4

VOCABULARY

sample space (p. 441)
sample point (p. 441)
event (p. 442)
complement (p. 445)
odds (p. 445)
mutually exclusive (p. 447)
conditional probability (p. 450)
independent events (p. 451)
dependent events (p. 451)

1. A letter is selected at random from the letters A, B, and C, and a number is selected at random from the numbers 1, 2, and 3. List a sample space for this experiment and list the event that the letter is not C and the number is odd. *Obj. 1, p. 441*
2. An integer from 1 to 8 inclusive is selected at random. What is the probability the integer is greater than 6? *Obj. 2, p. 441*
3. In Test Item 2, what is the probability the integer is greater than 6 or else it is 3 or 5? *Obj. 3, p. 441*
4. Two dice are tossed. Let A be the event one die shows a one. Let B be the event that the sum of the numbers shown is 8. Are the events independent? Why or why not? *Obj. 4, p. 441*

Check your answers with those at the back of the book.

Chapter Summary

1. If finite sets A, B, and $A \cap B$ contain r, s, and t elements, respectively, then $A \cup B$ contains $r + s - t$ elements and $A \times B$ contains rs elements. From these fundamental counting principles you can derive a formula for the number of *permutations* of n elements, r at a time.
$$_nP_r = n(n-1)(n-2) \cdots (n-(r-1)).$$
2. The number of *combinations* of n things, r at a time, is given by $_nC_r = \frac{_nP_r}{r!}$. Also, $_nC_r = {_nC_{n-r}}$. The *coefficients of the expansion* $(a+b)^n$ can be expressed as numbers of combinations according to the Binomial Theorem. (See page 437.)
3. If there are h ways in which an event A can occur, in n possible outcomes, all of which are equally likely, $0 \le h \le n$, then the *probability* of A is $P(A) = \frac{h}{n}$. If $h < n$, the *odds* in favor of A are $\frac{h}{n-h}$. Probabilities may be discussed in terms of *sample spaces* and *events*.
4. The probability that at least *one* of the events A and B will occur is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. When A and B are *mutually exclusive*, $P(A \cap B) = 0$.

5. The probability that *two* events A and B will occur is given by $P(A \cap B)$ where $P(A \cap B) = P(A) \cdot P(B|A)$. $P(B|A)$ is the *conditional probability* that B will occur given that A has occurred.
6. Two events are *independent* if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

Chapter Review

1. $A = \{1, 2, 3\}$, $B = \{2, 4\}$. State the number of elements in $A \times B$. 12-1

   a. 1 b. 4 c. 5 d. 6

2. A contains 4 elements, B contains 3 elements, and $A \cap B$ contains 2 elements. State the number of elements in $A \cup B$.

   a. 7 b. 5 c. 9 d. 12

3. In how many ways can 4 books be arranged on a shelf? 12-2

   a. 24 b. 6 c. 4 d. 10

4. In how many ways can 5 people be seated at a circular table?

   a. 120 b. 15 c. 24 d. 5

5. Evaluate ${}_{10}P_4$.

   a. 5040 b. 151,200 c. 30,240 d. 604,800

6. State the number of distinguishable permutations of the letters in the word MISSISSIPPI. 12-3

   a. 69,300 b. 3150 c. 138,600 d. 34,650

7. Evaluate ${}_{36}C_{33}$. 12-4

   a. 3570 b. 7140 c. 14,280 d. 21,420

8. In how many different ways can 3 people be selected for the math team from a group of 7 people?

   a. 35 b. 840 c. 210 d. 21

9. In how many different ways can you select 3 blue and 4 red marbles from an urn containing 6 blue and 5 red marbles? 12-5

   a. 20 b. 25 c. 330 d. 100

10. Use the Binomial Theorem to find the fourth term in the expansion $(3x + 2)^7$. 12-6

    a. $840x^4$ b. $2923x^4$ c. $22{,}680x^4$ d. $210x^3$

11. Use Pascal's triangle to find the third term in the expansion $\left(\frac{x}{2} + 3\right)^5$. 12-7

    a. $\frac{45}{4}x^3$ b. $45x^3$ c. $\frac{19}{2}x^3$ d. $\frac{27}{4}x^3$

12. Four coins are tossed. How many elements are in the sample space? *12-8*

a. 4 b. 16 c. 15 d. 24

13. A number is selected at random from the numbers 1, 2, 3, . . . , 15. What is the probability that the number is prime? *12-9*

a. $\frac{7}{15}$ b. $\frac{2}{5}$ c. $\frac{1}{3}$ d. $\frac{8}{15}$

14. A marble is selected at random from an urn containing 6 black and 4 white marbles. What are the odds in favor of the marble being white?

a. 2 to 5 b. 2 to 3 c. 3 to 2 d. 1 to 3

15. Two marbles are drawn at random from an urn containing 3 black, 7 yellow, and 5 green marbles. What is the probability that at least one of them is black? *12-10*

a. $\frac{1}{22}$ b. $\frac{6}{11}$ c. $\frac{13}{35}$ d. $\frac{5}{17}$

16. A die is rolled and a coin is tossed. What is the probability that the die shows a 3 and the coin lands heads? *12-11*

a. $\frac{1}{6}$ b. $\frac{2}{3}$ c. $\frac{1}{4}$ d. $\frac{1}{12}$

Chapter Test

1. How many 3-letter code words can be formed from the letters in the word COMPLEX? *12-1*
2. How many different 6-digit license plates can be made using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, if no digit can be repeated? *12-2*
3. How many different permutations exist of all the letters in the word DIFFERENCE? *12-3*
4. Given a set with 10 elements. How many 4-element subsets are there? *12-4*
5. How many different ways can a hand consisting of 2 aces and 3 queens be chosen from a standard 52-card bridge deck? *12-5*
6. Find the eighth term of $(\frac{3}{4}x - 1)^{13}$. *12-6*
7. Use Pascal's triangle to expand $(y - 3)^5$. *12-7*
8. A coin is tossed and a die is rolled. List the sample space. *12-8*
9. An integer between 20 and 35 inclusive is selected at random. What is the probability it is a multiple of 3? *12-9*
10. Two marbles are drawn at random from an urn containing 8 black, 7 white, and 5 red marbles. What is the probability that 1 marble is black and the other is red? *12-10*
11. A drawer contains 7 blue socks, 3 green socks, and 6 brown socks. If you choose 2 socks at random, what is the probability that they are both blue? *12-11*

Cumulative Review (*Chapters 9–12*)

1. Simplify $6\sqrt{-\frac{2}{3}}$.

   a. $-2\sqrt{6}$ b. $-2i$ c. $2i\sqrt{6}$ d. $-3i\sqrt{6}$

2. Express $(3 - 2i)^2$ in the form $a + bi$.

   a. $13 - 12i$ b. $5 - 6i$ c. $5 - 12i$ d. $1 - 4i$

3. For $P(x) = x^4 - 3x^3 + 2x - 5$, use synthetic substitution to find $P(-i)$.

   a. $-4 - i$ b. $-6 - i$ c. $-4 - 5i$ d. $6 - 5i$

4. Given that $2i$ is a root of $x^4 + 3x^2 - 4 = 0$, find the complete solution set.

   a. $\{2i, -2i, -2, \frac{1}{2}\}$ b. $\{2i, -2i, 2, -1\}$ c. $\{2i, -2i, 1, -1\}$

5. Find the distance between $P_1(-2, 3)$ and $P_2(-5, -6)$.

   a. $\sqrt{58}$ b. $3\sqrt{10}$ c. $\sqrt{130}$ d. $3\sqrt{2}$

6. Find an equation of the line passing through $(-2, 1)$ that is perpendicular to the graph of $3y - 2x = 10$.

   a. $2y - 3x = -4$ b. $3x + 2y = -2$ c. $2y + 3x = -4$

Review Items 7 amd 8 refer to the circle with equation

$$x^2 - 8x + y^2 + 2y = 8.$$

7. Find the radius of the circle.

   a. 5 b. $2\sqrt{2}$ c. 4 d. 3

8. Find the coordinates of the center of the circle.

   a. $(4, 1)$ b. $(-4, 1)$ c. $(4, -1)$ d. $(-4, -1)$

9. Find the function of the form $y = a(x - h)^2 + k$ whose graph has vertex $(2, 4)$ and passes through the point $(1, 6)$.

   a. $y = 4(x - 2)^2 - 2$ b. $y = -2(x + 2)^2 - 1$ c. $y = 2(x - 2)^2 + 4$

10. Write an equation of an ellipse with foci at $(0, 2)$ and $(0, -2)$ whose x-intercepts are $(3, 0)$ and $(-3, 0)$.

    a. $\frac{x^2}{9} + \frac{y^2}{25} = 1$ b. $\frac{x^2}{9} + \frac{y^2}{13} = 1$ c. $\frac{x^2}{13} + \frac{y^2}{9} = 1$ d. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

11. Give the equations of the asymptotes to the hyperbola $\frac{x^2}{81} - \frac{y^2}{16} = 1$.

    a. $y = -\frac{16}{81}x,\ y = \frac{16}{81}x$ b. $y = \frac{4}{9}x,\ y = -\frac{4}{9}x$ c. $y = \frac{9}{4}x,\ y = -\frac{9}{4}x$

12. If x varies inversely as y and directly as z^2, and $x = 3$ when $y = 24$ and $z = 6$, find x when $y = 16$ and $z = 8$.

    a. 8 b. 16 c. $\frac{9}{8}$ d. $\frac{8}{9}$

13. Solve over $\mathcal{C}$: $y = x^2 + 2x + 3$
$5 = y - x$

a. $\{(6, 1)\}$ b. $\{(-3, 2)\}$ c. $\{(-2, 3), (1, 6)\}$ d. $\{(3, 2), (-1, 6)\}$

14. Simplify $\left(\frac{1}{4b^4}\right)^{-\frac{3}{2}}$

a. $8b^6$ b. $-8b^6$ c. $\frac{1}{8b^6}$ d. $\frac{-1}{8b^3}$

15. State the inverse of the function $y = 3x - 2$.

a. $y = \frac{x}{3} - \frac{2}{3}$ b. $y = \frac{x}{3} + \frac{2}{3}$ c. $y = \frac{x}{2} - \frac{3}{2}$ d. $y = \frac{x}{2} + \frac{3}{2}$

16. Solve $x^{\frac{3}{2}} = 216$ over $\mathcal{R}$.

a. $\{6\}$ b. $\{36\}$ c. $\{18\}$ d. $\{64\}$

17. Simplify $\log_4 32$.

a. 4 b. 5 c. $-\frac{1}{2}$ d. $\frac{5}{2}$

18. Solve $\log x + \log 6 = \log 3$ over $\mathcal{R}$.

a. $\{3\}$ b. $\{\frac{1}{2}\}$ c. $\{30\}$ d. $\{18\}$

19. Use Table 5 to find $\log(350 \div 126)$.

a. 1.4407 b. 0.4437 c. 4.4437 d. 0.4042

20. How many different codes of a single letter followed by 2 digits are there?

a. 6760 b. 2340 c. 2600 d. 5790

21. How many different permutations exist of the letters in CAREER?

a. 120 b. 180 c. 720 d. 360

22. In how many different ways can a committee of 4 be selected from a group of 7 people?

a. 105 b. 35 c. 210 d. 21

23. Two marbles are drawn at random from a bag containing 4 red, 3 black, and 2 green marbles, with the first marble not being replaced before the second is drawn. What is the probability that both are green?

a. $\frac{4}{81}$ b. $\frac{2}{9}$ c. $\frac{1}{36}$ d. $\frac{2}{81}$

24. Which of the following is equal to $64 - 144k + 108k - 27k^3$?

a. $(3 - 4k)^3$ b. $(4 - 3k)^4$ c. $(4 - 3k)^3$ d. $(3 + 4k)^3$

25. Find the fourth term in the binomial expansion $(a - 2b)^4$.

a. $-32ab^3$ b. $-8ab^3$ c. $8a^3b^3$ d. $-32a^3b^3$

This experimental rail vehicle being developed in Japan has achieved speeds of more than three hundred kilometers per hour. By means of a magnetic levitation device the vehicle will ride one hundred millimeters above the central coil.

13 Matrices

Basic Properties of Matrices

***OBJECTIVES** for Sections 13-1 through 13-5:*
1. Find sums and differences of matrices.
2. Find the product of a scalar and a matrix.
3. Solve certain matrix equations.
4. Find the product of two matrices.

13-1 Matrices and Their Sums

In Chapter 4 you saw that it is sometimes convenient to name a number by means of a square array of numerals. Thus the *determinant* $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ names the number $ad - bc$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The array $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ itself is called a *matrix*. Notice that brackets [ ] are used to distinguish the matrix from the determinant. In general, a rectangular array such as that shown in red at the right is a **matrix** (plural, **matrices**).

Each numeral in the array is an **entry** of the matrix. The number of (horizontal) rows and the number of (vertical) columns of entries in the matrix are its **dimensions**. For example, the matrix displayed has three rows and two columns and is called a

$$\begin{array}{cc} & \text{column} \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{r} \text{row } 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 5 & 2 \end{bmatrix} \end{array}$$

3×2 (read "three by two") matrix.

Notice that the number of rows is given first and then the number of columns. In this book, we shall use only real numbers for entries of a matrix. An entry of a matrix is referred to by giving its row and column numbers. In matrix A, below, the entry in the third row and second column is 3.

Capital letters, such as A, B, C, are used to denote matrices, and sometimes subscripts, as in $A_{3\times 2}$, are used to represent the dimensions of a matrix. Thus, for

$$A = \begin{bmatrix} 4 & 1 & 7 \\ 2 & 0 & 6 \\ 5 & 3 & 2 \end{bmatrix}, \qquad B = [1 \quad 5], \qquad \text{and} \qquad C = \begin{bmatrix} -1 \\ 7 \\ 8 \end{bmatrix},$$

you might write $A_{3\times 3}$, $B_{1\times 2}$, and $C_{3\times 1}$. Matrices, such as B and C, which have only one row or one column are called *row matrices* and *column matrices* (or *row vectors* and *column vectors*), respectively. Since the matrix $A_{3\times 3}$ has 3 rows and 3 columns, it is said to be a *square matrix of order* 3. Similarly, we have square matrices of order 2, square matrices of order 4, and so on.

For two matrices of the same dimensions, the entries in the same row and same column are said to be *corresponding entries.*

Two matrices are *equal* if and only if they have the same dimensions and all their corresponding entries are equal. Notice that matrices of different dimensions are never said to be equal, even if corresponding entries, as far as they extend, are equal. For instance,

$$[4 \quad 5] \neq \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE Find values of x, y, and z so that

$$\begin{bmatrix} x - 2 & y + 5 \\ z + 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 0 & 2 \end{bmatrix}.$$

SOLUTION By the definition of equal matrices, this matrix equation is true if and only if

$$x - 2 = 4, \qquad y + 5 = 5, \qquad \text{and} \qquad z + 3 = 0,$$

or

$$x = 6, \qquad y = 0, \qquad \text{and} \qquad z = -3.$$

$\therefore$ the value for x is 6, for y is 0, and for z is -3. Answer.

If two matrices A and B have the same dimensions, then their *sum*, denoted by

$$A + B,$$

is a matrix of the same dimensions, whose entries are the sums of the corresponding entries of A and B. For example,

$$[2 \quad -3] + [-5 \quad 4] = [2 + (-5) \quad -3 + 4] = [-3 \quad 1]$$

and

$$\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 + (-6) & 4 + 2 \\ -3 + (-5) & 2 + 7 \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ -8 & 9 \end{bmatrix}.$$

For any given natural numbers m and n, in the set $\mathcal{S}_{m\times n}$ of $m \times n$ matrices with real number entries, the **zero matrix,** denoted by

$$O_{m\times n} \text{ (or simply } O\text{)},$$

is the $m \times n$ matrix each of whose entries is 0. For example, some zero matrices are:

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the sum of the zero matrix $O_{m\times n}$ and any other matrix $A_{m\times n}$ is $A_{m\times n}$, the zero matrix $O_{m\times n}$ is also called the *identity matrix for addition* in the set $\mathcal{S}_{m\times n}$. For example,

$$[a \quad b] + [0 \quad 0] = [a + 0 \quad b + 0] = [a \quad b].$$

The **negative** of the $m \times n$ matrix A, denoted by

$$-A \text{ (read "the negative of } A\text{")},$$

is the $m \times n$ matrix whose entries are the negatives of the corresponding entries in A. For example, if

$$A = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \qquad \text{then} \qquad -A = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

and if

$$B = \begin{bmatrix} 3 & -7 \\ -4 & 2 \end{bmatrix}, \qquad \text{then} \qquad -B = \begin{bmatrix} -3 & 7 \\ 4 & -2 \end{bmatrix}.$$

Because the sum of $A_{m\times n}$ and $-A_{m\times n}$ is $O_{m\times n}$, we call $-A_{m\times n}$ the *additive inverse* of $A_{m\times n}$. For example, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{then} \qquad -A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix},$$

and

$$A + (-A) = \begin{bmatrix} a - a & b - b \\ c - c & d - d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If two matrices have the same dimensions, then they have a *difference,* denoted by

$$A - B,$$

which is defined to be the sum $A + (-B)$ and is therefore a matrix. Thus, to obtain the difference of two matrices of the same dimensions, you simply subtract their corresponding entries. For example,

$$\begin{bmatrix} 2 & 8 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} 2 - (-3) & 8 - 2 \\ -1 - (-5) & 4 - (-1) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}.$$

Oral Exercises

State the dimension of each matrix.

1. $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 1 \\ -2 & 3 \\ 4 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

4. $\begin{bmatrix} 6 & 2 & 0 & 4 \\ 7 & -1 & 9 & 0 \\ -3 & 2 & 5 & -2 \end{bmatrix}$

State the specified entry in the sum.

$$\begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 7 & -5 \\ 8 & -2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

5. a

6. b

7. c

8. d

If the second matrix is *subtracted* from the first in the addition example above, state the value of the specified entry in the *difference*.

9. a

10. b

11. c

12. d

State the value of the specified variable so that

$$\begin{bmatrix} 0 & x + 2 \\ 5 & z - 1 \end{bmatrix} = \begin{bmatrix} w & -3 \\ y + 1 & -2 \end{bmatrix}$$

13. w

14. x

15. y

16. z

17. Explain why there is not just one identity matrix for addition. In other words, explain why there is not just one zero matrix.

Written Exercises

Find the indicated sum or difference.

A

1. $\begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 4 & -2 \\ -5 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -7 & -1 \\ 0 & -4 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 2 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} -3 & 4 \\ 2 & -7 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 2 & 4 \end{bmatrix}$

5. $[3 \quad -1 \quad 7] - [-2 \quad -3 \quad 4]$

6. $\begin{bmatrix} 11 & 5 & -6 \\ -4 & 0 & 3 \\ 2 & 12 & 10 \end{bmatrix} + \begin{bmatrix} 6 & 20 & -3 \\ 4 & -1 & -5 \\ 9 & -12 & 4 \end{bmatrix}$

Find the values of the variables for which the given statement is true.

7. $\begin{bmatrix} w + 2 & 1 - x \\ 5 & 2y \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ z + 3 & 12 \end{bmatrix}$

8. $\begin{bmatrix} w - 4 & 7 \\ 3 & 2z + 1 \end{bmatrix} = \begin{bmatrix} 5 & 3x - 2 \\ y + 5 & -9 \end{bmatrix}$

9. $[x + 2 \quad x - y] = [-3 \quad 2]$

10. $\begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$

11. $[w - x \quad -4 \quad 3] = [5 \quad x - y \quad w + y]$

12. $\begin{bmatrix} w+x & w-y \\ y+2 & z+y \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$

13. $[w \quad y \quad x] - [x \quad -w \quad y] = [1 \quad -1 \quad -4]$

14. $\begin{bmatrix} w+2 & x \\ y-1 & z \end{bmatrix} + \begin{bmatrix} x-1 & -z \\ 3 & 2z \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & -1 \end{bmatrix}$

B 15. Let $A = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ -4 & -6 \end{bmatrix}$. Find the determinants of A, B, and $A + B$, and show that $|A| + |B| \neq |A + B|$, where $|A|$ denotes the determinant of A.

16. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} rc & rd \\ ta & tb \end{bmatrix}$, then it will be true that $|A| + |B| = |A + B|$.

C 17. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then $|A + B| = |A| + |B| + \begin{vmatrix} a & b \\ y & z \end{vmatrix} + \begin{vmatrix} w & x \\ c & d \end{vmatrix}$.

13-2 Properties of Matrix Addition

With the definitions of the equality and the sum of two matrices (page 462), for any given natural numbers m and n we can state some addition properties of the set $\mathcal{S}_{m\times n}$ of all $m \times n$ matrices with real-number entries as follows:

Theorem. For any given natural numbers m and n, let A, B, and C be members of the set $\mathcal{S}_{m\times n}$ of all $m \times n$ matrices with real-number entries. Then:

I. $A + B \in \mathcal{S}_{m\times n}$.	Closure Property
II. $A + B = B + A$.	Commutative Property
III. $(A + B) + C = A + (B + C)$.	Associative Property
IV. There exists in $\mathcal{S}_{m\times n}$ a unique element O such that for each $A \in \mathcal{S}_{m\times n}$, $O + A = A$ and $A + O = A$.	Additive-Identity Property
V. For each $A \in \mathcal{S}_{m\times n}$ there exists a unique element $-A$ in $\mathcal{S}_{m\times n}$ such that $A + (-A) = O$ and $(-A) + A = O$.	Additive-Inverse Property

Compare the properties listed in the preceding theorem with the axioms of addition in $\mathcal{R}$ given in Chapter 1. We shall consider the proofs of these various properties only for the representative case $m = 2$, $n = 2$. All the properties are direct applications of the definitions that we made for the equality and the sum of two matrices together with the corresponding properties of real numbers.

PROOF OF CLOSURE FOR $\mathcal{S}_{2\times 2}$

By definition, the entries of $A + B$ are the sums of the corresponding entries in A and B. Since the sum of two real numbers is a real number, the entries of $A + B$ are real numbers, and therefore $A + B$ is a member of $\mathcal{S}_{2\times 2}$.

PROOF OF THE COMMUTATIVE PROPERTY FOR $\mathcal{S}_{2\times 2}$

Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \end{bmatrix} \quad \text{and} \quad B + A = \begin{bmatrix} c_1 + a_1 & d_1 + b_1 \\ c_2 + a_2 & d_2 + b_2 \end{bmatrix}.$$

But by the commutative property of addition for real numbers,

$$\begin{aligned} a_1 + c_1 &= c_1 + a_1, & \quad b_1 + d_1 &= d_1 + b_1, \\ a_2 + c_2 &= c_2 + a_2, & \quad b_2 + d_2 &= d_2 + b_2. \end{aligned}$$

Since each entry of the matrix $A + B$ is equal to the corresponding entry of the matrix $B + A$, by the definition of equality for matrices you have $A + B = B + A$.

Proofs of Parts III, IV, and V, for the representative case $m = 2$, $n = 2$, are similar and are left as exercises (Exercises 10–12, page 468). By using Part II, you need to establish only one of the equations in IV and one in V.

By using the substitution principle and the properties of equality (Chapter 1), you can also solve certain matrix equations.

EXAMPLE Solve $X + \begin{bmatrix} 4 & 2 & 0 \\ 3 & -1 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 2 \\ -1 & 5 & 4 \end{bmatrix}$ over $\mathcal{S}_{2\times 3}$.

SOLUTION Assume that there is a matrix $X \in \mathcal{S}_{2\times 3}$ such that

$$X + \begin{bmatrix} 4 & 2 & 0 \\ 3 & -1 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 2 \\ -1 & 5 & 4 \end{bmatrix}.$$

Then by the substitution principle, you have

$$\left(X + \begin{bmatrix} 4 & 2 & 0 \\ 3 & -1 & 7 \end{bmatrix}\right) + \begin{bmatrix} -4 & -2 & 0 \\ -3 & 1 & -7 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 6 & 2 \\ -1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -2 & 0 \\ -3 & 1 & -7 \end{bmatrix}$$

By the associative property and additive identity property of matrices:

$$X + \left(\begin{bmatrix} 4 & 2 & 0 \\ 3 & -1 & 7 \end{bmatrix} + \begin{bmatrix} -4 & -2 & 0 \\ -3 & 1 & -7 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -3 & 6 & 2 \\ -1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -2 & 0 \\ -3 & 1 & -7 \end{bmatrix}$$
$$X + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 4 & 2 \\ -4 & 6 & -3 \end{bmatrix}$$
$$X = \begin{bmatrix} -7 & 4 & 2 \\ -4 & 6 & -3 \end{bmatrix}$$

Check: Replacing X with $\begin{bmatrix} -7 & 4 & 2 \\ -4 & 6 & -3 \end{bmatrix}$ in the original equation, you get

$$\begin{bmatrix} -7 & 4 & 2 \\ -4 & 6 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 0 \\ 3 & -1 & 7 \end{bmatrix} = \begin{bmatrix} -7+4 & 4+2 & 2+0 \\ -4+3 & 6-1 & -3+7 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 6 & 2 \\ -1 & 5 & 4 \end{bmatrix}$$

$\therefore$ the solution set is $\left\{\begin{bmatrix} -7 & 4 & 2 \\ -4 & 6 & -3 \end{bmatrix}\right\}$. Answer.

Oral Exercises

Exercises 1–5 list the steps you might use to solve the matrix equation $X - [2 \quad -3 \quad 5] = [7 \quad -1 \quad -2]$. Give a reason for each step.

1. $X + (-[2 \quad -3 \quad 5]) = [7 \quad -1 \quad -2]$
2. $(X + (-[2 \quad -3 \quad 5])) + [2 \quad -3 \quad 5] = [7 \quad -1 \quad -2] + [2 \quad -3 \quad 5]$
3. $X + (-[2 \quad -3 \quad 5] + [2 \quad -3 \quad 5]) = [9 \quad -4 \quad 3]$
4. $X + [0 \quad 0 \quad 0] = [9 \quad -4 \quad 3]$
5. $X = [9 \quad -4 \quad 3]$
6. Suppose the addition of 1×2 matrices were defined as follows:

$$[a \quad b] + [c \quad d] = [a \quad b + d].$$

Which of the properties in the theorem on page 465 would hold?

Written Exercises

Solve for the matrix X.

A 1. $X + \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

2. $X + \begin{bmatrix} -6 & 5 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 0 \end{bmatrix}$

3. $X - \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}$

4. $X - \begin{bmatrix} 5 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -7 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 7 \\ -3 & 8 \end{bmatrix} - X = \begin{bmatrix} 2 & 9 \\ 5 & -1 \end{bmatrix}$

6. $\begin{bmatrix} -2 & 0 \\ 1 & 6 \end{bmatrix} - X = \begin{bmatrix} -5 & 3 \\ -2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -3 & 7 \\ 8 & 4 & -5 \end{bmatrix} + X = \begin{bmatrix} -1 & 0 & 5 \\ 2 & -3 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 8 & -11 & 0 \\ -7 & 6 & 2 \end{bmatrix} - X = \begin{bmatrix} 6 & -4 & 2 \\ 3 & 0 & 5 \end{bmatrix}$

B 9. Solve for X in terms of a, b, c, and d and give a reason for each step of your solution: $[a \quad b] - X = [c \quad d]$.

In Exercises 10–15, prove the given statement when A, B, and C are 2×2 matrices.

10. $(A + B) + C = A + (B + C)$

11. $A + O = A$

12. $A + (-A) = O$

13. If $A = -A$, then $A = O$.

14. $(A + B) - A = B$

15. If $A + B = A$, then $B = O$.

C 16. Find all ordered triples (x, y, z) for which the following is true.

$$\begin{bmatrix} 4 & 4x \\ y^2 - 2y & 5z \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 8 & -(6 + z^2) \end{bmatrix} = O_{2\times 2}$$

13-3 Product of a Scalar and a Matrix

In dealing with matrices, we often refer to real numbers as **scalars**. We define the **product** of a scalar c and a matrix A, denoted by

$$cA,$$

as the matrix of the same dimensions as A whose entries are the products of c and the corresponding entries of A. For example, if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -4 & 7 \end{bmatrix}$, then

$$5A = 5\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 5(2) & 5(3) \\ 5(1) & 5(0) \\ 5(-4) & 5(7) \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 5 & 0 \\ -20 & 35 \end{bmatrix}$$

and

$$-2A = -2\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} -2(2) & -2(3) \\ -2(1) & -2(0) \\ -2(-4) & -2(7) \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ -2 & 0 \\ 8 & -14 \end{bmatrix}$$

Notice that the product of a *scalar* and a *matrix* is a *matrix*.

Products of scalars and matrices have a number of basic properties which follow from the definition above and the properties of real numbers. These basic properties are given in the following theorem.

Theorem. If $A \in \mathcal{S}_{m\times n}$ and $B \in \mathcal{S}_{m\times n}$, where m and n are given natural numbers, and if $c \in \mathcal{R}$ and $d \in \mathcal{R}$, then:

I. $cA \in \mathcal{S}_{m\times n}$

II. $c(dA) = (cd)A$

III. $(c + d)A = cA + dA$

IV. $c(A + B) = cA + cB$

V. $1 \cdot A = A$

VI. $(-1)A = -A$

VII. $0 \cdot A = O$

VIII. $cO = O$

As in Section 13-2, we shall consider proofs only for the representative case $m = 2$, $n = 2$. For the following sample proof we let

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

PROOF OF PART VI FOR $\mathcal{S}_{2\times 2}$

$$\begin{aligned} (-1)A &= (-1)\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \\ &= \begin{bmatrix} (-1)a_1 & (-1)b_1 \\ (-1)a_2 & (-1)b_2 \end{bmatrix} \\ &= \begin{bmatrix} -a_1 & -b_1 \\ -a_2 & -b_2 \end{bmatrix} = -A. \end{aligned}$$

You should be able to supply a reason for each step given above.

Proofs of the remaining parts of this theorem for the representative case $m = 2$, $n = 2$ are similar to the one we have given. Writing them is left to you (Exercises 19–25 on page 471).

You can use parts of the foregoing theorem to help you solve some equations involving matrices.

EXAMPLE Solve over $\mathcal{S}_{3\times 2}$: $5X - 3\begin{bmatrix} 0 & -2 \\ 1 & 2 \\ -4 & 7 \end{bmatrix} = 2\begin{bmatrix} 5 & -7 \\ 1 & -3 \\ -9 & 2 \end{bmatrix}$

SOLUTION Assuming that there is a 3×2 matrix X satisfying the equation, you first simplify the products; thus,

$$5X - \begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ 2 & -6 \\ -18 & 4 \end{bmatrix}.$$

Then you add $\begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix}$ to each member, as shown on page 470.

$$5X - \begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ 2 & -6 \\ -18 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix}$$

$$5X + O = \begin{bmatrix} 10 & -20 \\ 5 & 0 \\ -30 & 25 \end{bmatrix}$$

$$X = \frac{1}{5}\begin{bmatrix} 10 & -20 \\ 5 & 0 \\ -30 & 25 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & 0 \\ -6 & 5 \end{bmatrix}$$

Check: Replacing X with $\begin{bmatrix} 2 & -4 \\ 1 & 0 \\ -6 & 5 \end{bmatrix}$ in the original equation, you get

$$5\begin{bmatrix} 2 & -4 \\ 1 & 0 \\ -6 & 5 \end{bmatrix} - 3\begin{bmatrix} 0 & -2 \\ 1 & 2 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & -20 \\ 5 & 0 \\ -30 & 25 \end{bmatrix} - \begin{bmatrix} 0 & -6 \\ 3 & 6 \\ -12 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -14 \\ 2 & -6 \\ -18 & 4 \end{bmatrix} = 2\begin{bmatrix} 5 & -7 \\ 1 & -3 \\ -9 & 2 \end{bmatrix}$$

$\therefore$ the solution set is $\left\{\begin{bmatrix} 2 & -4 \\ 1 & 0 \\ -6 & 5 \end{bmatrix}\right\}$. Answer.

Oral Exercises

In Exercises 1–14, find the value of the variable so that the equation in which it appears is true.

$$3\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad w\begin{bmatrix} -3 & x \\ y & 2 \end{bmatrix} = \begin{bmatrix} -12 & 16 \\ -4 & z \end{bmatrix}$$

1. a **2.** b **3.** c **4.** d **5.** w **6.** x **7.** y **8.** z

$$2\begin{bmatrix} -1 & 3 & 0 \\ 2 & 4 & -3 \end{bmatrix} + \begin{bmatrix} 5 & -2 & -4 \\ l & m & n \end{bmatrix} = \begin{bmatrix} o & p & q \\ -1 & 3 & 4 \end{bmatrix}$$

9. l **10.** m **11.** n **12.** o **13.** p **14.** q

Written Exercises

In Exercises 1–12, find the 2×2 matrix X that satisfies the equation, if $A = \begin{bmatrix} 3 & -9 \\ 21 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -11 \\ -1 & 10 \end{bmatrix}$.

A **1.** $X = 3A$ **2.** $X = -4B$ **3.** $X = 2A + B$ **4.** $X = A - 2B$

5. $3X = A$ 6. $\frac{1}{2}X = B$ 7. $X + 2A = B$ 8. $X - 3A = B$

9. $\frac{1}{3}X = A - B$ 10. $3X = A + 3B$ 11. $5X = A + B$ 12. $2X - B = 3A$

In Exercises 13–18, find the 2×2 matrix X that satisfies the given equation, if $A = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 13 \\ -5 & -8 \end{bmatrix}$.

13. $X + 3A = B$ 14. $5A - X = 2B$ 15. $\frac{1}{2}X + 4A = B$

16. $3X - 2A = 2B$ 17. $4A = 5X - 3B$ 18. $2X + 5A = 3B$

In Exercises 19–27, prove the statement for all 2×2 matrices A, B, and X and all real numbers c and d.

B 19. $cA \in \mathcal{S}_{2\times 2}$ 20. $c(dA) = (cd)A$ 21. $(c + d)A = cA + dA$

22. $c(A + B) = cA + cB$ 23. $1 \cdot A = A$ 24. $0 \cdot A = O$

25. $cO_{2\times 2} = O_{2\times 2}$ 26. If $cX = B$, then $X = \frac{1}{c}B$.

27. If $cX + B = A$, then $X = \frac{1}{c}(A - B)$.

C 28. Find all ordered pairs (x, y) for which the following matrix equation is true:

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - \begin{bmatrix} x \\ 4y \end{bmatrix} - 6\begin{bmatrix} 1 \\ 2 \end{bmatrix} = O_{2\times 1}$$

programming in BASIC

Earlier we used variables with single subscripts in handling lists. BASIC also provides variables with two subscripts for use in handling tables. Try this program:

```
10  FOR R=1 TO 3
20  FOR C=1 TO 4
30  READ A(R,C)
40  PRINT A(R,C);
50  NEXT C
60  PRINT
70  NEXT R
80  DATA 10,11,12,13,14,15,16,17,18,19,20,21
90  END
```

(DIMension statements are needed if the table is to be more than 10 by 10.)

Insert these lines in the preceding program and run it again.

```
84  PRINT
85  PRINT "A(2,3) =";A(2,3)
86  PRINT "A(3,2) =";A(3,2)
```

Change the following lines in the program and run it again.

```
10  FOR R=1 TO 4
20  FOR C=1 TO 3
```

When a table, or array, is used as a matrix, BASIC provides several special statements that save a great deal of time. When the MATrix statements, however, are used on some systems, there must be a DIMension statement for every matrix that appears in the program.

In the preceding program make the changes

```
10  DIM A(3,4)
20  MAT READ A
30  MAT PRINT A;
40
50
60
70
```

and run it again. The semicolon in line 30 will make the entries of the matrix print close together. If the semicolon is omitted, the entries will print in the five regular zones. (Lines 40, 50, 60, 70 are deleted by typing only the line number followed by RETURN.) Notice that DATA for a matrix is listed row by row.

To illustrate multiplication of a matrix by a scalar, delete lines 84, 85, 86 and make these changes:

```
10  DIM A(3,4),P(3,4)
40  PRINT "    4*A"
50  MAT P=(4)*A
60  MAT PRINT P;
```

(Note that the scalar in line 50 must be in parentheses.)

To illustrate addition and subtraction of two matrices, try this program:

```
10  DIM A(3,4),B(3,4),C(3,4)
20  MAT READ A,B
30  PRINT "    A"
40  MAT PRINT A;
50  PRINT "    B"
60  MAT PRINT B;
70  MAT C=A+B
80  PRINT "    A+B"
90  MAT PRINT C;
100  MAT C=A-B
110  PRINT "    A-B"
120  MAT PRINT C;
130  DATA 21,20,19,18,17,16,15,14,13,12,11,10
140  DATA 10,11,12,13,14,15,16,17,18,19,20,21
150  END
```

Exercises

1. Find the sum and the difference of each pair of matrices given in Written Exercises 1–6 on page 464.
2. Find the additive inverse of matrix A given in the program given at the bottom of page 472, and verify that $A + (-A) = 0$.
3. Use the computer to check your solutions of Written Exercises 1–4 on page 470.

13-4 Product of Two Matrices

Before introducing the definition of the product of two matrices in general, let us consider the two matrices

$$A_{1\times 3} = [a \quad b \quad c] \qquad \text{and} \qquad B_{3\times 2} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}.$$

Suppose that the numbers a, b, and c represent the price per package of frozen strawberries, peaches, and apricots charged by a supermarket, while x_1, y_1, and z_1 represent the number of packages of each sold the first week, respectively, and x_2, y_2, z_2 the numbers sold the second week. How much money would the supermarket collect for these items? For the first week, the total amount collected for the strawberries is $a \cdot x_1$, for the peaches $b \cdot y_1$, and for the apricots $c \cdot z_1$. By adding these products, you obtain the total amount collected, $ax_1 + by_1 + cz_1$. Similarly, for the second week, you have a total collection of $ax_2 + by_2 + cz_2$.

The process of adding the products obtained by multiplying the elements of a row in one matrix by the corresponding elements of a column in another matrix suggests a fruitful way of defining the product of two matrices. We may say that the product of $A_{1\times 3}$ and $B_{3\times 2}$ shown above is

$$C_{1\times 2} = [ax_1 + by_1 + cz_1 \qquad ax_2 + by_2 + cz_2].$$

The **product** of two 2×2 matrices A and B, denoted by

$$A \times B \qquad \text{or} \qquad A \cdot B \qquad \text{or} \qquad AB,$$

is defined as follows:

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$, then:

$$A \times B = AB = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \times \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_1c_1 + b_1c_2 & a_1d_1 + b_1d_2 \\ a_2c_1 + b_2c_2 & a_2d_1 + b_2d_2 \end{bmatrix}.$$

Notice that the entries of a given row of A, say the ith row, are multiplied by the entries of a given column of B, say the jth column, in order, and these products are then added to obtain the entry in the ith row and jth column of $A \times B$. Thus, the multiplication of two matrices can be described as "row by column" multiplication.

As an example, let us find the product

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix}$$

by displaying the computation of each entry, one at a time, in red. Notice that we may omit the times sign between the matrices.

1. First row, first column:

$$\begin{bmatrix} 3 & 2 \\ & \end{bmatrix}\begin{bmatrix} -2 & \\ 1 & \end{bmatrix} = \begin{bmatrix} 3 \times (-2) + 2 \times 1 & \\ & \end{bmatrix} = \begin{bmatrix} -6 + 2 & \\ & \end{bmatrix} = \begin{bmatrix} -4 & \\ & \end{bmatrix}$$

2. First row, second column:

$$\begin{bmatrix} 3 & 2 \\ & \end{bmatrix}\begin{bmatrix} & 3 \\ & 6 \end{bmatrix} = \begin{bmatrix} -4 & 3 \times 3 + 2 \times 6 \\ & \end{bmatrix} = \begin{bmatrix} -4 & 9 + 12 \\ & \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ & \end{bmatrix}$$

3. Second row, first column:

$$\begin{bmatrix} & \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -2 & \\ 1 & \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ 5 \times (-2) + 4 \times 1 & \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -10 + 4 & \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -6 & \end{bmatrix}$$

4. Second row, second column.

$$\begin{bmatrix} & \\ 5 & 4 \end{bmatrix}\begin{bmatrix} & 3 \\ & 6 \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -6 & 5 \times 3 + 4 \times 6 \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -6 & 15 + 24 \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -6 & 39 \end{bmatrix}$$

Putting Steps 1–4 together, we have

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ -6 & 39 \end{bmatrix}.$$

Ordinarily, of course, the steps are not all shown in detail.

In general, the product AB of any two matrices A and B, where A has the same number of *columns* as B has *rows*, can be defined through "row by column" multiplication. The number of rows in the product matrix AB will be the same as the number of rows in A, and the number of columns in AB will be the same as the number of columns in B. Thus,

$$A_{m \times p} \times B_{p \times n} = C_{m \times n}.$$

If the number of *columns* in A is not equal to the number of *rows* in B, then the product of A and B is not defined. Notice that $B_{p\times n} \times A_{m\times p}$ is not defined unless $n = m$.

In particular, the product of a 2×2 and a 2×1 matrix is defined as follows:

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $B = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix}.$$

As you can see, the product of a 2×2 matrix and a 2×1 matrix is a 2×1 matrix. Later in this chapter, we shall use products of 2×2 and 2×1 matrices in connection with systems of equations.

As with numbers, for square matrices A we use the symbol A^2 to mean $A \times A$.

Oral Exercises

State the value of the specified letter in the computation of the product.

$$\begin{bmatrix} 4 & -5 & 2 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 + (-5)2 + a(-4) \\ -2\cdot b + c\cdot 2 + 3\cdot d \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

1. a 2. b 3. c 4. d 5. e 6. f

Written Exercises

Find the given product.

A 1. $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & -3 \\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -1 \\ 3 & 0 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ -4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} -2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -3 & -4 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

8. $\begin{bmatrix} 5 & 2 & -4 \\ 3 & -1 & 6 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 0 & 5 & -2 \\ 2 & 4 & 2 \end{bmatrix}$

For $A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$, find a 2×2 matrix equal to the given product.

9. AB 10. BA 11. A^2 12. B^2 13. $(A + B)^2$

14. Show that if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then $AB = BA$.

Solve for x and y, or for x, y, and z.

B 15. $\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

16. $\begin{bmatrix} 4 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} -3 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \end{bmatrix}$

18. $[x \quad y]\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = [12 \quad 13]$

19. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 6 \end{bmatrix}$

In Exercises 21 and 22, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

21. Show that AB is in the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$.

22. What is the relation between AB and BA?

C 23. Let the matrix $A = \begin{bmatrix} 17 & 19 \\ 2 & 3 \end{bmatrix}$ represent the fact that wristwatch Model I has 17 jewels and 2 straps, while Model II has 19 jewels and 3 straps. Let the matrix $B = \begin{bmatrix} 30 & 40 & 50 \\ 20 & 15 & 10 \end{bmatrix}$ represent the fact that a factory produced 30 Model I wristwatches on Monday, 40 on Tuesday, and 50 on Wednesday, and on the same days produced 20, 15, and 10 sets of the Model II wristwatch, respectively. Simplify the product AB and tell what its entries represent.

24. Let the matrix $A = [120 \quad 15]$ represent the fact that a sporting-goods company produces 120 standard and 15 deluxe tennis rackets each week. Let the matrix $B = \begin{bmatrix} 4 & 0.8 \\ 6 & 1.0 \end{bmatrix}$ represent the fact that the standard racket requires 4 h of labor for assembly and 0.8 h of labor for lamination whereas the deluxe model requires 6 h of labor for assembly and 1 h of labor for lamination. Let matrix $C = \begin{bmatrix} 10.50 \\ 15.00 \end{bmatrix}$ represent the fact that the costs per hour of labor are \$10.50 for fabrication and \$15 for lamination. Simplify the product ABC and specify what its entry represents.

13-5 Properties of Matrix Multiplication

Although multiplication of matrices with real-number entries has some of the properties of multiplication of real numbers, there are some important differences. We shall illustrate these by considering multiplication in the set $\mathcal{S}_{2\times 2}$.

EXAMPLE 1 For $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}$, find (a) AB, and (b) BA.

SOLUTION a. $AB = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -2+2 & -6+6 \\ 4-4 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Answer.

b. $BA = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -2+12 & -1+6 \\ 4-24 & 2-12 \end{bmatrix}$

$= \begin{bmatrix} 10 & 5 \\ -20 & -10 \end{bmatrix}$. Answer.

Notice in Example 1 that $A \neq O$ and $B \neq O$ but $AB = O$. Thus, in $\mathcal{S}_{2\times 2}$ the product of two matrices can be the zero matrix without either factor being the zero matrix!

Notice also in Example 1 that $BA \neq AB$. Therefore, multiplication in $\mathcal{S}_{2\times 2}$ *is not commutative.* For this reason, you must be careful to specify the order of the factors in matrix multiplication. To specify the product AB, for example, you say that you **right-multiply** A by B or that you **left-multiply** B by A. Some special matrix products, however, do not depend on the order of the factors.

EXAMPLE 2 If $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, simplify (a) $I_{2\times 2}A$, and (b) $AI_{2\times 2}$.

SOLUTION a. $I_{2\times 2}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1+0 & b_1+0 \\ 0+a_2 & 0+b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Answer.

b. $AI_{2\times 2} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1+0 & 0+b_1 \\ a_2+0 & 0+b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Answer.

As Example 2 demonstrates, not only do the products of *some* matrices AB in $\mathcal{S}_{2\times 2}$ commute, but also the product of any 2×2 matrix A and

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

always is the matrix A. Thus, $I_{2\times 2}$ is an **identity matrix for multiplication** in $\mathcal{S}_{2\times 2}$:

$$I_{2\times 2}A = AI_{2\times 2} = A$$

This property and some other properties of multiplication of 2×2 matrices are listed in the following theorem.

Theorem. If $A \in \mathcal{S}_{2\times 2}$, $B \in \mathcal{S}_{2\times 2}$, $C \in \mathcal{S}_{2\times 2}$, and $a \in \mathcal{R}$, then:

I. $AB \in \mathcal{S}_{2\times 2}$ II. $(AB)C = A(BC)$

III. $A(B + C) = AB + AC$ IV. $(B + C)A = BA + CA$

V. $I_{2\times 2}A = AI_{2\times 2} = A$ VI. $a(AB) = (aA)B = A(aB)$

VII. $O_{2\times 2}A = AO_{2\times 2} = O_{2\times 2}$

EXAMPLE 3 Show that if $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}$, then

(a) $(A + B)(A - B) \neq A^2 - B^2$, but

(b) $(A + B)(A - B) = A^2 - AB + BA - B^2$.

SOLUTION a. First:

$$A + B = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -6 & -8 \end{bmatrix}$$

$$(A + B)(A - B) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -6 & -8 \end{bmatrix} = \begin{bmatrix} 15 & 20 \\ -30 & -40 \end{bmatrix}$$

Next:

$$A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -5 & -15 \\ 10 & 30 \end{bmatrix}$$

$$A^2 - B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -5 & -15 \\ 10 & 30 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ -10 & -30 \end{bmatrix}$$

Since $\begin{bmatrix} 15 & 20 \\ -30 & -40 \end{bmatrix} \neq \begin{bmatrix} 5 & 15 \\ -10 & -30 \end{bmatrix}$, you have

$$(A + B)(A - B) \neq A^2 - B^2. \quad \text{Answer.}$$

b. From Example 1 on page 477 you have

$$-AB + BA = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 10 & 5 \\ -20 & -10 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ -20 & -10 \end{bmatrix},$$

so that $A^2 - AB + BA - B^2 = A^2 - B^2 - AB + BA$

$$= \begin{bmatrix} 5 & 15 \\ -10 & -30 \end{bmatrix} + \begin{bmatrix} 10 & 5 \\ -20 & -10 \end{bmatrix} = \begin{bmatrix} 15 & 20 \\ -30 & -40 \end{bmatrix}.$$

Also since $(A + B)(A - B) = \begin{bmatrix} 15 & 20 \\ -30 & -40 \end{bmatrix}$, you have

$$(A + B)(A - B) = A^2 - AB + BA - B^2. \quad \text{Answer.}$$

To understand the results in Example 3, notice that by the distributive property you have

$$(A + B)(A - B) = A(A - B) + B(A - B) = A^2 - AB + BA - B^2.$$

Oral Exercises

In Exercises 1–5, let A be any 2×2 matrix. State a simpler name for each of these.

1. $AI_{2\times 2}$
2. $A + O_{2\times 2}$
3. $A - A$
4. $A^2O_{2\times 2}$
5. $I^2{}_{2\times 2}A$

Written Exercises

In Exercises 1–16 let $A = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 2 \\ 10 & 4 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$. Compute the value of the expressions and state whether or not they are equal.

A

1. AB and BA
2. AC and CA
3. AD and DA
4. BC and CB
5. CD and DC
6. BD and DB
7. $A(B + C)$ and $AB + AC$
8. $A(BC)$ and $(AB)C$
9. $A(BC)$ and $(BC)A$
10. $A(CD)$ and $(CD)A$
11. $C(A + B)$ and $(A + B)C$
12. $(AB)(CD)$ and $(CD)(AB)$

Compute each of the three expressions and tell which of the three are the same.

B

13. a. $(A + B)^2$ b. $A^2 + 2AB + B^2$ c. $A^2 + AB + BA + B^2$
14. a. $(A - B)^2$ b. $A^2 - AB - BA + B^2$ c. $A^2 - 2AB + B^2$
15. a. $(A + B)(C + D)$ b. $AC + BC + AD + BD$ c. $C(A + B) + D(A + B)$
16. a. $(A - B)(C + D)$ b. $AC - BC - BD + AD$ c. $A(C + D) - B(C + D)$

Prove each of the following for all 2×2 matrices A, B, and C, and all real numbers a.

17. $AB \in \mathcal{S}_{2\times 2}$
18. $(AB)C = A(BC)$
19. $A(B + C) = AB + AC$
20. $(B + C)A = BA + CA$
21. $a(AB) = (aA)B$
22. $a(AB) = A(aB)$
23. $O_{2\times 2}A = O_{2\times 2}$
24. $AO_{2\times 2} = O_{2\times 2}$

C

25. Let $|A|$ denote the determinant of A. Show that for all 2×2 matrices A and B and all real numbers c.

    a. $|cA| = c^2|A|$ b. $|AB| = |A||B|$

Self-Test 1

VOCABULARY

matrix (p. 461)
entry of a matrix (p. 461)
dimensions of a matrix (p. 461)
negative of a matrix (p. 463)
zero matrix (p. 463)
identity matrix for addition (p. 463)
scalar (p. 468)
identity matrix for multiplication (p. 477)

1. Simplify: *Obj. 1, p. 461*

a. $\begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ -8 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} -2 & 0 & 5 \\ 4 & -1 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -7 \\ 6 & -2 & 1 \end{bmatrix}$

2. Simplify $3A - 2B$ if $A = \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ -5 & -3 \end{bmatrix}$. *Obj. 2, p. 461*

3. Using the matrices A and B in Test Item 2, solve the following equations for X. *Obj. 3, p. 461*

a. $X - A = B$ **b.** $3X + 4B = 5A$

4. Simplify $\begin{bmatrix} -4 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$. *Obj. 4, p. 461*

Check your answers with those at the back of the book.

programming in BASIC

BASIC also provides for multiplication of matrices. Study this program:

```
10  DIM A(3,3),B(3,3),P(3,3)
20  MAT READ A,B
30  PRINT "    A"
40  MAT PRINT A;
50  PRINT "    B"
60  MAT PRINT B;
70  MAT P=A*B
80  PRINT "   A*B"
90  MAT PRINT P;
100 MAT P=B*A
110 PRINT "   B*A"
120 MAT PRINT P;
130 DATA 11,12,13,14,15,16,17,18,19
140 DATA 39,38,37,36,35,34,33,32,31
150 END
```

Exercises

1. Rewrite the given program to find the products of two 4×4 matrices. Supply thirty-two two-digit numbers as DATA.
2. Matrices can also be input. Try this program:

```
10  DIM A(3,4)
20  MAT INPUT A
30  PRINT
40  MAT PRINT A;
50  END
```

   The computer will print only one question mark, but you are to respond by typing in 12 numbers for the 3×4 matrix.
3. Rewrite the text program so that you can INPUT the two matrices to be multiplied.
4. Use the computer to check your answers to Exercises 1–12 on page 475.

Matrices and Linear Systems

***OBJECTIVES** for Section 13-6:*

1. Determine the inverse of a nonsingular square matrix of order 2.
2. Use the inverse of a matrix to solve a matrix equation.
3. Write a linear system of equations in matrix form.
4. Use a matrix equation to solve a linear system.

13-6 Matrix Solution of a Linear System

Every nonzero real number r has a multiplicative inverse. That is, for each $r \in \mathcal{R}$, $r \neq 0$, there is a number $r^{-1} \in \mathcal{R}$ for which

$$rr^{-1} = 1 \qquad \text{and} \qquad r^{-1}r = 1.$$

Does every nonzero 2×2 matrix A with real-number entries also have a multiplicative inverse? That is, for each $A \in \mathcal{S}_{2\times 2}$, $A \neq 0$, is there a matrix $A^{-1} \in \mathcal{S}_{2\times 2}$ for which

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I?$$

You can readily see, for example, that the matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ has a multiplicative inverse $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, since

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In general, for 2×2 matrices, given $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, let us try to find $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ such that

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix} = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This is true if and only if

$$\begin{aligned} a_1x_1 + b_1x_2 = 1, &\quad a_1y_1 + b_1y_2 = 0, \\ a_2x_1 + b_2x_2 = 0, &\quad a_2y_1 + b_2y_2 = 1. \end{aligned}$$

Recall that in Chapter 4 we used determinants to solve systems of equations. If the determinant of coefficients in this system, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is not equal to zero, that is, if $a_1b_2 - a_2b_1 \neq 0$, then these equations can be solved for x_1, x_2, y_1, and y_2 to produce

$$x_1 = \frac{b_2}{a_1b_2 - a_2b_1}, \quad y_1 = \frac{-b_1}{a_1b_2 - a_2b_1},$$

$$x_2 = \frac{-a_2}{a_1b_2 - a_2b_1}, \quad y_2 = \frac{a_1}{a_1b_2 - a_2b_1}.$$

You can check that with these values for x_1, x_2, y_1, and y_2, not only do you have $AA^{-1} = I_{2\times 2}$, but also (somewhat surprisingly, since products AB of matrices do not ordinarily commute) you have $A^{-1}A = I_{2\times 2}$. (See Exercises 25 and 26, page 486.)

With each square matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ we associate a particular real number, namely, the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, and we write

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The pairing of each square matrix with its determinant constitutes a function, since associated with each such matrix is one and only one real number. The symbol "det A" (read "determinant of A") represents the element in the range of the function det associated with the matrix A in its domain.

Because each denominator in the preceding equations for x_1, x_2, y_1, and y_2 is det A, you can see that

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

If det $A = 0$, then the equations for x_1, x_2, y_1, and y_2 have no solution (see

Exercise 27, page 486) and so such a matrix A (called a **singular matrix**) has no inverse. If $\det A \neq 0$, then matrix A is said to be **nonsingular** or **invertible.**

Notice that, to find A^{-1} from A, you interchange a_1 and b_2, replace a_2 and b_1 with their negatives, and multiply by the reciprocal of $\det A$.

EXAMPLE 1 If $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$, find A^{-1}.

SOLUTION Note first that $\det A = 3 \times 5 - 2 \times 6 = 15 - 12 = 3$, so that $\det A \neq 0$. Then

$$A^{-1} = \tfrac{1}{3}\begin{bmatrix} 5 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & -2 \\ -\frac{2}{3} & 1 \end{bmatrix}.$$

Check: $$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \frac{5}{3} & -2 \\ -\frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ \frac{10}{3} - \frac{10}{3} & -4+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore A^{-1} = \begin{bmatrix} \frac{5}{3} & -2 \\ -\frac{2}{3} & 1 \end{bmatrix}$. Answer.

Matrix equations in the form $AX = B$ may be solved using inverse matrices in the following way:

$$AX = B, \quad A^{-1}AX = A^{-1}B, \quad IX = A^{-1}B, \quad X = A^{-1}B.$$

EXAMPLE 2 Solve the following equation for the matrix X.

$$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} X = \begin{bmatrix} 8 & -6 \\ 14 & -8 \end{bmatrix}$$

SOLUTION 1. First, to find the inverse of $\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$, note that its determinant is $-10 + 12$, or 2. Then

$$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}^{-1} = \tfrac{1}{2}\begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -2 & 1 \end{bmatrix}.$$

2. Next, left-multiply each member of the given equation by this inverse.

$$\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} X = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 \\ 14 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

Check: $$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2+6 & 6-12 \\ 4+10 & 12-20 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 14 & -8 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$. Answer.

We shall now consider a method of solving the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

by using matrices. By the definition of matrix equality, this system may be written in matrix notation as

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

You saw on page 475 that

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, you can rewrite the matrix equation as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

which represents the linear system in the simple matrix form

$$AX = B,$$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is called the **coefficient matrix**, and X and B are 2×1 matrices.

If the coefficient matrix is invertible, the components of the single member of the solution set are the entries in $A^{-1}B$; if it is not invertible, the equations in the system are either dependent or inconsistent.

Cramer's Rule for solving equations using determinants, which was presented in Chapter 5, may be derived from this method.

EXAMPLE 3 Use matrices to find the solution set of the system:
$$\begin{aligned} -x + 2y &= -6 \\ 3x + 4y &= 8 \end{aligned}$$

SOLUTION 1. First, write the matrix equation:

$$\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

2. Next, find the inverse of the coefficient matrix:

$$\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{10}\begin{bmatrix} 4 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{10} & \frac{2}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

3. Then: $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4}{10} & \frac{2}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}\begin{bmatrix} -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Checking, you find that the values $x = 4$, $y = -1$ satisfy the given equations.

$\therefore$ the solution set is $\{(4, -1)\}$. Answer.

Systems of three linear equations in three variables, and also larger systems, can similarly be represented in the simple matrix form $AX = B$, with the unique solution $X = A^{-1}B$ in the case A is nonsingular.

Oral Exercises

Let $A = \begin{bmatrix} 9 & 5 \\ 3 & 2 \end{bmatrix}$. If $A^{-1} = \frac{1}{a}\begin{bmatrix} b & c \\ d & e \end{bmatrix}$, give the value of the specified variable.

1. a **2.** b **3.** c **4.** d **5.** e

6. State the matrix equation $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ as a system of linear equations.

If the linear system $\begin{aligned} 3x + y &= 5 \\ -2x - 4y &= 0 \end{aligned}$ is written as a matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$, state the value of the specified letter.

7. a **8.** b **9.** c **10.** d **11.** e **12.** f

Written Exercises

Find the inverse of the given matrix if it is nonsingular. If the matrix is singular, so state.

A

1. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$ **7.** $\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ **9.** $\begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ **10.** $\begin{bmatrix} -4 & 7 \\ -2 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, and $D = \begin{bmatrix} 4 & 3 \\ -2 & -2 \end{bmatrix}$.
Solve each of the following matrix equations for X.

11. $AX = B$ **12.** $AX = C$ **13.** $BX = A$ **14.** $BX = D$

15. $CX = D$ **16.** $CX = A$ **17.** $DX = C$ **18.** $DX = A$

Find the solution set of the given system by using a matrix equation.

B

19. $\begin{aligned} 3x + y &= 8 \\ 2x + y &= 5 \end{aligned}$ **20.** $\begin{aligned} 2x + y &= 2 \\ 5x + 3y &= -1 \end{aligned}$ **21.** $\begin{aligned} 5x + 3y &= -7 \\ -3x - 2y &= 4 \end{aligned}$

22. $\begin{aligned} 3x - 4y &= 1 \\ 2x - 2y &= 2 \end{aligned}$ **23.** $\begin{aligned} 6x - 4y &= 3 \\ 2x - 2y &= 1 \end{aligned}$ **24.** $\begin{aligned} 2x + 5y &= 7 \\ x + 4y &= 8 \end{aligned}$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove the given statement.

C 25. $A \cdot \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

26. $\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

27. Use the result of Exercise 25(b) of Written Exercises 13-5 and the fact that $AA^{-1} = I$ to show that if $\det A = 0$, then A has no inverse. (*Hint:* Assume A has an inverse and show that this leads to a contradiction.)

Self-Test 2

VOCABULARY singular matrix (p. 483) invertible matrix (p. 483) coefficient matrix of a linear system (p. 484)

1. Find the multiplicative inverse of $\begin{bmatrix} 3 & 5 \\ -3 & -4 \end{bmatrix}$. *Obj. 1, p. 481*

2. Solve for X: $\begin{bmatrix} 6 & 5 \\ 4 & 3 \end{bmatrix} X = \begin{bmatrix} 2 & -4 \\ -6 & 4 \end{bmatrix}$. *Obj. 2, p. 481*

3. Write in the matrix form $AX = B$: $\begin{aligned} 3x - 4y &= 2 \\ 5x + y &= 11 \end{aligned}$ *Obj. 3, p. 481*

4. Use a matrix equation to solve the system: $\begin{aligned} 2x - 5y &= 3 \\ x - 3y &= 1 \end{aligned}$ *Obj. 4, p. 481*

Check your answers with those at the back of the book.

programming in BASIC

It has been shown in the text that a system of linear equations can be expressed in matrix form as $AX = B$. If A is invertible, then the solution can be expressed as $X = A^{-1}B$. In BASIC the inverse of a matrix is found by a statement such as:

```
MAT V = INV(A)
```

The program below finds the solution of the system:

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + y - z &= 5 \\ x + 2y + z &= 4 \end{aligned}$$

The solution is printed as

```
 1
 2
-1
```

which may be interpreted as: $x = 1$, $y = 2$, $z = -1$.

Notice that B(3) and X(3) in line 10 are matrices with dimensions of 3 rows and 1 column.

```
10  DIM A(3,3),B(3),V(3,3),X(3)
20  MAT READ A,B
30  MAT PRINT A;B
40  PRINT
50  PRINT
60  MAT V=INV(A)
70  MAT X=V*B
80  MAT PRINT X
90  DATA 1,1,2
100  DATA 2,1,-1
110  DATA 1,2,1
120  DATA 1,5,4
130  END
```

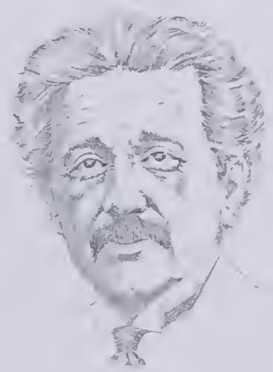
If A is singular, the computer will report that. To see what happens, try the program with these DATA changes:

```
100  DATA 2,2,-1
110  DATA 2,2,1
```

Exercises

1. Rewrite the text program so that you can INPUT the matrices.
2. Use the computer to check your answers to Exercises 5–10 on page 158.

Albert Einstein
1879–1955



Albert Einstein's career as a student had not been impressive, but while working as an examiner for a Swiss patent office he thought a great deal about a wide variety of unexplained experiments which had to do with light and motion. In 1905, Einstein atempted to account for these experiments in his Special Theory of Relativity. Today many observations are in complete accord with special relativity. Among the observations predicted by the special theory is the equivalence of mass and energy related by the equation $E = mc^2$, where c is the speed of light.

Another paper Einstein published in 1905 helped to establish the quantum mechanics branch of physics. This work contributed to the discovery of the electric eye, which, in turn, led to other inventions such as television. In 1910 Einstein included the effects of acceleration among various observers in what is called the General Theory of Relativity. In 1921 Einstein was awarded the Nobel prize in physics for his paper on quanta.

Einstein left Europe shortly before World War II for a lifetime appointment at the Institute for Advanced Study in Princeton, New Jersey.

Transformations of the Plane

***OBJECTIVES** for Sections 13-7 and 13-8:*

1. *Determine an equation of a translation of the plane when the image of one point under the translation is given.*
2. *Find the coordinates of the image of a point, and also the coordinates of the preimage of a point, under a given translation of the plane.*
3. *Find the coordinates of the image of a point under a given linear transformation of the plane.*
4. *Find the coordinates of the preimage of a point under a given nonsingular transformation of the plane.*

13-7 Transformations by Matrix Addition

Imagine a triangular piece of cardboard resting on a coordinate plane and having vertices at $A(-4, -3)$, $B(3, -1)$, and $C(2, 3)$, as shown in Figure 1. You can think of sliding the cardboard **2** units in the x-direction and **1** unit in the y-direction to the position shown in Figure 2. In their new positions, the vertices will be at $A'(-4 + \mathbf{2}, -3 + \mathbf{1})$, $B'(3 + \mathbf{2}, -1 + \mathbf{1})$, and $C'(2 + \mathbf{2}, 3 + \mathbf{1})$, or $A'(-2, -2)$, $B'(5, 0)$, and $C'(4, 4)$. By such a sliding, you can think of each point $P(x, y)$ of the plane as being *mapped* on a corresponding point $P'(x + \mathbf{2}, y + \mathbf{1})$.

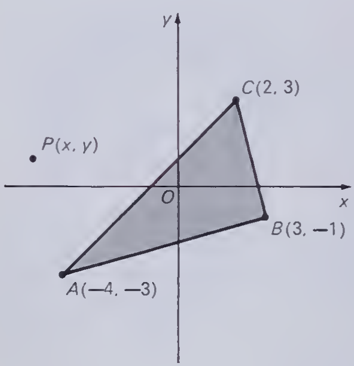


Figure 1

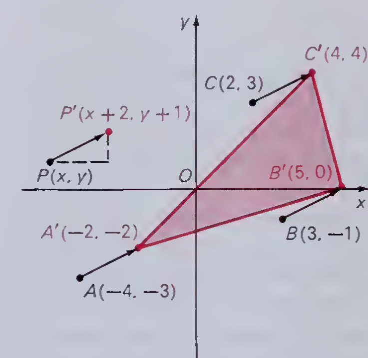


Figure 2

The sliding described above is an example of a **transformation,** or **mapping,** of the plane called a **translation.** In the translation of amount h in the x-direction and k in the y-direction, the point $P(x, y)$ is mapped on the point $P'(x', y')$, where

$$x' = x + \mathrm{h} \qquad \text{and} \qquad y' = y + \mathbf{k}.$$

We say that P' is the **image** of P, and that P is the **preimage** of P', under the mapping.

It is convenient to use matrices in working with transformations of the plane and to designate the coordinates (x, y) by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Thus the translation given above is written simply as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}. \tag{1}$$

If h and k are given, then Equation (1) can be considered as defining a function,

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$$

with domain the set of 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathcal{R}$, and range the set of corresponding matrices $\begin{bmatrix} x' \\ y' \end{bmatrix}$.

Of course, the function

$$(x, y) \rightarrow (x', y'),$$

whose domain and range are sets of ordered pairs, describes the same transformation. Matrices are used because they are very convenient in studying linear transformations (Section 13-8).

EXAMPLE In a translation of the plane, the image of $P(5, 4)$ is $P'(8, -2)$.

a. Find a matrix equation of the transformation.

b. Find the image of $Q(-4, 0)$ under the transformation.

c. Find the preimage of $R'(3, 2)$ under the transformation.

SOLUTION a. The matrix equation of the translation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

is satisfied when

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

Hence:

$$\begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} 8 - 5 \\ -2 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$\therefore$ the transformation is given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix}$. Answer.

(*Solution continued on page 490.*)

b. To find the image of $Q(-4, 0)$, you substitute $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix}$ in the equation of the transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

$\therefore$ the image of $Q(-4, 0)$ under the transformation is $Q'(-1, -6)$. **Answer.**

c. To find the preimage of $R'(3, 2)$ you substitute $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for $\begin{bmatrix} x' \\ y' \end{bmatrix}$ in the equation of the transformation:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 - 3 \\ 2 - (-6) \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$\therefore$ the preimage of $R'(3, 2)$ under the transformation is $R(0, 8)$. **Answer.**

Parts (b) and (c) of the Example illustrate the fact that under a translation every point of the plane has a unique image, and also every point has a unique preimage. Therefore such a transformation is a one-to-one mapping of the entire plane onto itself.

Oral Exercises

Find the images of the given points under the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

1. $P(4, 2)$ **2.** $Q(3, -7)$ **3.** $R(-4, 5)$ **4.** $S(-8, -8)$ **5.** $T(0, 0)$

Written Exercises

Find the coordinates of the image P' of the given point P under the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$ for the given matrix $\begin{bmatrix} h \\ k \end{bmatrix}$.

A **1.** $P(-5, 2)$; $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ **2.** $P(-7, 10)$; $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$ **3.** $P(-4, -9)$; $\begin{bmatrix} -2 \\ 12 \end{bmatrix}$

4. $P(0, 7)$; $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$ **5.** $P(a, b)$; $\begin{bmatrix} -a \\ b \end{bmatrix}$ **6.** $P(a, a + b)$; $\begin{bmatrix} a \\ a - b \end{bmatrix}$

Find the coordinates of the preimage P of the given point P' under the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$ for the given matrix $\begin{bmatrix} h \\ k \end{bmatrix}$.

7. $P'(3, -5)$; $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$
8. $P'(0, 5)$; $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$
9. $P'(-2, -6)$; $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$
10. $P'(8, -3)$; $\begin{bmatrix} -3 \\ 9 \end{bmatrix}$
11. $P'(0, 0)$; $\begin{bmatrix} a \\ b \end{bmatrix}$
12. $P'(a, -b)$; $\begin{bmatrix} a + b \\ a - b \end{bmatrix}$

Find a matrix equation of the translation of the plane that transforms the given point P into the given point P'.

13. $P(3, 7)$; $P'(5, 1)$
14. $P(-2, 6)$; $P'(5, -1)$
15. $P(4, -3)$; $P'(9, -1)$
16. $P(-2, -7)$; $P'(-8, 5)$
17. $P(a, b)$; $P'(c, d)$
18. $P(a + b, a - b)$; $P'(a, b)$

Find the image of the point X under the translation that transforms the given point P into the given point P'.

B 19. $X(5, 4)$; $P(0, -2)$; $P'(8, -5)$
20. $X(-3, 7)$; $P(-1, 4)$; $P'(7, -1)$
21. $X(6, 5)$; $P(-4, -9)$; $P'(-4, 2)$
22. $X(a, b)$; $P(c, d)$; $P'(a, -b)$

Find the preimage of X' under the translation that transforms the given point P into the given point P'.

23. $X'(3, -1)$; $P(8, 0)$; $P'(5, -2)$
24. $X'(5, 7)$; $P(-1, 4)$; $P'(3, 7)$
25. $X'(-2, -5)$; $P(3, 6)$; $P'(-1, 8)$
26. $X'(a, b)$; $P(c, d)$; $P'(a, 2b)$

C 27. Let $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h_1 \\ k_1 \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h_2 \\ k_2 \end{bmatrix}$ define two translations of the plane. What is the image of $P(a, b)$ under the translations applied in succession?
28. If the order of the translations in Exercise 27 is reversed, what is the image of $P(a, b)$?
29. On the basis of Exercises 27 and 28, is the application of successive translations of the plane commutative?

13-8 Transformations by Matrix Multiplication

The system of linear equations

$$\begin{aligned} x' &= 3x - 2y \\ y' &= 7x - 5y \end{aligned}$$

can be written in the matrix form $X' = AX$ as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

For each x, $y \in \mathcal{R}$, this matrix equation yields just one $\begin{bmatrix} x' \\ y' \end{bmatrix}$. For example,

$$\text{If } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \text{ then } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$

Accordingly, Equation (1) can be considered as defining a mapping function, or transformation of the plane,

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix},$$

with domain set of 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathcal{R}$, and range the set of corresponding matrices. (See Figure 3.)

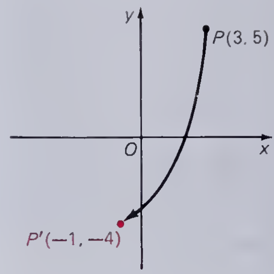


Figure 3

EXAMPLE 1 Under the transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ determine **(a)** the image of $P(3, -5)$; and **(b)** the preimage of $Q'(7, -1)$.

SOLUTION a. Substituting $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix}$ in the equation for the transformation, you have:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$\therefore$ the image of $P(3, -5)$ is $P'(7, 9)$. Answer.

b. Substituting $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ for $\begin{bmatrix} x' \\ y' \end{bmatrix}$ in the equation for the transformation, you obtain:

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The multiplicative inverse of $\begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix}$ is:

$$\frac{1}{-12 + 2}\begin{bmatrix} -3 & -1 \\ 2 & 4 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{10} & -\frac{4}{10} \end{bmatrix}$$

Left-multiplying both members of the equation by this multiplicative inverse, you obtain:

$$\begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{10} & -\frac{4}{10} \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{10} & -\frac{4}{10} \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Checking that these values satisfy the original equation is left to you.

$\therefore$ the preimage of $Q'(7, -1)$ is $Q(2, -1)$. Answer.

In general, any transformation of the form

$$X' = AX$$

where $X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $a_1, b_1, a_2, b_2 \in \mathcal{R}$, is called a **linear transformation of the plane.**

Each point in the plane has an image under such a transformation, as illustrated in Example 1(a). In particular, since $AO = O$, the image of the origin is the origin under every linear transformation.

Further, as illustrated in Example 1(b), if $\det A \neq 0$ then each point also has a unique preimage because the matrix A has a unique inverse. In this case, the transformation is a one-to-one mapping of the entire plane onto itself, and is said to be *nonsingular.*

EXAMPLE 2 Describe the mapping of the plane onto itself under the linear transformation of the plane, $X' = AX$, for which **(a)** $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; and **(b)** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

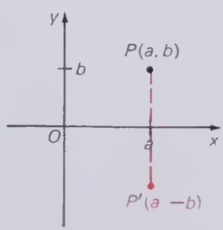
SOLUTION a. Substituting $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ in

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

you have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}, \text{ or } (x', y') = (x, -y).$$



Thus, for each $P(a, b)$ the image is the reflection $P'(a, -b)$ of P in the x-axis, as shown in the figure.

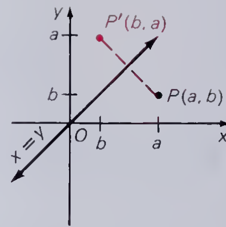
$\therefore$ the transformation is a *reflection in the x-axis.* Answer.

(*Solution continued on page 494.*)

b. Substituting $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, you have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \text{ or } (x', y') = (y, x).$$



Thus for each $P(a, b)$ the image is the reflection $P'(b, a)$ of P in the line $y = x$, as discussed on page 388 and as shown in the figure.

$\therefore$ the transformation is a *reflection in the line* $y = x$. Answer.

If $\det A = 0$ then the linear transformation $X' = AX$ is said to be *singular.*

EXAMPLE 3 Describe the mapping of the plane under the singular linear transformation for which **(a)** $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; and **(b)** $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

SOLUTION **a.** For all $x, y \in \mathcal{R}$, you have:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus the image of each $P(a, b)$ is $P'(0, 0)$.

$\therefore$ each point of the plane is mapped onto the origin under this transformation. Answer.

b. For any x and $y \in \mathcal{R}$, you have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 4x + 2y \end{bmatrix},$$

so that $x' = 2x + y, y' = 4x + 2y$. Since $4x + 2y = 2(2x + y)$, you have $y' = 2x'$. Therefore the map of the entire plane lies on the line $y' = 2x'$. Further, each point, say $(c, 2c)$ on this line is the image of many different points; for example, $(c, 2c)$ is the image of $\left(\frac{c}{2}, 0\right)$ and also of $\left(\frac{c}{3}, \frac{c}{3}\right)$.

$\therefore$ under this transformation the plane is mapped onto the line $y' = 2x'$. Answer.

Examples 2 and 3 illustrate the theorem on page 495.

Theorem. The linear transformation $AX = X'$, where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $a_1, b_1, a_2, b_2 \in \mathcal{R}$ is a one-to-one mapping of the plane onto itself if $\det A \neq 0$.

If $A = O$, then the entire plane is mapped onto the origin.

If $A \neq O$ but $\det A = 0$, then the entire plane is mapped onto a line through the origin.

Written Exercises

Find the coordinates of the image P' of the given point P under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for the given matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

A

1. $P(1, 2)$; $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
2. $P(-2, 3)$; $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$
3. $P(5, -1)$; $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$
4. $P(1, -2)$; $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$
5. $P(-1, -4)$; $\begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}$
6. $P(3, 5)$; $\begin{bmatrix} -4 & 3 \\ 5 & -2 \end{bmatrix}$

Find the coordinates of the preimage P of the given point P' under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for the given matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

7. $P'(3, 12)$; $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$
8. $P'(6, -4)$; $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$
9. $P'(-2, 5)$; $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$
10. $P'(4, -1)$; $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$
11. $P'(8, 6)$; $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$
12. $P'(6, -3)$; $\begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$

For each of the following linear transformations the image P' of each point $P(x, y)$ lies on a line with equation $y' = mx'$. Find x' and y' in terms of x and y, and find m.

B

13. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
14. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
15. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 & -10 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
16. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Find the image of the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1) under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for the given matrix.

17. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
18. $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$
19. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
20. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Describe the mapping of the plane onto itself under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for the given matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

C **21–24.** The matrices in Exercises 17–20 on page 495.

25. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **27.** $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ **28.** $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

29. Show that the images of the points (0, 0), (1, 0), (1, 1), and (0, 1) are the vertices of a parallelogram under any nonsingular linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

30. Show that if $X' = AX$ and $X' = BX$ are two linear transformations, then the image of point $P(x, y)$ under the transformations applied in succession (B, then A) is the same as the image of $P(x, y)$ under the product transformation AB. That is, show that $A(BX) = (AB)X$. $\left(\text{Use } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.\right)$

Self-Test 3

VOCABULARY translation (p. 488) image (p. 488) preimage (p. 488) linear transformation (p. 493) nonsingular linear transformation (p. 493)

1. Find an equation of the translation of the plane that transforms the point $P(-1, 4)$ into the point $P'(5, 3)$. *Obj. 1, p. 488*
2. Find the coordinates of the image P' of the point $P(-2, 7)$ under the translation of the plane given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. *Obj. 2, p. 488*
3. Find the preimage P of the point $P'(4, -9)$ under the translation given in Test Item 2 above.
4. Find the coordinates of the image P' of the point $P(-1, -6)$ under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. *Obj. 3, p. 488*
5. Find the coordinates of the preimage P of the point $P'(2, 4)$ under the linear transformation given in Test Item 4 above. *Obj. 4, p. 488*

Check your answers with those at the back of the book.

Chapter Summary

1. Two matrices are *equal* if and only if they have the same dimensions and all their corresponding entries are equal.
2. If two matrices have the same dimensions, then their *sum* is a matrix of the same dimensions, whose entries are the sums of the corresponding entries of the given matrices.
3. In the set $\mathcal{S}_{m\times n}$ of all $m \times n$ matrices with real-number entries, the *identity matrix for addition* is the zero matrix $O_{m\times n}$ all of whose entries are zero. The *additive inverse,* or *negative,* of the matrix $A_{m\times n}$ is the $m \times n$ matrix $-A_{m\times n}$ each of whose entries is the negative of the corresponding entry of $A_{m\times n}$. The *difference* $A_{m\times n} - B_{m\times n}$ is defined to be the sum $A_{m\times n} + (-B_{m\times n})$.
4. The set $\mathcal{S}_{m\times n}$ of all $m \times n$ matrices with real-number entries has the same addition properties as the set of all real numbers: closure, commutative, associative, additive-identity, and additive-inverse.
5. In dealing with matrices, we often refer to real numbers as *scalars.* The *product* of a scalar c and a matrix A is denoted by cA; it is the matrix of the same dimensions as A whose entries are the products of c and the corresponding entries of A. Basic properties of these products follow from the definition and the properties of real numbers.
6. The *product* AB of the matrices A and B can be described as "row by column" multiplication. The product is defined only if the number of columns in A is equal to the number of rows in B. The product matrix then has the same number of rows as A and the same number of columns as B. Thus: $A_{m\times p} \times B_{p\times n} = C_{m\times n}$.
7. The set $\mathcal{S}_{2\times 2}$ is closed under matrix multiplication, and the associative and distributive properties hold for matrix multiplication. The *identity element for multiplication* is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But multiplication in $\mathcal{S}_{2\times 2}$ *is not commutative,* though certain products do commute. The product of two matrices can be the zero matrix without either factor being the zero matrix.
8. The determinant having the same entries as a given square matrix A is the *determinant* of the matrix. The matrix A is *singular* if $\det A = 0$; otherwise A is nonsingular. A has a *multiplicative inverse,* that is, there is a square matrix A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$, if and only if A is nonsingular.
9. A transformation, or mapping of the plane defined by an equation of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$, where $h, k \in \mathcal{R}$, is called a *translation.* Under the transformation, the point $P'(x', y')$ is called the *image* of the point $P(x, y)$, and P is called the *preimage* of P'.

10. An equation of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, where $a, b, c, d \in \Re$, defines a *linear transformation*. Such a transformation is one-to-one if and only if its matrix is *nonsingular*.

Chapter Review

1. What are the dimensions of the matrix $\begin{bmatrix} -1 & 2 & 3 \\ -2 & 0 & 2 \end{bmatrix}$? 13-1

a. 3×2 b. 2×3 c. 1×3 d. 3×1

2. Find the sum: $\begin{bmatrix} -2 & 3 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & -7 \end{bmatrix}$.

a. $\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix}$ c. $\begin{bmatrix} -4 & 0 \\ -2 & 2 \end{bmatrix}$ d. $\begin{bmatrix} -4 & 0 \\ 2 & -2 \end{bmatrix}$

3. Solve for the matrix X: $X + \begin{bmatrix} -3 & -2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -2 & 0 \end{bmatrix}$. 13-2

a. $\begin{bmatrix} 3 & -3 \\ -2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 9 & 1 \\ -2 & -6 \end{bmatrix}$ c. $\begin{bmatrix} 3 & 2 \\ 0 & -6 \end{bmatrix}$ d. $\begin{bmatrix} 9 & 3 \\ -2 & 6 \end{bmatrix}$

In Review Items 4 and 5, find the 2×2 matrix X that satisfies the equation for $A = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -9 \\ 5 & -3 \end{bmatrix}$.

4. $X = 2A + B$ 13-3

a. $\begin{bmatrix} 8 & -10 \\ 8 & -7 \end{bmatrix}$ b. $\begin{bmatrix} 14 & -17 \\ 13 & -10 \end{bmatrix}$ c. $\begin{bmatrix} 10 & -17 \\ 11 & -10 \end{bmatrix}$ d. $\begin{bmatrix} 10 & -11 \\ 11 & -11 \end{bmatrix}$

5. $X + A = B$

a. $\begin{bmatrix} 4 & -8 \\ 2 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 8 & -10 \\ 8 & -7 \end{bmatrix}$ c. $\begin{bmatrix} 8 & -8 \\ 2 & -7 \end{bmatrix}$ d. $\begin{bmatrix} 4 & -10 \\ 2 & -7 \end{bmatrix}$

6. Find the 2×2 matrix equal to $\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ 0 & 6 \end{bmatrix}$. 13-4

a. $\begin{bmatrix} -6 & 4 \\ 8 & 0 \end{bmatrix}$ b. $\begin{bmatrix} -6 & -10 \\ 9 & 0 \end{bmatrix}$ c. $\begin{bmatrix} -6 & 4 \\ 9 & -12 \end{bmatrix}$ d. $\begin{bmatrix} -6 & 4 \\ 9 & 0 \end{bmatrix}$

7. If $A \in \mathcal{S}_{2\times 2}$ and $B \in \mathcal{S}_{2\times 2}$, then $(A + B)(A - B) = \underline{\ ?\ }$ 13-5

a. $A^2 - B^2$ b. $A^2 - AB + BA - B^2$ c. $A^2 - 2AB - B^2$ d. $A^2 + B^2$

8. Find the inverse of $\begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix}$. 13-6

a. $\begin{bmatrix} -1 & 2 \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$ b. $\begin{bmatrix} \frac{3}{4} & -2 \\ -\frac{1}{4} & 1 \end{bmatrix}$ c. $\begin{bmatrix} \frac{3}{4} & 2 \\ \frac{1}{4} & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & -2 \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$

9. If $AX = B$, then $X = \underline{\ ?\ }$

a. AB b. $A^{-1}B$ c. BA^{-1} d. $B^{-1}A$

In Review Items 10–11, use the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

10. Find the image P' of $P(4, -7)$. 13-7

a. $P'(2, -2)$ **b.** $P'(-6, -2)$ **c.** $P'(2, -12)$ **d.** $P'(-6, -12)$

11. Find the preimage Q of the point $Q'(1, -1)$.

a. $Q(3, -6)$ **b.** $Q(-1, 4)$ **c.** $Q(-1, -6)$ **d.** $Q(3, 4)$

12. Find the image R' of the point $R(2, -1)$ under the transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. 13-8

a. $R'(-5, -6)$ **b.** $R'(-1, 2)$ **c.** $R'(1, -2)$ **d.** $R'(-5, 6)$

Chapter Test

1. Simplify: $\begin{bmatrix} 5 & 3 \\ -2 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -3 \\ 6 & 9 \end{bmatrix}$ **2.** Simplify: $\begin{bmatrix} -8 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 3 & 6 \end{bmatrix}$ 13-1

3. Solve: $[x - y \quad x + y] = [6 \quad -2]$

4. Solve over $\mathcal{S}_{2\times3}$: $X - \begin{bmatrix} -3 & 2 & 0 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -2 \\ 0 & -5 & 1 \end{bmatrix}$ 13-2

5. Solve over $\mathcal{S}_{2\times2}$: $3X - \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} = 4X - \begin{bmatrix} 3 & -2 \\ 0 & -5 \end{bmatrix}$ 13-3

6. Solve for x and y: $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ 13-4

7. For $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$, compute $(2A)B$ and $A(2B)$ and determine whether or not $(2A)B = A(2B)$. 13-5

8. Find the inverse of the matrix $\begin{bmatrix} 3 & -2 \\ 4 & 0 \end{bmatrix}$. 13-6

9. Solve over $\mathcal{S}_{2\times2}$: $\begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}X = \begin{bmatrix} -10 & -3 \\ 3 & 4 \end{bmatrix}$.

10. Use a matrix equation to solve the system: $\begin{aligned} 2x - 3y &= 6 \\ x + 2y &= -4 \end{aligned}$

11. A translation of the plane maps $P(-3, 6)$ on $P'(-2, 1)$. 13-7

a. Find an equation of the translation.
b. Find the image of $Q(-3, 0)$.
c. Find the preimage of $R'(-1, 4)$.

12. Consider the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. 13-8

a. Find the coordinates of the image P' of $P(0, 2)$.
b. Find the coordinates of the preimage Q of $Q'(1, 3)$.

The undersea habitat shown is located at Great Lameshur Bay, St. John, U.S. Virgin Islands. It served as home base for four scientist-aquanauts who spent a record-breaking two months doing research while living on the ocean floor.

14 Trigonometric and Circular Functions

Angles and Their Measurement

***OBJECTIVES** for Sections 14-1 and 14-2:*
1. *Find the distance traveled during a given number of revolutions by a point on the rim of a wheel.*
2. *Convert radian measure to degree measure, and vice versa.*

14-1 Rotations and Angles

How can you measure the distance traveled by a wheeled vehicle in going from one point to another? One way may be described as follows: First, count the number of times a wheel on which the vehicle travels has rotated through a complete revolution in making the trip. Count partial revolutions as fractions of complete revolutions. Then multiply this number of revolutions by the circumference of the wheel. The result will be (in the absence of slippage, of course) the distance the wheel, and hence the vehicle, has moved.

Figure 1

EXAMPLE A tractor wheel has a diameter of 2 m. How far will the tractor travel as the wheel makes 18.6 revolutions?

SOLUTION To find the circumference C of the wheel, we use $C = \pi d$ where $\pi \approx 3.14$. Thus,

$$C \approx 3.14(2) = 6.28.$$

Then the distance the tractor travels in 18.6 revolutions is

$$s \approx 6.28(18.6) \approx 116.8.$$

$\therefore$ the distance traveled is approximately 117 m. Answer.

To study rotating objects, we need to reconsider the concept of *angle*. An *angle* may have been defined in your geometry course as the union of two noncollinear rays that have the same endpoint, as suggested by $\angle AOB$ pictured in Figure 2. We shall now define a **directed angle** as an *ordered pair* of rays with a common endpoint, one ray called the *initial* side of the angle and the other called the *terminal* side of the angle, together with a *rotation* from the initial side to the terminal side (see Figure 3). For this definition we drop the restriction that the rays be noncollinear.

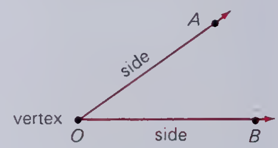


Figure 2

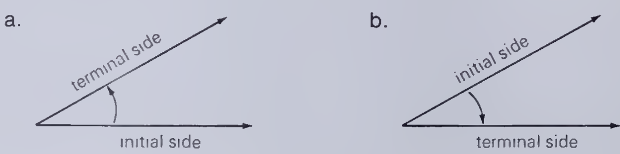


Figure 3

The rotation of the directed angle pictured in Figure 3b is *clockwise;* the rotation pictured in Figure 3a is *counterclockwise.* The angle in Figure 4 has a rotation of $\frac{1}{4}$ of a revolution *counterclockwise.* An angle having a rotation of $\frac{1}{4}$ of a revolution is a **right angle.**

Figure 4

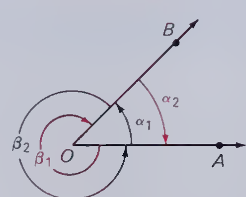


Figure 5

The sides of any geometric angle form the sides of many different directed angles. (Greek letters such as α (alpha) and β (beta) are often used to name angles.) In Figure 5, the sides of $\angle AOB$ are those of angle α_1, which has a counterclockwise rotation of $\frac{1}{8}$ of a revolution; angle α_2,

which has a clockwise rotation of $\frac{1}{8}$ of a revolution; angle β_1, which has a clockwise rotation of $\frac{7}{8}$ of a revolution; and angle β_2, which has a counterclockwise rotation of $\frac{7}{8}$ of a revolution.

Angles that have the same initial side and the same terminal side are called **coterminal angles.** In Figure 5, α_1 and β_1 are coterminal angles, as are α_2 and β_2.

To study an angle, it is convenient to consider it as placed on a rectangular coordinate system with the vertex, O, of the angle at the origin and the initial side of the angle as the positive part of the horizontal axis, as shown in Figure 6. The angle is then said to be in **standard position.**

wherever the terminal side lies determines the quadrant the angle is in

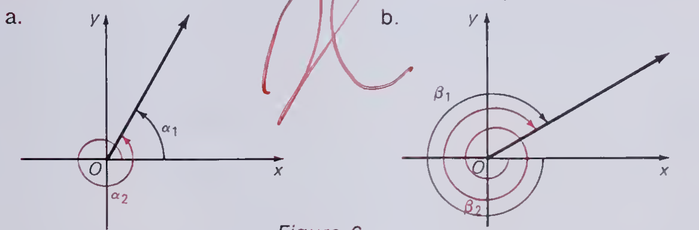


Figure 6

In Figure 6a, angle α_1 and angle α_2 are coterminal. Angle α_1 has a counterclockwise rotation of $\frac{1}{6}$ of a revolution, and angle α_2 has a counterclockwise rotation of $1\frac{1}{6}$ revolutions.

In Figure 6b, angles β_1 and β_2 are coterminal, with angle β_1 having a clockwise rotation of $\frac{11}{12}$ of a revolution and angle β_2 having a clockwise rotation of $1\frac{11}{12}$ of a revolution.

If the terminal side of an angle in standard position lies in a given quadrant, then the angle is said to lie *in* that quadrant. If the terminal side of an angle coincides with a coordinate axis, then the angle is called a **quadrantal angle.** The angles in Figure 6 all lie in the first quadrant. Angle α_1 in Figure 7a lies in the second quadrant; angle α_2 lies in the third quadrant. Angle β_1 in Figure 7b is a quadrantal angle having $\frac{1}{2}$ of a revolution; its terminal side lies in the negative x-axis. It is called a **straight angle.** Angle β_2 is a quadrantal angle having $1\frac{3}{4}$ revolutions; its terminal side lies in the negative y-axis.

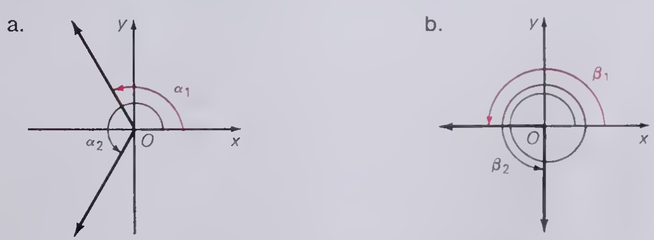


Figure 7

1-5

Oral Exercises

How many revolutions does the minute hand of a clock make from 4:00 P.M. to the given time?

1. 4:15 P.M. 2. 4:30 P.M. 3. 4:50 P.M. 4. 5:00 P.M. 5. 5:20 P.M.

1-7 odd

Written Exercises

9-20 all

Find the distance traveled by a wheel of the given diameter as it makes the given number of revolutions. Use $\pi \approx 3.14$.

A

1. 20 cm; 1.5 revolutions
2. 3.5 m; 2.2 revolutions
3. 8 m; 7.7 revolutions
4. 10 m; 0.4 revolutions

Find the radius of a wheel such that the wheel will travel the given distance in the given number of revolutions. Use $\pi \approx 3.14$.

5. 628 cm; 5 revolutions
6. 56.52 m; 6 revolutions
7. 94.2 m; 8 revolutions
8. 70.65 m; 4.5 revolutions

Sketch the following angles in standard position, indicating the rotation with a curved arrow.

9. $\frac{3}{4}$ revolution counterclockwise
10. $\frac{7}{8}$ revolution counterclockwise
11. $\frac{2}{3}$ revolution counterclockwise
12. $1\frac{1}{4}$ revolutions clockwise
13. $\frac{5}{6}$ revolution clockwise
14. $2\frac{3}{8}$ revolutions counterclockwise

15–20. Express each angle in Exercises 9–14 above as a number of revolutions in the opposite direction, clockwise or counterclockwise.

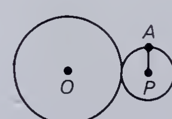
In Exercises 21–23, use $\pi \approx 3.14$.

B

21. How far does a point on the outer tip of the minute hand of a clock travel in 5 min if the hand is 21 cm long?
22. How far does a point on the outer tip of the hour hand of a clock travel in 5 min if the hand is 6.3 cm long?
23. A satellite circles the earth once every 6 h. If the satellite is 7000 km from the center of the earth, what is its speed?

C

24. The radius of the wheel with center O is 4 times that of the wheel with center P. If wheel O is stationary and wheel P rolls around it once, how many revolutions does radius $\overline{PA}$ make?



25. In Exercise 24 above, if wheel P were rolling *inside* wheel O, how many revolutions would radius $\overline{PA}$ make?

14-2 Measurement of Angles

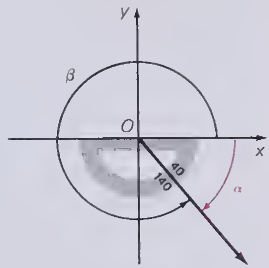


Figure 8

In Section 14-1, angles were described in terms of complete revolutions and fractions of revolutions. However, in order to make effective use of angles, we need some system of measurement that is based on a smaller unit than a complete revolution. Two such systems will be described in this section.

In one system a complete revolution is divided into 360 equal parts, each of which is called a **degree of rotation** or simply a **degree.** If the rotation is counterclockwise, the measure is ordinarily taken as positive. If the rotation is clockwise, the measure is negative. In Figure 8, the measure of angle α is negative 40 degrees, written as $-40°$, and the measure of angle β is $320°$.

For more precise measurements, a degree is subdivided into minutes and seconds as follows:

$$1° = 60 \text{ minutes (written } 60')$$
$$1' = 60 \text{ seconds (written } 60'')$$

The equals sign is used to indicate that we have written two names for the same amount of rotation.

EXAMPLE 1 Find the measure in the degree system of an angle formed by a rotation of:

a. $2\frac{1}{4}$ revolutions clockwise. b. $\frac{17}{54}$ revolution counterclockwise.

SOLUTION a. $2\frac{1}{4} \times (-360°) = -(\frac{9}{4} \times 360°) = -810°$. Answer.

b. $\dfrac{17}{54} \times 360° = \dfrac{340°}{3} = 113\frac{1}{3}° = 113°20'$. Answer.

Several notations exist for recording angle measurements. If angle α has 35°, you may write

$m°(\alpha) = 35$, read "the measure in degrees of α is 35" or
$m(\alpha) = 35°$, read "the measure of α is 35°."

If angle α has 35°23′, you may write

$$m°(\alpha) = 35\tfrac{23}{60} \qquad \text{or} \qquad m(\alpha) = 35°23'.$$

If α and β are coterminal angles, you may write

$$m°(\beta) = m°(\alpha) + k \cdot 360,\ k \in \{\text{the integers}\}.$$

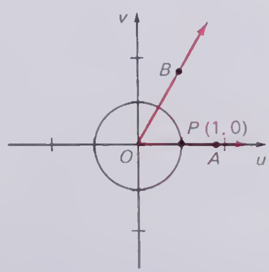


Figure 9

For a second system of measuring angles, consider $\angle AOB$ placed in standard position on the uv-coordinate axes (Figure 9). On the same set of axes, picture the **unit circle,** that is, the circle with center at (0, 0) and radius 1.

Notice that any angle in standard position is determined by the point of intersection of its terminal side with the unit circle. We will measure angles by measuring the length of the arc of the unit circle intercepted by the angle.

Imagine a flexible x number line tangent to the unit circle at $P(1, 0)$, with origin at P and the same scale, or unit length, as on the u and v axes. Think of the number line as being wound around the unit circle as a thread would be wound on a spool. The positive ray would be wound in the counterclockwise direction, and the negative ray in the clockwise direction (Figure 10). This winding procedure pairs each real number (with graph on the flexible number line) with one and only one point on the unit circle. For example, 2 is paired with point S.

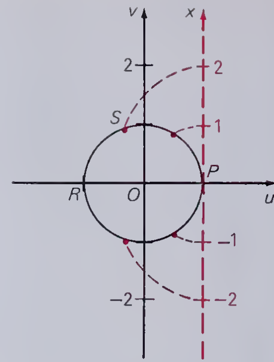


Figure 10

As the flexible number line is wound around the unit circle, more than one number will be paired with the same point. Thus, both π and $-\pi$ are paired with point R. Moreover, S is paired with 2, $2 + 2\pi$, $2 - 2\pi$, and so on, and P with 0, 2π, -2π, 4π, and so on.

Now we can assign a measure to any angle in standard position as follows: If the *length of an arc* on the unit circle measured from point P is a units, then the *measure of the angle* intercepting that arc is said to be a **radians.** For example, the measure of the angle α in Figure 11 is 2 radians, written 2^{R}. The angle for a complete revolution is measured by $2\pi^{\mathrm{R}}$.

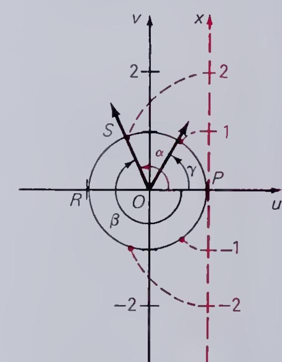


Figure 11

A notation similar to that for degree measure is used. Thus, you may write

$$m^{\mathrm{R}}(\alpha) = 2 \qquad \text{or} \qquad m(\alpha) = 2^{\mathrm{R}}.$$

Also, in Figure 11,

$$m^{\mathrm{R}}(\beta) = 2 - 2\pi \qquad \text{or} \qquad m(\beta) = (2 - 2\pi)^{\mathrm{R}}.$$

In general, if α and β are coterminal angles, you may write:

$$m^{\mathrm{R}}(\beta) = m^{\mathrm{R}}(\alpha) + 2k\pi,\ k \in \{\text{the integers}\}.$$

Since an angle for a complete revolution is also measured by 360°, you have

$$360° = 2\pi^{\mathrm{R}},$$

or

$$180° = \pi^{\mathrm{R}}.$$

Thus,

$$1° = \frac{\pi^{\mathrm{R}}}{180} \qquad \text{and} \qquad 1^{\mathrm{R}} = \frac{180°}{\pi}.$$

To change from degrees to radians, or from radians to degrees, we use the above to get **conversion equations:**

$$m^\circ(\alpha) = \frac{180}{\pi} m^R(\alpha) \qquad m^R(\alpha) = \frac{\pi}{180} m^\circ(\alpha)$$

Since $\pi \approx 3.14159$, you have $1^\circ \approx 0.01745^R$ and $1^R \approx 57°17'45''$. In Figure 11, γ (Greek *gamma*) measures 60°. Hence $m(\gamma) = 60^\circ = \frac{\pi}{180} \cdot 60^R = \frac{\pi}{3}^R$.

Thus, γ is slightly greater than an angle that measures 1^R.

EXAMPLE 2 a. If $m^R(\alpha) = \frac{5\pi}{6}$, find $m^\circ(\alpha)$. b. If $m^\circ(\beta) = 240$, find $m^R(\beta)$.

SOLUTION a. $m^\circ(\alpha) = \frac{180}{\pi} m^R(\alpha) = \frac{180}{\pi}\left(\frac{5\pi}{6}\right) = 150.$

b. $m^R(\beta) = \frac{\pi}{180} m^\circ(\beta) = \frac{\pi}{180}(240) = \frac{4\pi}{3}.$

EXAMPLE 3 Find the length of the arc intercepted by a central angle with measure $\frac{7}{3}\pi^R$ in a circle whose radius is 18 cm. Use $\pi \approx \frac{22}{7}$.

SOLUTION $C = 2\pi r = 2\pi(18) \approx 2(\frac{22}{7})(18) = \frac{792}{7} \approx 113.$

The angle contains $\dfrac{\frac{7\pi}{3}}{2\pi}$ or $\frac{7}{6}$ of a revolution, so the length of the arc is $\frac{7}{6}$ of the circumference. $\frac{7}{6}(113) \approx 132.$

$\therefore$ the length of the arc is approximately 132 cm. Answer.

In general, you can show (Exercise 33, page 508) that in a circle of radius r, the length, s, of the arc intercepted by a central angle α is given by the formula

$$s = r \cdot m^R(\alpha).$$

1-6

Oral Exercises

Find the degree measure of an angle of rotation through the given number of revolutions in the given direction.

Oral

1. $\frac{2}{3}$ revolution counterclockwise
2. $1\frac{3}{4}$ revolutions clockwise
3. $3\frac{1}{2}$ revolutions counterclockwise
4. $\frac{5}{6}$ revolution counterclockwise
5. $2\frac{3}{5}$ revolutions counterclockwise
6. $1\frac{5}{8}$ revolutions clockwise

1-30 odd

Written Exercises

Express each degree measure as a radian measure using π.

A 1. $120°$ 2. $-135°$ 3. $210°$ 4. $-45°$ 5. $300°$

6. $-315°$ 7. $150°$ 8. $330°$ 9. $-144°$ 10. $108°$

Express each radian measure as a degree measure.

11. $\frac{3\pi}{2}^{\mathrm{R}}$ 12. $\frac{4\pi}{3}^{\mathrm{R}}$ 13. $-\frac{\pi}{4}^{\mathrm{R}}$ 14. $-\frac{5\pi}{6}^{\mathrm{R}}$ 15. $\frac{11\pi}{6}^{\mathrm{R}}$

16. $-\frac{3\pi}{5}^{\mathrm{R}}$ 17. $\frac{7\pi}{4}^{\mathrm{R}}$ 18. $\frac{13\pi}{6}^{\mathrm{R}}$ 19. $\frac{5\pi}{12}^{\mathrm{R}}$ 20. $-\frac{7\pi}{9}^{\mathrm{R}}$

Find the length of the arc on a circle with the given radius that is intercepted by a central angle of the given measurement. Use $\pi \approx \frac{22}{7}$.

21. 28 cm; $270°$ 22. 2.1 cm; $390°$ 23. 10.5 cm; $150°$

24. 714 mm; $\frac{5\pi}{3}^{\mathrm{R}}$ 25. 49 cm; $\frac{\pi}{4}^{\mathrm{R}}$ 26. 0.56 cm; $\frac{9\pi}{4}^{\mathrm{R}}$

Find the radius of a circle in which the arc of given length is intercepted by the angle of given degree measure.

EXAMPLE $\widehat{AB}$: 8π; $m°(\alpha) = 120$

SOLUTION Since $m^{\mathrm{R}}(\alpha) = \frac{\pi}{180} m°(\alpha)$, you have

$$m^{\mathrm{R}}(\alpha) = \frac{\pi}{180} \cdot 120 = \frac{2\pi}{3}.$$

Then, since $s = r \cdot m^{\mathrm{R}}(\alpha)$, you have

$$8\pi = r \cdot \frac{2\pi}{3} \text{ and } r = 8\pi \cdot \frac{3}{2\pi} = 12. \quad \text{Answer.}$$

27. $\widehat{AB}$: 15π; $m°(\alpha) = 300$ 28. $\widehat{AB}$: 21π; $m°(\alpha) = 150$

29. $\widehat{AB}$: $\frac{9\pi}{4}$; $m°(\alpha) = 135$ 30. $\widehat{AB}$: $\frac{8\pi}{5}$; $m°(\alpha) = 270$

31. $\widehat{AB}$: $\frac{5\pi}{12}$; $m°(\alpha) = 330$ 32. $\widehat{AB}$: $\frac{j\pi}{k}$; $m°(\alpha) = c$

B 33. Derive the formula $s = r \cdot m^{\mathrm{R}}(\alpha)$.

C 34. Write an arithmetic sequence such that each of its terms expresses the measure in radians of an angle coterminal with $\frac{\pi}{2}$.

Self-Test 1

VOCABULARY directed angle (p. 502)
right angle (p. 502)
coterminal angles (p. 503)
standard position (p. 503)
quadrantal angle (p. 503)
straight angle (p. 503)
degree (p. 505)
unit circle (p. 505)
radians (p. 506)

1. How far will a point on the rim of a wheel of radius 21 cm travel in 4.5 revolutions? Use $\pi \approx \frac{22}{7}$. *Obj. 1, p. 501*
2. Convert to radian measure: 420°. *Obj. 2, p. 501*
3. Convert to degree measure: $\frac{11\pi^R}{4}$.

Check your answers with those at the back of the book.

programming in BASIC

Exercises

1. Write a program to change an angle measurement in degrees and minutes to the corresponding measurement in radians. (Use $\pi = 3.14159$.)
2. Write a program to change an angle measurement in radians to the corresponding measurement in degrees and minutes.
3. You can use a defined function to round a result to a given number of decimal places. For example, to round to tenths, use:

   ```
   DEF FNR(X) = (INT (10*X + 0.5))/10
   ```

   Add this function to the program of Exercise 2 and give the measurement correct to tenths of a minute. That is, if D is the integral number of degrees and M the number of minutes, then use:

   ```
   PRINT D; " DEGREES"; FNR(M); " MINUTES"
   ```

4. Write a program to change an angle measurement expressed in radians to the corresponding measurement in decimal degrees. For example $\frac{\pi^R}{8} = 22.5°$. (Use $\pi = 3.14159$.)

The Sine and Cosine Trigonometric and Circular Functions

***OBJECTIVES** for Sections 14-3 through 14-8:*

1. *Determine the sine and cosine of an angle in standard position given the coordinates of a point other than the origin on the terminal side of the angle.*
2. *Determine one of the values sin α or cos α given the other value and the quadrant in which α lies.*
3. *Find values or approximate values for cos α and sin α for specified angles α.*
4. *Sketch the graph of* $\{(x, y){:}\ y = A \sin Bx\}$ *and* $\{(x, y){:}\ y = A \cos Bx\}$ *for given constants A and B.*

14-3 The Sine and Cosine Functions

Figure 12 shows an angle α in standard position on a uv-coordinate system. $P(u_1, v_1)$ and $R(u_2, v_2)$ are two distinct points different from the origin on the terminal side of α. $\overline{PA}$ and $\overline{RB}$ are perpendicular to the u-axis. Since right triangles OAP and OBR share a common acute angle, α, they are similar. Therefore their sides are proportional, that is:

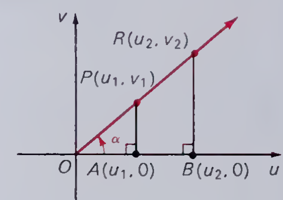


Figure 12

$$\frac{v_1}{\sqrt{u_1^2 + v_1^2}} = \frac{v_2}{\sqrt{u_2^2 + v_2^2}} \quad \text{and} \quad \frac{u_1}{\sqrt{u_1^2 + v_1^2}} = \frac{u_2}{\sqrt{u_2^2 + v_2^2}}.$$

Although the terminal side of angle α is pictured in Quadrant I in Figure 12, the preceding proportions hold for an angle in any quadrant (see Figure 13, for example) and for quadrantal angles as well. The ratios above are independent of the points chosen in the terminal side of α, and so for each angle α there are unique values

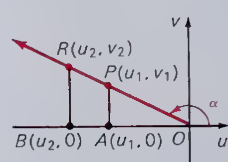


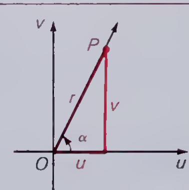
Figure 13

$$\frac{v_1}{\sqrt{u_1^2 + v_1^2}} \quad \text{and} \quad \frac{u_1}{\sqrt{u_1^2 + v_1^2}}.$$

Thus, we can define two functions as follows:

If α is an angle in standard position, with $P(u, v)$ any point other than the origin on the terminal side of α, and if $\sqrt{u^2 + v^2} = r$, then:

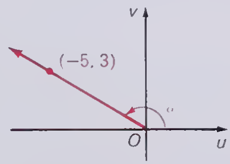
sine: $\alpha \to \dfrac{v}{r}$ cosine: $\alpha \to \dfrac{u}{r}$



To represent function values of these functions, this notation is used:

$\sin \alpha$ (read "sine of α") $\cos \alpha$ (read "cosine of α")

The preceding definitions tell you what is meant by the sine and the cosine of any angle *in standard position.* But any angle can be put into standard position, and so *the domain of each of these functions is the set of all angles.* Because of their relation to triangles, these functions are called **trigonometric functions,** where the word "trigonometric" comes from the Greek (*trigonon,* meaning *triangle,* and *metron,* meaning *measure*). There are several other trigonometric functions, which will be defined later in this chapter.



EXAMPLE If α is an angle in standard position and its terminal side contains the point $(-5, 3)$, find $\sin \alpha$ and $\cos \alpha$.

SOLUTION Using -5 for u and 3 for v in these definitions of the functions, and noting that

$$\sqrt{u^2 + v^2} = \sqrt{(-5)^2 + 3^2} = \sqrt{34},$$

you have

$$\sin \alpha = \frac{3}{\sqrt{34}}$$

and

$$\cos \alpha = -\frac{5}{\sqrt{34}}.$$

By computation, you find that for the angle α in the example above,

$$\sin \alpha \approx 0.5145 \qquad \text{and} \qquad \cos \alpha \approx -0.8575.$$

The definitions of $\sin \alpha$ and $\cos \alpha$ make it clear that these values are positive or negative depending on the quadrant in which the terminal side of α lies. Thus, since $u > 0$ and $v < 0$ in the fourth quadrant (and you always have $r > 0$ except at the origin), it follows that in the fourth quadrant,

$$\sin \alpha = \frac{v}{r} < 0$$

and

$$\cos \alpha = \frac{u}{r} > 0.$$

These inequalities and corresponding ones for the other quadrants are shown in Figure 14.

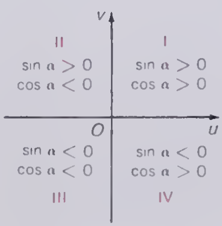


Figure 14

Let us now find the ranges of the sine and cosine functions. Suppose that $R(u, v)$ is the point of intersection of the terminal side of an angle α in standard position with the unit circle (Figure 15). Then you have

$$u^2 + v^2 = 1.$$

$$\sin \alpha = \frac{v}{\sqrt{u^2 + v^2}} = \frac{v}{1} = v,$$

$$\cos \alpha = \frac{u}{\sqrt{u^2 + v^2}} = \frac{u}{1} = u.$$

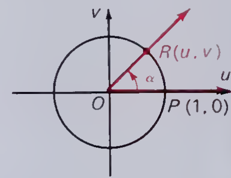


Figure 15

This means that the coordinates of R are $(\cos \alpha, \sin \alpha)$. Thus, the ranges of the sine and cosine functions are given, respectively, by

$$-1 \le \sin \alpha \le 1 \qquad \text{and} \qquad -1 \le \cos \alpha \le 1.$$

Since the coordinates of R must satisfy $u^2 + v^2 = 1$, you have

$$(\sin \alpha)^2 + (\cos \alpha)^2 = 1,$$

which is usually written

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

This statement is one of the **fundamental trigonometric identities.** (Recall that an identity is an equation which is true for all real values of the variables.)

Theorem. For every angle α,

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

There is an alternative way to define sine and cosine functions, a way in which the elements of the domains of the functions are real numbers instead of angles. Recall that when we set up the system of radian measure, each real number was paired with one and only one point on the unit circle. Thus, with each real number $m^R(\alpha)$, you can pair a value $\sin \alpha$ and a value $\cos \alpha$ to define functions

$$\text{sine: } m^R(\alpha) \to \sin \alpha \qquad \text{cosine: } m^R(\alpha) \to \cos \alpha$$

which each have $\mathcal{R}$ as domain. Each range is, as before,

$$-1 \le \sin \alpha \le 1 \qquad \text{and} \qquad -1 \le \cos \alpha \le 1.$$

Although $m^R(\alpha)$ was used in defining these functions, it is customary to represent them by using a single variable, x, y, z, etc., in place of $m^R(\alpha)$ and writing, for example:

$$\text{sine: } x \to \sin x \qquad \text{cosine: } x \to \cos x$$

Because, in this context, values in the domains of these functions can be pictured as lengths of arcs on the unit circle (Figure 16), these functions are sometimes called **circular functions** to distinguish them from the trigonometric functions with the set of angles as domains.

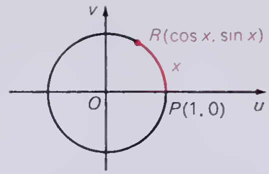


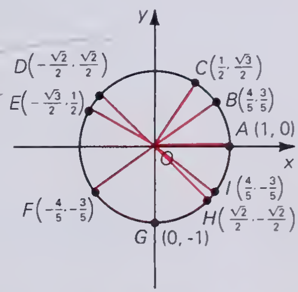
Figure 16

The fact that each angle α has one and only one measure in radians ensures that the circular functions and angle functions have identical properties. Since it is sometimes convenient (and easier) to study properties of one of these kinds of functions in preference to the other, we shall, hereafter, use whichever seems best suited for our purposes, with the understanding that fundamental properties developed using one kind are equally applicable to the other.

Oral Exercises

State the sine and cosine of the angle.

1. $\angle AOB$
2. $\angle AOC$
3. $\angle AOD$
4. $\angle AOE$
5. $\angle AOF$
6. $\angle AOG$
7. $\angle AOH$
8. $\angle AOI$



Written Exercises

Sketch the angle α whose terminal side in standard position passes through the given point, and find $\sin \alpha$ and $\cos \alpha$. Leave your answers in fractional form.

A

1. (8, 6)
2. (−3, 4)
3. (12, −5)
4. (8, 15)
5. (0, 2)
6. (−3, −3)
7. (2, 4)
8. (−2, 3)
9. (−3, −4)
10. (−5, 0)

In Exercises 11–18, find $\sin \alpha$ or $\cos \alpha$, whichever is not given, for α in the given quadrant.

EXAMPLE $\sin \alpha = \frac{4}{5}$; II

SOLUTION Since $\sin^2 \alpha + \cos^2 \alpha = 1$ for every angle α, you have $(\frac{4}{5})^2 + \cos^2 \alpha = 1$.
$\cos^2 \alpha = 1 - \frac{16}{25} = \frac{9}{25}$ and $\cos \alpha = \frac{3}{5}$ or $\cos \alpha = -\frac{3}{5}$
Since α is in the second quadrant, $\cos \alpha < 0$.
$\therefore$ choose $\cos \alpha = -\frac{3}{5}$. Answer.

11. $\sin \alpha = -\dfrac{5}{13}$; IV
12. $\sin \alpha = -\dfrac{12}{13}$; III

13. $\cos\alpha = \frac{8}{17}$; I

14. $\cos\alpha = -\frac{24}{25}$; II

15. $\cos\alpha = \frac{1}{3}$; IV

16. $\sin\alpha = -\frac{\sqrt{2}}{2}$; III

17. $\sin\alpha = \frac{\sqrt{3}}{2}$; II

18. $\cos\alpha = \frac{\sqrt{10}}{10}$; I

Find $\sin\alpha$ and $\cos\alpha$ if $R(u, v)$ is the point where the terminal side of α in standard position intersects the unit circle and u and v satisfy the given conditions.

EXAMPLE $u = v$, $u > 0$

SOLUTION Since $R(u, v)$ is on the unit circle, $u = \cos\alpha$ and $v = \sin\alpha$ and you have $u^2 + v^2 = 1$. Replacing v with u,

$$u^2 + u^2 = 1$$
$$2u^2 = 1$$
$$u^2 = \tfrac{1}{2}$$
$$u = \frac{1}{\sqrt{2}} \quad \text{or} \quad u = -\frac{1}{\sqrt{2}}$$

Since $u > 0$, choose the positive value.

$$\therefore \sin\alpha = \frac{1}{\sqrt{2}} \text{ and } \cos\alpha = \frac{1}{\sqrt{2}}.$$

B 19. $u = 2v$, $u < 0$

20. $3u = 4v$, $u > 0$

21. $u = -3v$, $u < 0$

22. $v = -\sqrt{3}u$, $u < 0$

C 23. $v = 2u^2 - 1$, $u > 0$

24. $16v = 15u^2$, $u > 0$

25. Show that if $P(a, b)$ is a point on the terminal side of an angle α in standard position, then its image P' under the linear transformation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is on the x-axis the same distance from the origin as P itself.

14-4 Special Values of Sine and Cosine

We use shortened notation to refer to the values of trigonometric functions for specific angles. The notation "sin 30°" means "the value of the sine of the angle whose measure is 30°." "Cos $\frac{\pi}{4}^{R}$" means "the value of the cosine of the angle whose measure is $\frac{\pi}{4}^{R}$."

In geometry, you learned that if one acute angle of a right triangle measures 30° $\left(\text{or } \frac{\pi}{6}^{\mathrm{R}}\right)$, then the lengths of the sides of the triangle are in the ratio

$$1 : \sqrt{3} : 2,$$

and that in a right triangle with an acute angle measuring 45° $\left(\text{or } \frac{\pi}{4}^{\mathrm{R}}\right)$, the lengths of the sides are in the ratio

$$1 : 1 : \sqrt{2}.$$

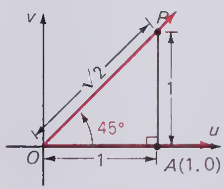
You can use these facts to find $\sin \alpha$ and $\cos \alpha$ when the measure of α is a multiple of 30° $\left(\text{or } \frac{\pi}{6}^{\mathrm{R}}\right)$ or of 45° $\left(\text{or } \frac{\pi}{4}^{\mathrm{R}}\right)$.

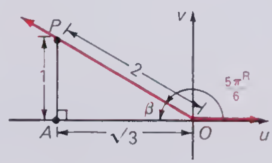
EXAMPLE Find **(a)** $\sin 45°$ and **(b)** $\cos \frac{5\pi}{6}^{\mathrm{R}}$.

SOLUTION In each case, sketch an angle α in standard position with the given measure.

a. Since the measure of the angle is 45°, take point $A(1, 0)$ on the u-axis, and complete the right triangle AOP with dimensions as shown at the left below.

$$\therefore \sin 45° = \frac{v}{r} = \frac{1}{\sqrt{2}}. \quad \text{Answer.}$$





b. Since the measure of the given angle is $\frac{5\pi}{6}^{\mathrm{R}}$, the measure of angle β in the diagram is

$$\left(\pi - \frac{5\pi}{6}\right)^{\mathrm{R}}, \text{ or } \frac{\pi}{6}^{\mathrm{R}}.$$

Take point P at a distance of 2 units from the origin on the terminal side of the given angle, and complete the right triangle AOP with the dimensions as shown at the right above. Since $\cos \alpha$ is negative in the second quadrant,

$$\cos \frac{5\pi}{6}^{\mathrm{R}} = -\frac{\sqrt{3}}{2}. \quad \text{Answer.}$$

For quadrantal angles, that is, angles with measures which are multiples of 90° $\left(\text{or } \frac{\pi}{2}^R\right)$, the definitions of sin α and cos α are sufficient to provide these values by inspection. Thus, you can see from a particular point P on the unit circle (Figure 17) that:

$\cos 0° = 1$	$\cos 90° = 0$	$\cos 180° = -1$	$\cos 270° = 0$
$\sin 0° = 0$	$\sin 90° = 1$	$\sin 180° = 0$	$\sin 270° = -1$

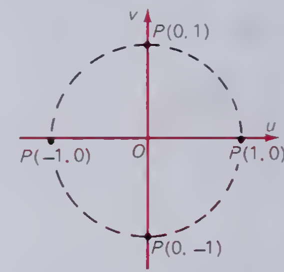


Figure 17

Using these function values for quadrantal angles and procedures such as those supplied in the previous Example, you can construct the table shown below.

You can use this table for circular functions also, since

$$\sin m^R(\alpha) = \sin \alpha, \qquad \cos m^R(\alpha) = \cos \alpha.$$

For example,

$$\text{if } x = \frac{\pi}{3}, \text{ then } \sin x = \frac{\sqrt{3}}{2} \text{ and } \cos x = \frac{1}{2}.$$

On the other hand, for values of x between 0 and 2π,

$$\text{if } \sin x = \frac{1}{\sqrt{2}}, \text{ then } x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}.$$

Measure of α		sin α	cos α	Measure of α		sin α	cos α
0°	0^R	0	1	180°	π^R	0	-1
30°	$\frac{\pi}{6}^R$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	210°	$\frac{7\pi}{6}^R$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
45°	$\frac{\pi}{4}^R$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	225°	$\frac{5\pi}{4}^R$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
60°	$\frac{\pi}{3}^R$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	240°	$\frac{4\pi}{3}^R$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
90°	$\frac{\pi}{2}^R$	1	0	270°	$\frac{3\pi}{2}^R$	-1	0
120°	$\frac{2\pi}{3}^R$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	300°	$\frac{5\pi}{3}^R$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
135°	$\frac{3\pi}{4}^R$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	315°	$\frac{7\pi}{4}^R$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
150°	$\frac{5\pi}{6}^R$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	330°	$\frac{11\pi}{6}^R$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
180°	π^R	0	-1	360°	$2\pi^R$	0	1

Since coterminal angles differ in measure by multiples of 360°, or 2π radians, it follows that for any angle α

$$\sin(\alpha + k \cdot 360°) = \sin\alpha, \qquad \sin(\alpha + 2k\pi^{R}) = \sin\alpha,$$
$$\cos(\alpha + k \cdot 360°) = \cos\alpha, \qquad \cos(\alpha + 2k\pi^{R}) = \cos\alpha.$$

Sine and cosine are examples of *periodic* functions. A function f is **periodic** if there is some nonzero constant p such that $f(x + p) = f(x)$ for all x in the domain of f. p is called a **period** of the function. If there is a smallest positive constant p for which f is periodic, then p is called the **fundamental period** of f. Sine and cosine both have fundamental periods of 360° or $2\pi^{R}$. If their domains are each $\mathcal{R}$, the fundamental period is 2π.

Oral Exercises

Find the given function values.

1. $\sin 60°$
2. $\sin 420°$
3. $\cos -30°$
4. $\sin \frac{5\pi^{R}}{2}$

Written Exercises

Find the given function values.

A

1. $\cos 585°$
2. $\sin(-750°)$
3. $\cos 855°$
4. $\sin 570°$
5. $\sin \frac{11\pi^{R}}{4}$
6. $\sin \frac{7\pi^{R}}{2}$
7. $\cos\left(-\frac{5\pi^{R}}{3}\right)$
8. $\cos\left(-\frac{5\pi^{R}}{6}\right)$
9. $\sin\left(-\frac{7\pi^{R}}{3}\right)$
10. $\cos \frac{17\pi^{R}}{6}$
11. $\cos(-7\pi^{R})$
12. $\sin \frac{17\pi^{R}}{4}$

Find all values of α for which the terminal side of α in standard position passes through the given point, where (a) α is in degrees, and (b) α is in radians.

EXAMPLE $(-5, 5)$

SOLUTION a. Using -5 for u and 5 for v, you have

$$\sqrt{u^2 + v^2} = \sqrt{(-5)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}.$$

Then $\sin\alpha = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $\cos\alpha = \frac{-5}{5\sqrt{2}} = -\frac{1}{\sqrt{2}}$.

From the table on page 516, $\alpha = 135° + k \cdot 360°$. Answer.

b. In radians, $\alpha = \frac{3\pi}{4} + 2k\pi$. Answer.

13. $(\sqrt{3}, 1)$
14. $(4, -4)$
15. $(-2, -2\sqrt{3})$
16. $(7, 7)$
17. $(-\sqrt{6}, -\sqrt{2})$
18. $(-\sqrt{3}, 3)$
19. $(-2\sqrt{3}, -2\sqrt{3})$
20. $(\frac{1}{3}, -\frac{1}{3}\sqrt{3})$

B 21. Show that the line defined by $y = \sqrt{3}x$ makes an angle of $60°$ with the positive x-axis.

C 22. What is the fundamental period of the function $y = \sin nx$ where n is a positive integer?

14-5 Using Tables

In the preceding section, you computed values of sine and cosine for some special angles. Values in general can be computed to any desired accuracy by the methods of advanced mathematics. Some of these values have been collected in the tables given at the back of this book.

Table 6, at the back of the book, gives approximate values of the sine and cosine functions (together with those for other functions that you will study later) for angles whose measures run from $0°$ to $90°$ in multiples of $10'$. (Methods of finding function values for angles outside this range will be discussed in the next section.) The left-hand column, beginning at the top of page 640 and continuing to the bottom of page 644, lists such angle measures from $0°0'$ to $45°0'$. The right-hand column, beginning at the bottom of page 644 and continuing to the top of page 640, lists angle measures from $45°0'$ to $90°0'$. Most of the values are approximate and are given correct to four significant digits. Of course, the values of $\sin 0°$, $\cos 0°$, $\sin 30°$, $\cos 60°$, $\sin 90°$, and $\cos 90°$ are exact.

To find a four-significant-digit approximation for the value of $\sin \alpha$ or $\cos \alpha$ for an angle α whose measure is listed in Table 6:

1. a. If $0° \le m(\alpha) \le 45°$, find $m(\alpha)$ in the left-hand column, and read across the top to identify the column containing values of the specified function.
   b. If $45° \le m(\alpha) \le 90°$, find $m(\alpha)$ in the right-hand column, and read across the bottom to identify the column containing values of the specified function.
2. The intersection of the row containing the angle measure and the column for the specified function contains the desired value.

EXAMPLE 1 Find $\sin 61°10'$.

SOLUTION In the right-hand column of Table 6 you find "$61°10'$." In the row opposite this, *above* the label $\sin \alpha$ at the bottom of the page, you find "0.8760." $\therefore \sin 61°10' \approx 0.8760$. Answer.

If you wish to find the approximate value of a given function for an angle measure that is not a multiple of $10'$, you may use the process of *linear interpolation* as you did for logarithms on page 400. In so doing,

you must be careful to take into account whether the values of the given function increase or decrease when $m(\alpha)$ increases from $0°$ to $90°$.

EXAMPLE 2 Find $\cos 37°18'$.

SOLUTION $\cos 37°10' > \cos 37°18' > \cos 37°20'$. Using a convenient vertical arrangement for the linear interpolation, you can write:

$$
\begin{array}{r|l}
m(\alpha) & \cos\alpha \\
\hline
10'\left[\begin{array}{l} 8'\left[\begin{array}{l} 37°10' \\ 37°18' \end{array}\right. \\ \quad 37°20' \end{array}\right. &
\left.\begin{array}{l} \left.\begin{array}{r} 0.7969 \\ \underline{\;?\;} \end{array}\right] d \\ 0.7951 \end{array}\right] -0.0018 \left\{\begin{array}{l}\text{A negative number}\\ \text{because}\\ \cos 37°20' < \cos 37°10'\end{array}\right.
\end{array}
$$

$$\frac{8}{10} \approx \frac{d}{-0.0018}; \quad d \approx \frac{8}{10}(-0.0018) \approx -0.0014$$

$\cos 37°18' \approx 0.7969 + (-0.0014)$, or 0.7955. Answer.

You can also use Table 6 to approximate to the nearest minute the measure of an angle α with measure between $0°$ and $90°$ when you are given the value for a trigonometric function of α.

EXAMPLE 3 Find $m(\alpha)$, where $0° \leq m(\alpha) \leq 90°$, if $\sin\alpha = 0.3891$.

SOLUTION You first locate the nearest values given for $\sin\alpha$ that are above and below 0.3891. Then you arrange the values as follows:

$$
\begin{array}{r|l}
m(\alpha) & \sin\alpha \\
\hline
10'\left[\begin{array}{l} d\left[\begin{array}{l} 22°50' \\ \underline{\;?\;} \end{array}\right. \\ \quad 23°00' \end{array}\right. &
\left.\begin{array}{l} \left.\begin{array}{r} 0.3881 \\ 0.3891 \end{array}\right] 0.0010 \\ 0.3907 \end{array}\right] 0.0026 \left\{\begin{array}{l}\text{Positive numbers}\\ \text{because}\\ \sin 23°0' > \sin 22°50'\end{array}\right.
\end{array}
$$

$$\frac{d}{10} \approx \frac{0.0010}{0.0026} = \frac{10}{26}; \; d \approx \frac{10}{26}(10) \approx 4$$

$m(\alpha) \approx 22°50' + 4'$, or $22°54'$. Answer.

Table 7 gives values of circular functions for $0 \leq x \leq \frac{\pi}{2}$, together with values of trigonometric functions for angles such that $0 \leq m^{\mathrm{R}}(\alpha) \leq \frac{\pi}{2}$ at intervals of 0.01.

EXAMPLE 4 Find $\cos 0.42^{\mathrm{R}}$.

SOLUTION Locate "0.42" in the left-hand column on page 645. In the row opposite this, below the label "$\cos x$ or $\cos\alpha$," you find "0.9131."

$\therefore \cos 0.42^{\mathrm{R}} \approx 0.9131$. Answer.

By using linear interpolation in Table 7, you can find approximate function values for x (or $m^R(\alpha)$) given as multiples of 0.001.

Oral Exercises

State each function value from Tables 6 and 7 at the back of the book.

1. $\sin 15°30'$
2. $\cos 24°10'$
3. $\cos 56°50'$
4. $\sin 73°40'$
5. $\cos 42°$
6. $\sin 42°20'$
7. $\sin 0.63^R$
8. $\cos 1.25^R$
9. $\sin 1.08^R$
10. $\cos 1^R$
11. $\sin 1.46^R$
12. $\cos 0.07^R$

Use Table 6 to state the value of α in degrees and minutes for which the given function has the given value.

13. $\sin \alpha = 0.1937$
14. $\cos \alpha = 0.7735$
15. $\sin \alpha = 0.8746$
16. $\cos \alpha = 0.2334$
17. $\sin \alpha = 0.9996$
18. $\cos \alpha = 0.6494$

Written Exercises

In Exercises 1–24 use Tables 6 and 7 at the back of the book and linear interpolation as necessary. Find a four-significant-digit approximation of the given function value.

A

1. $\cos 28°15'$
2. $\sin 3°37'$
3. $\sin 29°26'$
4. $\cos 45°59'$
5. $\sin 40°3'$
6. $\sin 68°21'$
7. $\cos 58°18'$
8. $\cos 87°42'$
9. $\sin 1.057^R$
10. $\cos 0.844^R$
11. $\cos 1.429^R$
12. $\sin 1.516^R$

Find the measure of α in degrees and minutes, to the nearest minute, for the first-quadrant angle with the given function value.

13. $\sin \alpha = 0.5040$
14. $\sin \alpha = 0.3807$
15. $\cos \alpha = 0.9670$
16. $\cos \alpha = 0.6663$
17. $\sin \alpha = 0.8799$
18. $\cos \alpha = 0.5609$

Find the measure of x in radians, to the nearest thousandth of a radian, for the first-quadrant angle with the given function value.

19. $\sin x = 0.5826$
20. $\sin x = 0.9377$
21. $\cos x = 0.9529$
22. $\cos x = 0.4740$
23. $\sin x = 0.1663$
24. $\cos x = 0.0498$

Find the first-quadrant angle (to the nearest ten minutes) whose terminal side in standard position passes through the given point.

B

25. $(3, 4)$
26. $(12, 5)$
27. $(8, 15)$
28. $(21, 20)$
29. $(1, 2\sqrt{2})$
30. $(\sqrt{5}, 2)$
31. $(3\sqrt{5}, 2)$
32. $(2, 2\sqrt{3})$

C

33. For what acute angle α does $\sin \alpha = 3 \cos \alpha$?

14-6 Reference Angles and Arcs

We shall now discuss how to find values for the sine and cosine functions for angles with measures outside the range

$$0° \leq m(\alpha) \leq 90°.$$

First consider an angle in each of Quadrants II, III, and IV, as in Figure 18. Let $P(a, b)$ be any point other than the origin on the terminal side of α. Let T be the point with coordinates $(|a|, |b|)$; thus, T lies in the first quadrant. Then let θ (Greek *theta*) be the angle whose terminal side contains T.

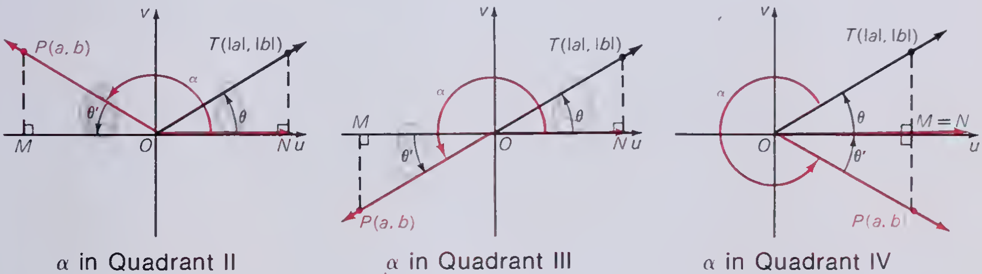


Figure 18

Now, notice that for α in Quadrant II, right triangles OMP and ONT are congruent and $\theta' \cong \theta$. Thus, the v-coordinates of P and T are equal, but their u-coordinates are negatives of each other; that is, $a = -|a|$. This means that, by definition,

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}} = \frac{|b|}{\sqrt{|a|^2 + |b|^2}} = \sin \theta;$$

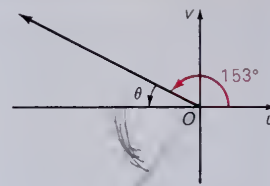
$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}} = \frac{-|a|}{\sqrt{|a|^2 + |b|^2}} = -\cos \theta.$$

By observing similar relationships between a and $|a|$ and b and $|b|$ in each quadrant, and noting that $\sqrt{a^2 + b^2} = \sqrt{|a|^2 + |b|^2}$, you can see that the sines and cosines of α and θ are related as shown in the table below. Either θ or θ' is called a **reference angle** for α, because you can use it to determine values for trigonometric functions of α. The *reference angle* will hereafter be denoted by θ.

Function value	Quadrant in which α lies			
	I	II	III	IV
$\sin \alpha$	$\sin \theta$	$\sin \theta$	$-\sin \theta$	$-\sin \theta$
$\cos \alpha$	$\cos \theta$	$-\cos \theta$	$-\cos \theta$	$\cos \theta$

EXAMPLE Find an approximation for **(a)** cos 153° and **(b)** sin 312°.

SOLUTION Use a sketch to help you picture the angle.



reference angle always positive

a. From the sketch you see that

$$m°(\theta) = 180 - m°(\alpha) = 180 - 153 = 27.$$

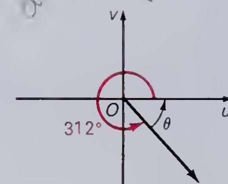
From the table on page 521,

$$\cos 153° = -\cos 27°.$$

Then, from Table 6, $\cos 27° = 0.8910$.

$\therefore \cos 153° = -0.8910$. Answer.

b. From the sketch you can see that



$$m°(\theta) = 360 - m°(\alpha) = 360 - 312 = 48.$$

From the table on page 521,

$$\sin 312° = -\sin 48°.$$

Then, from Table 6, $\sin 48° = 0.7431$.

$\therefore \sin 312° = -0.7431$. Answer.

For angles with measures outside the range $0° \leq m(\alpha) \leq 360°$, you use the periodic properties of the trigonometric functions (page 517) to find first an equivalent function value for an angle with measure in the range $0° \leq m(\alpha) < 360°$, and then use an appropriate reference angle. For example, to find sin 672°, you would first write

$$\sin 672° = \sin (672° - 360°) = \sin 312°,$$

and then proceed as in Example b, above.

When angle measures are given in radians, measures for reference angles can be approximated by using

$$\frac{\pi}{2} \approx 1.57,\ \pi \approx 3.14,\ \frac{3\pi}{2} \approx 4.71,\ \text{and } 2\pi \approx 6.28.$$

When finding values of circular functions, you may think of the real number x as the measure of an arc on the unit circle, and use a *reference arc* with measure x', as shown in Figure 19. For example,

$$\begin{aligned} \cos 4 &= -\cos (4 - \pi) \\ &\approx -\cos (4.00 - 3.14) \\ &\approx -\cos 0.86. \end{aligned}$$

Then, from Table 7,

$$\cos 0.86 \approx 0.6524.$$
$$\therefore \cos 4 \approx -0.6524.$$

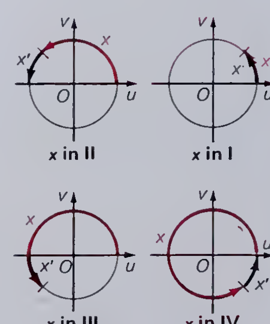


Figure 19

Oral Exercises

State the measure of the reference angle θ you would use to evaluate the given function.

1. $\sin 110°$ 2. $\cos 233°$ 3. $\cos 345°$ 4. $\sin 254°$

5. $\cos 207°$ 6. $\sin 168°$ 7. $\sin 291°10'$ 8. $\cos 212°40'$

9. $\cos 4.26^R$ 10. $\sin 5.89^R$ 11. $\cos 1.93^R$ 12. $\sin 2.76^R$

1–12

Written Exercises

A **1–12.** For each of the function values in Oral Exercises 1–12, **(a)** find the reference angle of the given angle, **(b)** make a sketch showing the given angle (labeled α) and the reference angle (labeled θ), and **(c)** evaluate the function using Tables 6 and 7.

Evaluate the function using Tables 6 and 7.

13. $\sin 138°40'$ **14.** $\sin 317°50'$ **15.** $\cos 224°10'$ **16.** $\cos 119°30'$

17. $\sin(-132°)$ **18.** $\cos(-245°)$ **19.** $\cos 381°$ **20.** $\sin 673°$

21. $\cos 816°$ **22.** $\sin 536°$ **23.** $\sin 955°$ **24.** $\cos 1014°$

25. $\cos(-3.5)^R$ **26.** $\sin(-2.6)^R$ **27.** $\sin 8.5^R$ **28.** $\cos 11.7^R$

In Exercises 29–36 use Table 6 to find the measure of α in degrees and minutes (to the nearest 10′) such that α satisfies the given conditions and $0° \le m(\alpha) < 360°$.

29. $\sin\alpha = -0.2924;\ \cos\alpha < 0$ **30.** $\cos\alpha = 0.9775;\ \sin\alpha < 0$

31. $\cos\alpha = -0.3201;\ \sin\alpha < 0$ **32.** $\sin\alpha = 0.6884;\ \cos\alpha < 0$

33. $\cos\alpha = -0.9426;\ \sin\alpha > 0$ **34.** $\sin\alpha = -0.7934;\ \cos\alpha < 0$

35. $\sin\alpha = 0.8542;\ \cos\alpha < 0$ **36.** $\cos\alpha = -0.0756;\ \sin\alpha < 0$

Find the measure of the angle α in degrees and minutes (to the nearest 10′) such that the terminal side of α in standard position passes through the given point and such that $0° \le \alpha < 360°$.

B **37.** $(-3, -4)$ **38.** $(12, -5)$ **39.** $(-8, 15)$ **40.** $(24, -7)$

41. $(-2, \sqrt{5})$ **42.** $(-21, -20)$ **43.** $(2\sqrt{10}, -4)$ **44.** $(-5, 2\sqrt{6})$

Follow the directions for Exercises 37–44 above but find α in radians (to the nearest hundredth of a radian) such that $0^R \le \alpha < 2\pi^R$.

C **45.** $(4, -3)$ **46.** $(-3, \sqrt{7})$ **47.** $(-1, -2\sqrt{6})$ **48.** $(5, -\sqrt{11})$

14-7 Graphs of Sine and Cosine I

The sine and cosine circular functions, when specified as sets of ordered pairs of real numbers, are represented respectively by

$$\text{sine} = \{(x, y)\colon\ y = \sin x\}$$

and

$$\text{cosine} = \{(x, y)\colon\ y = \cos x\}.$$

Each has a domain $\mathcal{R}$, and as range the set of real numbers $\{y\colon |y| \leq 1\}$. Therefore, each can be graphed in the coordinate plane. Because both functions are periodic with fundamental period 2π, we need to determine their graphs only over the interval $0 \leq x \leq 2\pi$; the pattern over this interval then repeats endlessly in both directions along the x-axis.

From the table on page 516, you can determine the ordered pairs in

$$\{(x, y)\colon\ y = \sin x\}$$

that have multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ as first components.

When all such ordered pairs with first components in the interval $0 \leq x \leq 2\pi$ are graphed, you obtain Figure 20. Assuming that the graph of sine is a smooth unbroken curve (as it is), you can connect the points shown in Figure 20 to produce the graph shown in Figure 21, which represents one fundamental period.

Figure 20

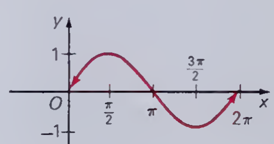


Figure 21

Now you need only to duplicate the pattern shown in Figure 21 over successive intervals of length 2π along the x-axis to obtain as much of the graph of the sine function as you wish. Figure 22 pictures the graph over $-4\pi \leq x \leq 4\pi$. Do you see why this graph is called a **sine wave?**

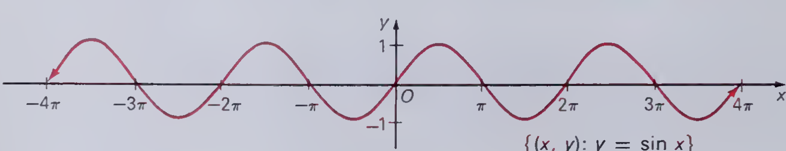


Figure 22

To graph

$$\{(x, y)\colon\ y = \cos x\},$$

you can use the same procedure as for the sine function. Graphing points for ordered pairs from the table on page 516, you obtain the

pattern shown in Figure 23, and, upon connecting the points, you have the graph of one fundamental period of the cosine function (Figure 24).

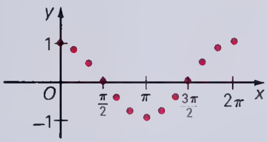


Figure 23

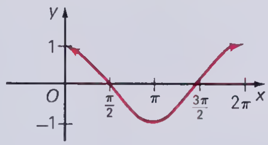


Figure 24

Drawing three more periods produces the graph of cosine over $-4\pi \leq x \leq 4\pi$, as shown in Figure 25. This graph is also an example of a sine wave, but one displaced along the x-axis $\frac{\pi}{2}$ to the left.

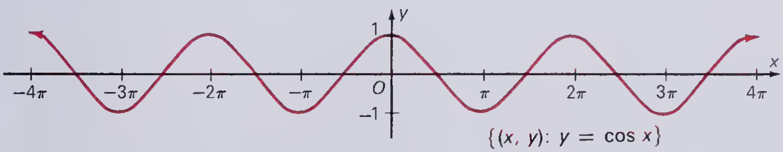


Figure 25

Notice that both curves (Figures 22 and 25) lie between the graphs of the *maximum* ordinate, 1, and the *minimum* ordinate, -1. When a periodic function attains a maximum value M and a minimum value m, you say that the function has an **amplitude** of

$$\frac{M - m}{2}.$$

Thus, the amplitude of sine and cosine is $\frac{1 - (-1)}{2}$, or 1.

Notice that the maximum ordinate in the graph of $y = \sin x$ occurs at $\frac{1}{4}$ of the distance across the fundamental periodic interval, $0 \leq x \leq 2\pi$ (see Figure 21), that is,

$$\sin \frac{\pi}{2} = 1,$$

and the minimum ordinate occurs at $\frac{3}{4}$ of the distance across that interval, that is,

$$\sin \frac{3\pi}{2} = -1.$$

Correspondingly, the maximum ordinates in the graph of $y = \cos x$ occur at the beginning and the end of the interval (see Figure 24), that is,

$$\cos 0 = 1 \qquad \text{and} \qquad \cos 2\pi = 1,$$

and the minimum at the midpoint of the interval, that is,

$$\cos \pi = -1.$$

Next consider

$$\{(x, y)\colon y = 2 \cos x\} \qquad \text{and} \qquad \{(x, y)\colon y = -2 \sin x\}.$$

Notice that the y-component of each solution of $y = 2 \cos x$ is 2 times the y-component of the corresponding solution of $y = \cos x$. Similarly, the y-component of $y = -2 \sin x$ is -2 times the y-component of the corresponding solution of $y = \sin x$. Their graphs are shown in Figures 26 and 27, respectively. The fundamental period of each of these functions is 2π, and the amplitudes are each 2. Notice also that the graph of

$$\{(x, y)\colon y = -2 \sin x\}$$

is a *reflection* with respect to the x-axis of the graph that would be drawn for

$$\{(x, y)\colon y = 2 \sin x\}.$$

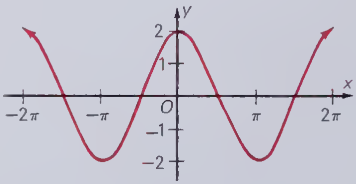


Figure 26

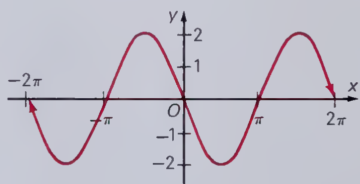


Figure 27

Hence, for $J = \{\text{the integers}\}$, in Figure 27

the *maximum* ordinates occur at $\sin\left(\frac{3\pi}{2} + 2k\pi\right) = 2,\ k \in J,$ $\frac{\pi}{2}$ 0, 1

and the *minimum* ordinates at $\sin\left(\frac{\pi}{2} + 2k\pi\right) = -2,\ k \in J.$ π −1, 0 2π 1, 0

Similarly, in Figure 26

the *maximum* ordinates occur at $\cos(2k\pi) = 2,\ k \in J,$ $\frac{3\pi}{}$
and the *minimum* ordinates at $\cos(\pi + 2k\pi) = -2,\ k \in J.$

0, −1

In general, the functions

$$\{(x, y)\colon y = A \sin x\} \qquad \text{and} \qquad \{(x, y)\colon y = A \cos x\},\ A \neq 0,$$

have fundamental period 2π and amplitude $|A|$.

Oral Exercises

State the amplitude of each function.

1. $y = 3 \sin x$
2. $y = 4 \cos x$
3. $y = \frac{1}{2} \sin x$
4. $y = \frac{3}{2} \cos x$
5. $y = -\cos x$
6. $y = -3 \sin x$

Written Exercises

Sketch the graph of each function on the interval $-2\pi \leq x \leq 2\pi$.

A **1–6.** Sketch the graphs of the functions in Oral Exercises 1–6.

7. $y = -\frac{2}{3}\sin x$ **8.** $y = -\frac{4}{3}\cos x$ **9.** $y = \frac{5}{2}\cos x$

In Exercises 10–13, give the maximum and minimum values of the function and state the amplitude.

10. $y = 2\sin x + 1$ **11.** $y = 3\cos x - 2$

12. $y = \frac{1}{2}\cos x + 3$ **13.** $y = -4\sin x + 1$

B **14–17.** Graph the functions in Exercises 10–13 above on the interval $-2\pi \leq x \leq 2\pi$.

C **18.** Graph the function $y = \sin\left(x + \frac{\pi}{2}\right)$ over the interval $-2\pi \leq x \leq 2\pi$. Can you recognize this graph as the graph of an equivalent function?

14-8 Graphs of Sine and Cosine II

In Section 14-7 you saw that the graphs of

$$\{(x, y)\}: y = A\sin x\} \qquad \text{and} \qquad \{(x, y): y = A\cos x\}$$

were sine waves with amplitude $|A|$ and period 2π.

Next consider

$$\{(x, y): y = \sin 2x\}.$$

Notice that as x varies from 0 to π, $2x$ varies from 0 to 2π. Hence, $\sin 2x$ will assume all the values of sine while in the interval $0 \leq x \leq \pi$. Thus, the fundamental period of this function is π rather than 2π. The graph is shown in Figure 28.

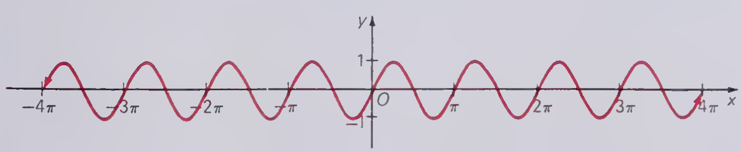


Figure 28

In general, the functions

$$\{(x, y): y = \sin Bx\} \qquad \text{and} \qquad \{(x, y): y = \cos Bx\},\ B \neq 0$$

have fundamental period $\frac{2\pi}{|B|}$ and amplitude 1.

Combining these results with those in Section 14-7, you have the result at the top of page 528.

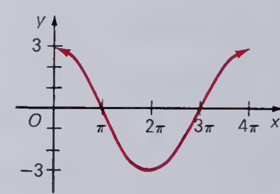
> The functions
>
> $$\{(x, y)\colon\ y = A \sin Bx\} \qquad (1)$$
> $$\{(x, y)\colon\ y = A \cos Bx\} \qquad (2)$$
>
> each have amplitude $|A|$ and fundamental period $\frac{2\pi}{|B|}$. Their graphs are sine waves.

These facts can be used to make a rapid sketch of such a graph.

EXAMPLE Sketch the graph of $\{(x, y)\colon\ y = 3 \cos \frac{1}{2}x\}$ over one fundamental period.

SOLUTION The amplitude is 3 and the fundamental period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. Then over the interval $0 \le x \le 4\pi$:

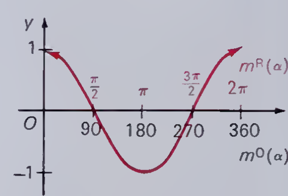
1. The maximum points are $(0, 3)$ and $(4\pi, 3)$.
2. The minimum point occurs when $x = \frac{1}{2} \times 4\pi = 2\pi$; it is $(2\pi, -3)$.
3. The intercepts are midway between the turning points, i.e., at $(\pi, 0)$ and $(3\pi, 0)$. These facts yield the curve shown at the right. Answer.



To graph the trigonometric functions sine and cosine, each having as domain the set of all angles, we represent each angle by its measure. If the angles are measured in radians, the graphs are the same as those of the associated circular functions. If the angles are measured in the degree system, the scale on the horizontal axis must be labeled accordingly. At the right is the graph of one fundamental period of

$$\{(\alpha, y)\colon\ y = \cos \alpha\},$$

where $m^R(\alpha)$ and $m^\circ(\alpha)$ are shown along the horizontal axis in red and black, respectively.



$\{(\alpha, y)\colon\ y = \cos \alpha\}$

Oral Exercises

State the amplitude $|A|$ and the fundamental period p of the function.

1. $y = \cos 2x$
2. $y = \sin 3x$
3. $y = 2 \sin \frac{1}{2}x$
4. $y = 3 \cos 2x$
5. $y = \frac{3}{2} \cos \frac{1}{3}x$
6. $y = -2 \sin 2x$

Written Exercises

Sketch the graph of the given function over the interval $0 \leq x \leq 2\pi$ if its fundamental period $p \leq 2\pi$. Otherwise, graph the function over the interval $0 \leq x \leq p$.

A **1–6.** Graph the functions in Oral Exercises 1–6.

7. $y = -\frac{1}{2}\cos 4x$ **8.** $y = 2\sin\frac{2}{3}x$ **9.** $y = 3\sin\frac{3}{2}x$

10. $y = 2\cos(-3x)$ **11.** $y = 3\sin(-2x)$ **12.** $y = \sin\pi x$

Graph each pair of functions on one set of axes over the given interval.

B **13.** $y = 3\sin 2x$; $y = \frac{1}{2}\sin 2x$; $0 \leq x \leq 2\pi$

14. $y = 4\sin\frac{1}{2}x$; $y = 4\sin\frac{1}{3}x$; $0 \leq x \leq 6\pi$

529
1-13 odd

15. $y = \cos\pi x$; $y = \cos\frac{\pi}{2}x$; $0 \leq x \leq 4$

16. $y = 2\cos 2x$; $y = 2\cos 3x$; $0 \leq x \leq 2\pi$

C **17.** $y = \sin 3x$; $y = 2 + \sin 3x$; $0 \leq x \leq 2\pi$

18. $y = \cos 2x$; $y = -1 + \cos 2x$; $0 \leq x \leq 2\pi$

19. $y = \sin\frac{1}{2}x$; $y = \sin\left(\frac{1}{2}x + \frac{\pi}{2}\right)$; $0 \leq x \leq 4\pi$

20. $y = \sin 4x$; $y = |\sin 4x|$; $0 \leq x \leq 2\pi$

Self-Test 2

VOCABULARY

sine of an angle (p. 510)
cosine of an angle (p. 510)
trigonometric function (p. 511)
circular function (p. 513)
periodic function (p. 517)
fundamental period (p. 517)
reference angle (p. 521)
amplitude (p. 525)

1. Find $\sin\alpha$ and $\cos\alpha$ if the terminal side of angle α in standard position passes through the point $(-7, 4\sqrt{2})$. *Obj. 1, p. 510*
2. Find $\cos\alpha$ if $\sin\alpha = -\frac{24}{25}$ and the terminal side of angle α is in quadrant IV. *Obj. 2, p. 510*
3. Find **(a)** $\sin 495°$, and **(b)** $\cos\frac{13\pi^R}{3}$. *Obj. 3, p. 510*
4. Use Tables 6 and 7 at the back of the book to find:

   a. $\sin 69°28'$ **b.** $\cos 0.652^R$ **c.** $\sin 236°10'$ **d.** $\cos 5.14^R$
5. Graph $y = 2\cos\frac{3}{2}x$ for $0 \leq x \leq 2\pi$. *Obj. 4, p. 510*

Check your answers with those at the back of the book.

Other Trigonometric and Circular Functions and Applications

***OBJECTIVES** for Sections 14-9 and 14-10:*
1. Find values for sec x, tan x, csc x, and cot x for certain specified values of x.
2. Solve and apply solutions of right triangles.

14-9 The Tangent, Cotangent, Secant, and Cosecant Functions

Several other trigonometric functions in common use are defined in terms of sine and cosine as follows:

tangent: $\alpha \to \dfrac{\sin \alpha}{\cos \alpha}$, $\cos \alpha \neq 0$

secant: $\alpha \to \dfrac{1}{\cos \alpha}$, $\cos \alpha \neq 0$

cotangent: $\alpha \to \dfrac{\cos \alpha}{\sin \alpha}$, $\sin \alpha \neq 0$

cosecant: $\alpha \to \dfrac{1}{\sin \alpha}$, $\sin \alpha \neq 0$

Values of these functions are denoted by:

tan α (read "tangent of α")
cot α (read "cotangent of α")
sec α (read "secant of α")
csc α (read "cosecant of α")

Notice that tangent and secant are not defined when $m(\alpha)$ is 90°, 270°, etc. $\left(or\ \frac{\pi}{2}^{R}, \text{etc.}\right)$, and cotangent and cosecant are not defined when $m(\alpha)$ is 0°, 180°, etc. (or π^R, etc.).

EXAMPLE If $\sin \alpha = \frac{3}{5}$ and $\cos \alpha = \frac{4}{5}$, find:

a. $\tan \alpha$ b. $\cot \alpha$ c. $\sec \alpha$ d. $\csc \alpha$

SOLUTION a. $\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha} = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{4}$.

b. $\cot \alpha = \dfrac{\cos \alpha}{\sin \alpha} = \dfrac{\frac{4}{5}}{\frac{3}{5}} = \dfrac{4}{3}$.

c. $\sec \alpha = \dfrac{1}{\cos \alpha} = \dfrac{1}{\frac{4}{5}} = \dfrac{5}{4}$.

d. $\csc \alpha = \dfrac{1}{\sin \alpha} = \dfrac{1}{\frac{3}{5}} = \dfrac{5}{3}$.

By using the definitions of sine and cosine on page 510,

$$\text{sine: } \alpha \to \frac{v}{r}, \quad \text{cosine: } \alpha \to \frac{u}{r},$$

you can verify the theorem stated at the top of the following page.

Theorem. If α is an angle in standard position, with $P(u, v)$ any point other than the origin on the terminal side of α, and if $\sqrt{u^2 + v^2} = r$, then:

tangent: $\alpha \to \dfrac{v}{u}$, $u \neq 0$ secant: $\alpha \to \dfrac{r}{u}$, $u \neq 0$

cotangent: $\alpha \to \dfrac{u}{v}$, $v \neq 0$ cosecant: $\alpha \to \dfrac{r}{v}$, $v \neq 0$

Do you see why tangent and cotangent are called **reciprocal functions?** Sine and cosecant are also reciprocal functions, as are cosine and secant.

Although the theorem above is stated for angles in standard position, the domain of each of the functions is the set of all angles, since any angle can be put into standard position.

We can now extend Figure 14 as shown in Figure 29.

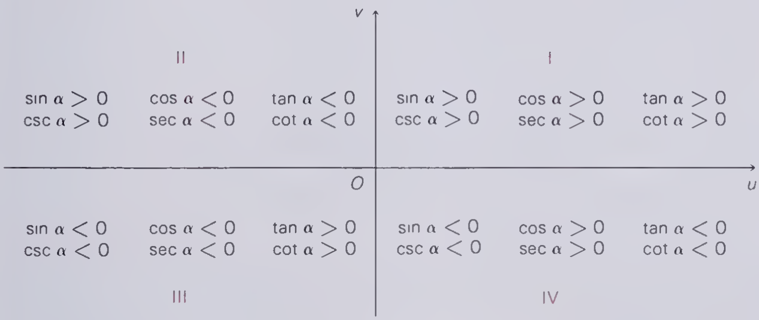


Figure 29

Circular functions tangent, cotangent, secant, and cosecant are defined in terms of arc length on the unit circle (Figure 30); since $\sin x = v$ and $\cos x = u$;

$$\tan x = \frac{v}{u},\ u \neq 0 \qquad \sec x = \frac{1}{u},\ u \neq 0$$

$$\cot x = \frac{u}{v},\ v \neq 0 \qquad \csc x = \frac{1}{v},\ v \neq 0$$

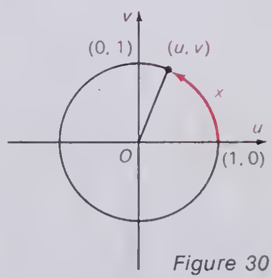


Figure 30

Special values for $\tan \alpha$, $\cot \alpha$, $\sec \alpha$, and $\csc \alpha$ can be computed from the values for $\sin \alpha$ and $\cos \alpha$ given in the table on page 516. You can use these values in sketching graphs of these functions, as shown in Figures 31–34 on page 532.

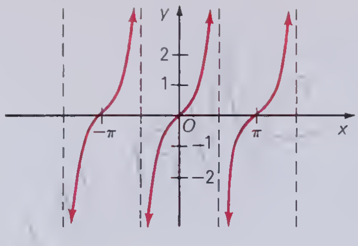


Figure 31 $\{(x, y): y = \tan x\}$

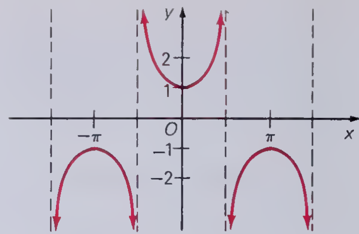


Figure 32 $\{(x, y): y = \sec x\}$

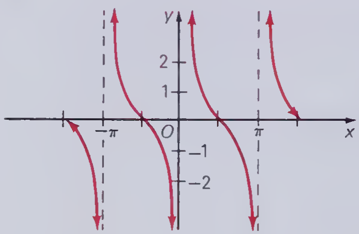


Figure 33 $\{(x, y): y = \cot x\}$

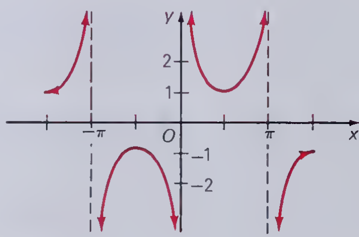


Figure 34 $\{(x, y): y = \csc x\}$

Compare the graph of Figure 31 with the graph of the reciprocal function in Figure 32. Also, compare Figure 33 with Figure 25 and Figure 34 with Figure 22.

We can now summarize the properties of the six circular functions that we have discussed ($J = \{\text{the integers}\}$).

sine: Domain: $\mathcal{R}$
Range: $\{y: y \in \mathcal{R}, |y| \leq 1\}$
Fundamental period: 2π

cosine: Domain: $\mathcal{R}$
Range: $\{y: y \in \mathcal{R}, |y| \leq 1\}$
Fundamental period: 2π

tangent: Domain: $\left\{x: x \in \mathcal{R}, x \neq \dfrac{(2k + 1)\pi}{2}, k \in J\right\}$
Range: $\mathcal{R}$
Fundamental period: π

cotangent: Domain: $\{x: x \in \mathcal{R}, x \neq k\pi, k \in J\}$
Range: $\mathcal{R}$
Fundamental period: π

secant: Domain: $\left\{x: x \in \mathcal{R}, x \neq \dfrac{(2k + 1)\pi}{2}, k \in J\right\}$
Range: $\{y: y \in \mathcal{R}, |y| \geq 1\}$
Fundamental period: 2π

cosecant: Domain: $\{x: x \in \mathcal{R}, x \neq k\pi, k \in J\}$
Range: $\{y: y \in \mathcal{R}, |y| \geq 1\}$
Fundamental period: 2π

Notice that the fundamental period of tangent and cotangent is π rather than 2π, and that each of the latter four functions has certain real numbers excluded from its domain.

Furthermore, as x increases from 0 to $\frac{\pi}{2}$, the function values vary as follows:

$\sin x$ increases, $\tan x$ increases, $\sec x$ increases,
$\cos x$ decreases, $\cot x$ decreases, $\csc x$ decreases.

Values of tangent, cotangent, secant, and cosecant can be found from Tables 6 and 7 at the back of the book.

EXAMPLE Find x where $0 \leq x \leq \frac{\pi}{2}$, if $\tan x \approx 1.305$.

SOLUTION Use Table 7.

$$\begin{array}{c|c} x & \tan x \\ \hline 0.010\left[\begin{array}{l} n\left[\begin{array}{c}0.910\\ ?\end{array}\right. \\ \quad\ 0.920\end{array}\right. & \left.\begin{array}{r}\left.\begin{array}{c}1.286\\ 1.305\end{array}\right]0.019\\ 1.313\qquad\ \ \end{array}\right]0.027 \left\{\begin{array}{l}\text{A positive number}\\ \text{because}\\ \tan 0.920 > \tan 0.910\end{array}\right. \end{array}$$

$$\frac{n}{0.010} \approx \frac{0.019}{0.027}, \quad n \approx 0.007$$

$\therefore x \approx 0.910 + 0.007 = 0.917$. Answer.

Oral Exercises

Use Figure 29 to predict whether these will be positive, negative, or zero.

1. $\tan 127°$ **2.** $\sec 314°$ **3.** $\csc 209°10'$ **4.** $\sec 113°40'$
5. $\cot 256°48'$ **6.** $\csc 298°4'$ **7.** $\sec 2.52^R$ **8.** $\csc 3.79^R$

Written Exercises

Use Tables 6 and 7 and linear interpolation as needed to find the given values to four significant digits. Make a sketch showing each angle in standard position.

A **1–8.** Find the values of the functions in Oral Exercises 1–8.

9. $\tan 5.83^R$ **10.** $\csc 6.01^R$ **11.** $\cot 4^R$ **12.** $\sec 0.733^R$

Use Table 6 and linear interpolation as needed to find α in degrees and minutes satisfying the given conditions, $0 \leq \alpha \leq 90°$.

13. $\tan \alpha = 0.3378$ **14.** $\sec \alpha = 1.853$ **15.** $\csc \alpha = 1.290$
16. $\sec \alpha = 1.261$ **17.** $\tan \alpha = 10.00$ **18.** $\cot \alpha = 4.470$

Use Table 7 and linear interpolation as needed to find α in radians to the nearest 0.001 radian satisfying the given conditions, $0 \le \alpha \le \frac{\pi}{2}$.

19. $\tan \alpha = 0.8423$
20. $\sec \alpha = 2.142$
21. $\csc \alpha = 1.844$
22. $\cot \alpha = 0.5059$
23. $\csc \alpha = 2.027$
24. $\tan \alpha = 0.1247$

Find the values of the six trigonometric functions of the angle α whose terminal side in standard position passes through the given point. If the function is not defined for the angle, so state.

25. $(8, 6)$
26. $(-12, 5)$
27. $(0, -2)$
28. $(-24, -7)$
29. $(-3, 0)$
30. $(2, -3)$
31. $(1, 2\sqrt{6})$
32. $(-6, \sqrt{13})$

Find the values of the other five trigonometric functions of the angle α in the given quadrant and having the given function value.

33. IV: $\sin \alpha = -\frac{1}{2}$
34. III: $\cos \alpha = -\frac{1}{\sqrt{2}}$
35. II: $\sin \alpha = \frac{2}{3}$
36. III: $\sec \alpha = -3$
37. II: $\csc \alpha = \frac{4}{\sqrt{7}}$
38. IV: $\cos \alpha = \frac{1}{\sqrt{13}}$

B 39. III: $\tan \alpha = \frac{2}{5}$
40. II: $\cot \alpha = -\frac{\sqrt{11}}{5}$

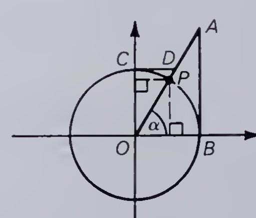
(*Hint:* In Exercises 39 and 40, let $u = \sin \alpha$ and $v = \cos \alpha$ and solve two simultaneous equations involving u and v.)

41. Use the theorem on page 531 to prove that for any angle α: $1 + \tan^2 \alpha = \sec^2 \alpha$.
42. Use the theorem on page 531 to prove that for any angle α: $1 + \cot^2 \alpha = \csc^2 \alpha$.

Graph each of the following functions over the given interval.

43. $y = \tan \frac{1}{2}x \ (-\pi < x < \pi)$
44. $y = 2 \csc x \ (-\pi < x < \pi)$
45. $y = -\tan 2x \ (-\pi < x < \pi)$
46. $y = -2 \sec x \ (0 < x < 2\pi)$
47. $y = \frac{1}{2} \csc 2x \ (0 < x < \pi)$
48. $y = 3 \cot 2x \ (0 < x < \pi)$

Use the diagram at the right, in which $P(u, v)$ is a point on the unit circle O, and $\overline{AB}$ and $\overline{CD}$ are tangents, to prove each of the following.



C 49. $AB = \tan \alpha$
50. $AO = \sec \alpha$
51. $CD = \cot \alpha$
52. $OD = \csc \alpha$

(*Hint:* In Exercises 49–52, use similar triangles.)

programming in BASIC

BASIC provides the function TAN(X) as well as SIN(X) and COS(X). It also provides the function ATN(X), which will give the measurement in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ of the angle whose tangent is X. For example, you can use this program to check the Example on page 533.

```
10  PRINT "IF TAN X = ";
20  INPUT T
30  PRINT "THEN X(R) = ";ATN(T)
40  END
```

Exercises

1. To see how ATN(X) works, try the program given above for values of tan x:

   0.6249 0.8693 1.150 2.145

   Round your results to 4 decimal places, and compare them with the corresponding values in Table 6.
2. Repeat Exercise 1 using the negatives of the values given.

ON THE CALCULATOR

The [sin], [cos], and [tan] keys are helpful in evaluating trigonometric functions. Most calculators have features allowing you to work with degrees or radians. All of these exercises are to be done in the degree mode.

EXAMPLE Evaluate $-2 \sin 30°$. **SOLUTION** 30 [sin] [×] 2 [+/−] [=] −1. Answer.

EXAMPLE Evaluate $\dfrac{1}{3 \cos 45°}$.

SOLUTION 45 [cos] [×] 3 [=] [1/x] 0.47140452. Answer.

Exercises

Evaluate. State your answers in decimal form.

1. $\frac{1}{2} \sin 45°$
2. $-\tan 38°$
3. $4 \cos 20°$
4. $\dfrac{5}{\tan 120°}$
5. $\sqrt{3} \sin 218°$
6. $-\dfrac{2}{3} \cos 91.5°$
7. $(\tan 25°)(\sin 115°)$
8. $\dfrac{7}{2 \cos 240°}$
9. $\sqrt{5} \tan 12.8°$
10. $3 \sin(-15°)$
11. $-\frac{3}{5} \cos(-120°)$
12. $15 \tan 140° + 4 \cos 140°$

14-10 Solving Right Triangles

To **solve a right triangle** means to find the measures (or approximations to these) of various parts (angles and sides) when measures of other parts are given. In working with right triangles, it is customary to use either capital letters or Greek letters to identify the angles, and the corresponding lower-case Roman letters to represent the lengths of the sides opposite these angles (see Figure 35). It is also common practice to label the vertex of the right angle as C. The trigonometric functions discussed earlier in this chapter can be used to solve right triangles.

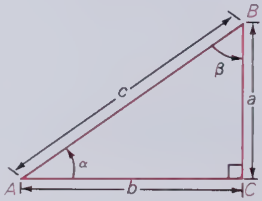


Figure 35

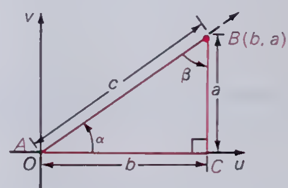


Figure 36

Any right triangle is congruent to a right triangle placed with an acute angle in standard position (Figure 36). If α of right triangle ABC is placed in standard position with side $\overline{AC}$ along the positive u-axis, then the definitions of the six trigonometric functions can be interpreted as follows:

If α is an acute angle of a right triangle, then:

$$\sin\alpha = \frac{\text{length of side opposite } \alpha}{\text{length of hypotenuse}} \qquad \cos\alpha = \frac{\text{length of side adjacent to } \alpha}{\text{length of hypotenuse}}$$

$$\tan\alpha = \frac{\text{length of side opposite } \alpha}{\text{length of side adjacent to } \alpha} \qquad \cot\alpha = \frac{\text{length of side adjacent to } \alpha}{\text{length of side opposite } \alpha}$$

$$\sec\alpha = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \alpha} \qquad \csc\alpha = \frac{\text{length of hypotenuse}}{\text{length of side opposite } \alpha}$$

From these statements, you can see that for right triangle ABC in Figure 35:

$$\sin\beta = \frac{b}{c} = \cos\alpha \qquad \cos\beta = \frac{a}{c} = \sin\alpha$$

$$\tan\beta = \frac{b}{a} = \cot\alpha \qquad \cot\beta = \frac{a}{b} = \tan\alpha$$

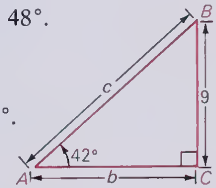
$$\sec\beta = \frac{c}{a} = \csc\alpha \qquad \csc\beta = \frac{c}{b} = \sec\alpha$$

Angles α and β are complementary angles, and the pairs of functions sine and cosine, tangent and cotangent, secant and cosecant are called **cofunctions.**

EXAMPLE 1 Solve the right triangle pictured, stating lengths of sides correct to the nearest hundredth of a unit of length and angle measures to the nearest minute.

SOLUTION Since $m(A) + m(B) = 90°$,

$$m(B) = 90° - m(A) = 90° - 42° = 48°.$$



To find c you can use

$$\csc A = \frac{c}{9} \quad \text{or} \quad c = 9 \csc 42°.$$

From Table 6, $\csc 42° \approx 1.494$, so that

$$c \approx 9(1.494) = 13.446 \approx 13.45.$$

To find b, you can use

$$\tan B = \frac{b}{9} \quad \text{or} \quad b = 9 \tan 48°.$$

From Table 6, $\tan 48° \approx 1.111$, so that

$$b \approx 9(1.111) = 9.999 \approx 10.00$$

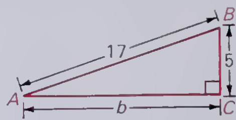
$\therefore$ the measures of the remaining parts of the right triangle are $m(B) = 48°$, $c \approx 13.45$, and $b \approx 10.00$. Answer.

In theoretical examples like Example 1, the given measures (in this case, the measures of side $\overline{BC}$, angle α, and the right angle at C) are taken to be exact. Notice also that the cotangent and cosecant were used instead of tangent and cosine to avoid long divisions.

In using Table 6 in the solution of triangles or in solving practical problems, lengths need be stated to no more than four significant digits, since this is the limit of the accuracy of the entries in this table. The following relationships between angle measure and length can be used as a guide in the exercises in this and later sections.

Angle measure to nearest	corresponds to	length to
1°	. . . .	2 significant digits
10′	. . . .	3 significant digits
1′	. . . .	4 significant digits

EXAMPLE 2 Solve the right triangle pictured.



SOLUTION Assume that the given measures are exact. Thus, with the tables in this book, you may compute b to 4 significant digits and the measures of angles A and B to the nearest $1'$, according to the chart given above. To find $m(A)$, you can use $\sin A = \frac{5}{17} \approx 0.2941$. From Table 6, you find that $m(A) \approx 17°6'$. Since $m(A) + m(B) = 90°$, you have

$$m(B) = 90° - m°(A) \approx 90° - 17°6' = 72°54'.$$

To find b, you can use

$$\cos A = \frac{b}{17} \quad \text{or} \quad b = 17 \cos 17°6'.$$

From Table 6, $\cos 17°6' \approx 0.9558$, so that

$$b \approx 17(0.9558) \approx 16.25.$$

$\therefore\ m(A) \approx 17°6'$, $m(B) \approx 72°54'$, and $b \approx 16.25$. Answer.

In many practical problems applying trigonometric function values, an angle is described as an *angle of elevation* or an *angle of depression* (see Figure 37). Since the point B is elevated with respect to the observer at A, $\angle CAB$, the angle between the horizontal ray AC through A and the line of sight, is an **angle of elevation.** The point T is depressed with respect to the observer at R; therefore, $\angle TRS$, the angle between the line of sight and the horizontal ray RS, is an **angle of depression.**

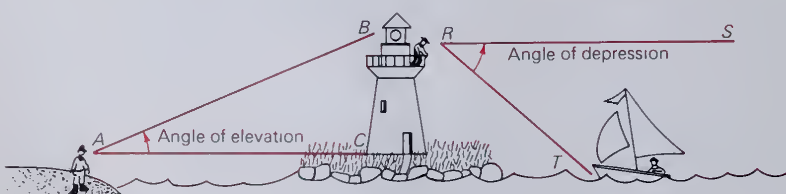
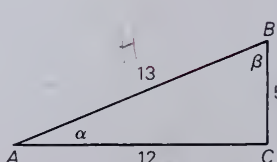


Figure 37

Oral Exercises

Give a fraction equal to the stated function evaluated at the given angle in the diagram.



1. $\sin \alpha$	**2.** $\cos \alpha$	**3.** $\tan \alpha$	**4.** $\csc \alpha$
5. $\sin \beta$	**6.** $\cos \beta$	**7.** $\sec \beta$	**8.** $\tan \beta$

Written Exercises

Solve the right triangle with the given parts. In each triangle $\angle C$ is a right angle. Make a sketch with sides and angles labeled according to the data. State the lengths of sides to three significant digits and the measures of angles to the nearest 10′.

A

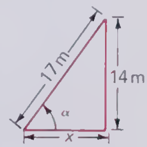
1. $m\angle B = 24°50'$; $c = 40$
2. $m\angle A = 22°20'$; $c = 80$
3. $m\angle A = 30°50'$; $b = 60$
4. $m\angle B = 77°20'$; $b = 440$
5. $m\angle A = 58°40'$; $a = 64$
6. $m\angle B = 43°$; $a = 15$
7. $m\angle B = 70°40'$; $a = 25$
8. $m\angle A = 37°$; $b = 120$
9. $m\angle A = 64°$; $c = 75$
10. $m\angle B = 82°$; $c = 250$
11. $a = 24$; $b = 15$
12. $a = 30$; $c = 50$
13. $b = 20$; $c = 29$
14. $a = 16$; $c = 65$
15. $a = 33$; $b = 56$
16. $b = 24$; $c = 66$
17. $a = 90$; $b = 400$
18. $a = 120$; $c = 130$
19. $b = 19.6$; $c = 27.8$
20. $a = 1.98$; $b = 2.36$

Problems

Give the measures of angles to the nearest 10′; give all other answers to three significant digits.

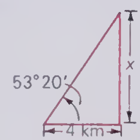
A

1. A fire ladder 17 m long will reach a height of 14 m measured along the side of a building. What angle does the ladder make with the ground? How far is the base of the ladder from the building?



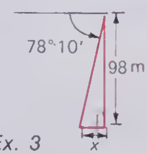
Ex. 1

2. Two ships start from the same point and sail on courses 53°20′ apart. After 0.5 h one ship is due north of the other, which has gone 4 km due east. How far apart are the ships at this time?



Ex. 2

3. The angle of depression measured from the top of a building 98 m high to the base of a building across the street is 78°10′. How wide is the street?

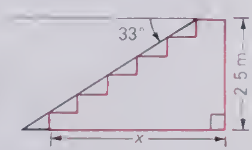


Ex. 3

4. A straight ski run down a mountain 1.1 km high is 5 km long. What angle does the ski run make with the horizontal?

5. From the top of a lookout tower 30 m high, what is the angle of depression to the top of a tree that is 14 m tall and whose trunk is 250 m from the base of the tower?

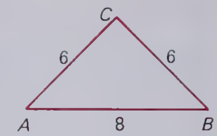
6. The top of a stairway is 2.5 m higher than the bottom. The angle of depression from the top of the stairway to the bottom is 33°. What is the horizontal length (x) of the staircase?



Ex. 6

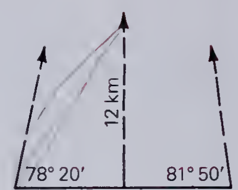
7. From a point A directly opposite a point B on the banks of a straight river, the angle CAB to a point C, 50 m upstream from B, measures $38°50'$. How many meters wide is the river?

8. Find the measure of angle A in the isosceles triangle pictured at the right.



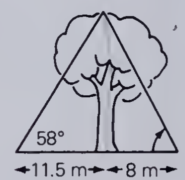
Ex. 8

B 9. When a rocket which has taken off vertically from a point on a straight line between two observers reaches a height of 12 km, the angles of elevation of the lines of sight of the observers are $81°50'$ and $78°20'$, respectively. How far apart are the two observers?



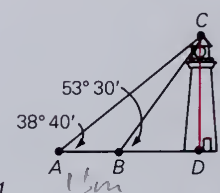
Ex. 9

10. At 6:00 A.M. on a certain day a tree casts a shadow 11.5 m long, and the angle of elevation from the tip of the shadow to the top of the tree is 58°. At 6:00 P.M. the tree casts a shadow 8 m long in the other direction. What is the angle of elevation from the tip of this shadow to the top of the tree?



Ex. 10

11. The angles of elevation from two ships at points A and B, which are on a straight line with the base of a lighthouse 20 m high, are $38°40'$ and $53°30'$, respectively, to the top of the lighthouse. How far apart are the ships?



Ex. 11

C 12. In Exercise 11, if the ships are exactly 15 m apart and the angles of elevation are the same as above, how tall is the lighthouse to the nearest tenth of a meter? (*Hint:* Let $BD = x$ and $CD = y$, and solve two equations in two variables.)

Self-Test 3

VOCABULARY tangent (p. 530), cotangent (p. 530), secant (p. 530), cosecant (p. 530), reciprocal functions (p. 531), cofunctions (p. 537)

Give the measures of angles to the nearest 10′; give all other answers to three significant digits.

1. Find the values for the six trigonometric functions of an angle whose terminal side in standard position contains the point $(-4, \sqrt{33})$. *Obj. 1, p. 530*

2. Solve the right triangle ABC in which $m\angle C = 90°$, $m\angle A = 58°20'$, and $b = 16$. *Obj. 2, p. 530*

Check your answers with those at the back of the book.

programming in BASIC

In BASIC you have these trigonometric functions to work with:

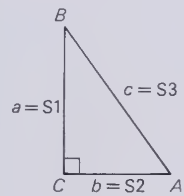
SIN(X), COS(X), TAN(X), ATN(X)

Exercises

Write a program to solve a right triangle *ABC* with angles measured in degrees when given:

1. $m(A)$, a 2. $m(A)$, b 3. $m(A)$, c 4. a, b 5. a, c

(Suggestions: Represent a, b, c by S1, S2, S3, and use conversion factors C1 = 3.14159 / 180 and C2 = 180 / 3.14159.)



6. Write a program that combines the preceding programs into one that solves a triangle when given angle, adjacent leg; angle, opposite leg; and so on. (Suggestions: To guide choices through the program, use questions that can be answered 1 for *yes*, 0 for *no*.)

Chapter Summary

1. A *directed angle* is an ordered pair of rays with a common endpoint, one ray called the *initial side* and the other the *terminal side* of the angle, together with a rotation from the initial side to the terminal side. Angle measures are ordinarily expressed in either *degrees* or *radians*. These measures are related by the *conversion equations:*

$$m^\circ(\alpha) = \frac{180}{\pi} m^R(\alpha) \qquad m^R(\alpha) = \frac{\pi}{180} m^\circ(\alpha)$$

2. If α is an angle in standard position, with $P(u, v)$ any point other than the origin on the terminal side of α, and if $\sqrt{u^2 + v^2} = r$, then the six *trigonometric functions* are:

sine: $\alpha \rightarrow \sin \alpha = \frac{v}{r}$ cosine: $\alpha \rightarrow \cos \alpha = \frac{u}{r}$

tangent: $\alpha \rightarrow \tan \alpha = \frac{v}{u}$, $u \neq 0$ cotangent: $\alpha \rightarrow \cot \alpha = \frac{u}{v}$, $v \neq 0$

secant: $\alpha \rightarrow \sec \alpha = \frac{r}{u}$, $u \neq 0$ cosecant: $\alpha \rightarrow \csc \alpha = \frac{r}{v}$, $v \neq 0$

3. Trigonometric and circular functions are *periodic*, and values for these functions can be found from tables in terms of *reference angles* or *reference arcs.*

4. Graphs of periodic functions consist of basic patterns repeated over each interval having a length of one fundamental period. The graphs of sine and cosine are *sine waves.* For a sine wave, the absolute value of one-half the difference of the minimum and maximum ordinates on the curve is called the *amplitude* of the wave.
5. Many practical problems can be solved by applying trigonometric function values.

Chapter Review

1. A wheel has a radius of 2.8 m. Find the distance traveled by the wheel as it makes 4.2 revolutions. Use $\pi \approx \frac{22}{7}$. *14-1*

   a. 73.92 m b. 36.96 m c. 72.93 m d. 36.39 m

2. If $m^{R}(\alpha) = \frac{3\pi}{5}$, find $m^{\circ}(\alpha)$. *14-2*

   a. 60° b. 300° c. 108° d. 226°

3. If $m^{\circ}(\beta) = 260°$, find $m^{R}(\beta)$.

   a. $\frac{4\pi^{R}}{3}$ b. $\frac{13\pi^{R}}{9}$ c. $\frac{15\pi^{R}}{7}$ d. $\frac{11\pi^{R}}{8}$

4. If $\sin \alpha = \frac{3}{4}$ and $\cos \alpha > 0$, find $\cos \alpha$. *14-3*

   a. $\frac{1}{4}$ b. $\frac{5}{16}$ c. $\frac{\sqrt{7}}{4}$ d. $\frac{7}{16}$

5. If β is an angle in standard position and its terminal side contains the point $(-3, 4)$, find $\sin \beta$.

   a. $-\frac{\sqrt{3}}{3}$ b. $\frac{4}{5}$ c. $-\frac{4}{5}$ d. $\frac{\sqrt{3}}{3}$

6. Use the table on page 516 to find sin 1305°. *14-4*

   a. $-\frac{\sqrt{3}}{3}$ b. $\frac{1}{\sqrt{2}}$ c. $-\frac{1}{2}$ d. $-\frac{1}{\sqrt{2}}$

7. Use the table on page 516 to find $\cos \frac{7\pi^{R}}{2}$.

   a. -1 b. $-\frac{\sqrt{3}}{2}$ c. 0 d. $\frac{\sqrt{3}}{2}$

8. Use Table 6 to find a four-significant-digit approximation for the value of sin 38° 15′. *14-5*

   a. 0.6191 b. 0.6676 c. 0.7815 d. 0.7423

9. Use Table 7 to find $\cos 1.27^{R}$.

   a. 0.9551 b. 1.047 c. 3.375 d. 0.2963

10. Find a four-significant-digit approximation for sin 575°. 14-6

a. -0.7071 b. -0.5736 c. -0.7854 d. 0.7071

11. Find the amplitude of the function $y = -\frac{3}{2}\cos x$. 14-7

a. $\frac{3}{2}$ b. $-\frac{3}{2}$ c. 1 d. -1

12. State the period of the function $y = \frac{3}{4}\cos 3x$. 14-8

a. $\frac{3}{4}\pi$ b. $\frac{3}{2}\pi$ c. $\frac{\pi}{3}$ d. $\frac{2\pi}{3}$

13. If $\sin \alpha = \dfrac{\sqrt{2}}{3}$ and $\cos \alpha > 0$, find $\tan \alpha$. 14-9

a. $\frac{2}{7}$ b. $\dfrac{\sqrt{2}}{\sqrt{7}}$ c. $\dfrac{\sqrt{7}}{\sqrt{2}}$ d. $\frac{7}{2}$

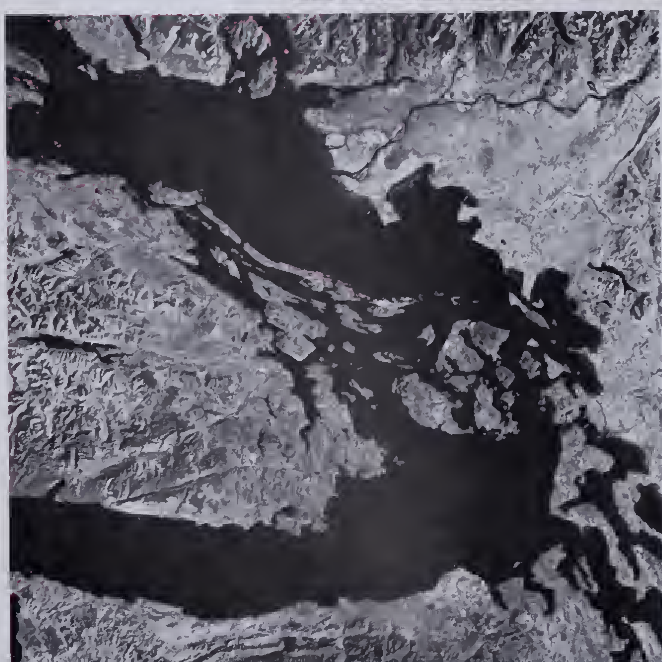
14. If ΔABC is a right triangle with $m\angle A = 52°$, $m\angle C = 90°$, and the length of $\overline{AC} = 10$, find the length of $\overline{AB}$. 14-10

a. 12.69 b. 12.8 c. 16.24 d. 7.813

Chapter Test

1. If a wheel travels 5.6 m as it makes 2.4 revolutions, find the diameter of the wheel to the nearest 0.01 m. (Use $\pi \approx 3.14$.) 14-1
2. Find the length of the arc on a circle with radius 12 cm which is intercepted by a central angle of $\frac{3\pi^R}{4}$. (Use $\pi \approx 3.14$.) 14-2
3. If $m°(\alpha) = 210$, find $m^R(\alpha)$.
4. If $\cos \alpha = \frac{3}{4}$ and $\sin \alpha < 0$, find $\sin \alpha$. 14-3
5. If $\frac{\pi}{2} < x < \pi$, and $\sin x = \frac{1}{2}$, find x. 14-4
6. Use the periodic properties of cosine to find cos 630°.
7. Find a four-significant-digit approximation of $\cos 1.362^R$. 14-5
8. Find the measure of α in degrees and minutes (to the nearest minute) for the first-quadrant angle such that $\sin \alpha = 0.6461$.
9. Make a sketch to help you find an approximation of sin 258°. 14-6
10. a. Sketch the graph of $y = \frac{1}{2}\sin x$ over the interval $-2\pi \le x \le 2\pi$. 14-7
    b. Find the amplitude.
11. Sketch the graph of $y = 2\cos \frac{1}{3}x$. Find the amplitude and the fundamental period. 14-8
12. If $\sin \alpha = \frac{1}{2}$ and $\cos \alpha > 0$, find 14-9

    a. $\tan \alpha$ b. $\csc \alpha$ c. $\sec \alpha$ d. $\cot \alpha$
13. ΔABC is a right triangle with $m\angle B = 90°$, the length of $\overline{AB} = 10$, the length of $\overline{AC} = 15$. Find $m\angle A$ to the nearest ten minutes. 14-10



These photographs of the Straits of Juan de Fuca were made through different filters. The photo above emphasizes cultural features such as metropolitan areas. The photo at left emphasizes geographical features such as land forms and the boundary between land and water.

15 Trigonometric Identities and Formulas

Identities

OBJECTIVE for Sections 15-1 and 15-2:
1. Prove simple trigonometric identities.

15-1 Fundamental Identities

Following are listed the eight fundamental trigonometric identities, some of which you have already met. You saw in Section 14-3 that the values of the sine and cosine functions are related by the identity

$$\sin^2 \alpha + \cos^2 \alpha = 1. \qquad (1)$$

In Section 14-9, the following statements were made as definitions:

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad \cos \alpha \neq 0 \quad (2) \qquad \cot \alpha = \frac{\cos \alpha}{\sin \alpha}, \quad \sin \alpha \neq 0 \quad (3)$$

$$\sec \alpha = \frac{1}{\cos \alpha}, \quad \cos \alpha \neq 0 \quad (4) \qquad \csc \alpha = \frac{1}{\sin \alpha}, \quad \sin \alpha \neq 0 \quad (5)$$

Since values of trigonometric functions are real numbers, the properties of real numbers can be used to find other relationships between the

function values. From (*2*) and (*3*) you have

$$\frac{\sin \alpha}{\cos \alpha} = \frac{1}{\frac{\cos \alpha}{\sin \alpha}}, \quad \sin \alpha \neq 0,\ \cos \alpha \neq 0,$$

that is,

$$\tan \alpha = \frac{1}{\cot \alpha}, \quad \tan \alpha \neq 0,\ \cot \alpha \neq 0. \qquad (6)$$

The last two of the eight fundamental trigonometric identities can be obtained from (*1*) as follows. For angles α for which $\cos \alpha \neq 0$, you have

$$\frac{\sin^2 \alpha}{\cos^2 \alpha} + \frac{\cos^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha},$$

or

$$\tan^2 \alpha + 1 = \sec^2 \alpha, \quad \cos \alpha \neq 0. \qquad (7)$$

Similarly, for angles α for which $\sin \alpha \neq 0$, you have

$$\frac{\sin^2 \alpha}{\sin^2 \alpha} + \frac{\cos^2 \alpha}{\sin^2 \alpha} = \frac{1}{\sin^2 \alpha},$$

or

$$1 + \cot^2 \alpha = \csc^2 \alpha, \quad \sin \alpha \neq 0. \qquad (8)$$

Of course, Identities (*1*)–(*8*) are equally valid for the circular functions. You know that for every real number x,

$$\sin^2 x + \cos^2 x = 1,$$

because $\sin x$ and $\cos x$ are coordinates of points on the unit circle. The fact that the remaining circular functions are defined in precisely the same way as the corresponding trigonometric functions of angles guarantees that every trigonometric identity is as valid for values of circular functions as it is for values of those functions with the same names whose domains are the set of angles.

EXAMPLE 1 Express $\tan \alpha \csc \alpha$ in terms of $\cos \alpha$, noting any restrictions.

SOLUTION The given expression is defined provided $\cos \alpha \neq 0$ and $\sin \alpha \neq 0$. Thus, from Identities (*2*) and (*5*), you have

$$\tan \alpha \csc \alpha = \frac{\sin \alpha}{\cos \alpha} \cdot \frac{1}{\sin \alpha} = \frac{1}{\cos \alpha}, \quad \cos \alpha \neq 0,\ \sin \alpha \neq 0. \quad \textsf{Answer.}$$

EXAMPLE 2 Express $\cos \alpha$ in terms of $\sin \alpha$.

SOLUTION From Identity (*1*),

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

$$\therefore \cos \alpha = \sqrt{1 - \sin^2 \alpha} \quad \text{or} \quad \cos \alpha = -\sqrt{1 - \sin^2 \alpha}. \quad \textsf{Answer.}$$

Notice that, in Example 2, the result is a disjunction of sentences. Since $\sqrt{1 - \sin^2 \alpha}$ always represents a nonnegative number, and

$-\sqrt{1-\sin^2\alpha}$ always represents a nonpositive number, it is customary to use the information in Figure 29 on page 531 and express this disjunction as follows:

$$\cos\alpha = \sqrt{1-\sin^2\alpha} \text{ if } \alpha \text{ lies in Quadrants I or IV.}$$
$$\cos\alpha = -\sqrt{1-\sin^2\alpha} \text{ if } \alpha \text{ lies in Quadrants II or III.}$$

Oral Exercises

State an equivalent expression in terms of the sine function only. Note any restrictions.

1. $1-\cos^2\alpha$ 2. $\cos\alpha\tan\alpha$ 3. $\csc^2\alpha$ 4. $\sec^2\alpha$ 5. $\cos\alpha\cot\alpha$ 6. $\sec\alpha\tan\alpha$

State an equivalent expression in terms of $\cos x$ only. Note any restrictions.

7. $\sin^2 x$ 8. $\csc x\tan x$ 9. $\dfrac{\sin^2 x}{\tan^2 x}$ 10. $\csc x\cot x$ 11. $\sin x\tan x$ 12. $\csc^2 x$

Written Exercises

Write an equivalent expression in simplest form, using the sine function only. Note any restrictions.

A 1. $\sin^2\alpha\cot^2\alpha$ 2. $\tan^2\alpha\cos^2\alpha\csc\alpha$ 3. $\csc x\sec x\cot x$

4. $\cot^2 x+1$ 5. $\sin x\cot x+\cos x$ 6. $\cos x\cot x+\sin x$

7. $\dfrac{\tan^2\alpha}{1+\tan^2\alpha}$ 8. $\dfrac{\cot^2\alpha}{1-\sin^2\alpha}$ 9. $\dfrac{\sec^2 x-1}{\cos^2 x-1}$

10–18. Rewrite the expressions in Exercises 1–9 as equivalent expressions in simplest form, using the cosine function only. Note any restrictions.

Write an equivalent expression in simplest form expressed in terms of a single function or constant.

19. $\sec\alpha\cos\alpha\tan^2\alpha$ 20. $\sec^2\alpha-\cos\alpha$ 21. $\sec x\csc x-\tan x$

22. $\dfrac{\csc^2\alpha-\cot^2\alpha}{1-\cos^2\alpha}$ 23. $\dfrac{\sec^2 x-\tan^2 x}{\cos^2 x}-1$ 24. $\dfrac{\csc x-\sin x}{\cos x}$

25. $\sin x(\cos x+\sin x\tan x)$ 26. $\sin\alpha(\tan\alpha-\sin\alpha\sec\alpha)$

Express in terms of $\sin x$ and $\cos x$ only, and simplify. Note any restrictions.

B 27. $\tan x+\cot x$ 28. $(\tan x+\sin x)(1-\cos x)$

29. $\sec^2 x+\csc^2 x$ 30. $(\cot x+\cos x)(\csc x+1)$

31. $\tan^2 x+\csc^2 x+1$ 32. $(\cos x\cot^2 x\sec x)(\tan^2 x\csc^2 x-1)$

33. If α is an angle in quadrant II, express $\cos\alpha$ in terms of $\tan\alpha$.

34. If α is an angle in quadrant III, express $\csc\alpha$ in terms of $\tan\alpha$.

35. If α is a positive acute angle and $\sin\alpha = \frac{1}{r}$, express $\tan\alpha$ in terms of r.

15-2 Proving Identities

Is the equation

$$\frac{1}{1+\sin x} + \frac{1}{1-\sin x} = 2\sec^2 x$$

true for every real number x for which both members are defined? That is, is it an identity? One method of proving that an equation is an identity is to transform the more complicated member to the form of the simpler member.

EXAMPLE 1 Prove that $\frac{1}{1+\sin x} + \frac{1}{1-\sin x} = 2\sec^2 x$ is an identity, noting any restrictions on values for x.

SOLUTION First notice that the left-hand member is not defined for numbers x for which $\sin x = 1$ or $\sin x = -1$, and the right-hand member is not defined for those x for which $\cos x = 0$. For all other values of x we can rewrite the left-hand member to obtain

$$\frac{1(1-\sin x)}{(1+\sin x)(1-\sin x)} + \frac{1(1+\sin x)}{(1-\sin x)(1+\sin x)}$$

$$= \frac{(1-\sin x)+(1+\sin x)}{1-\sin^2 x} = \frac{2}{1-\sin^2 x}$$

$$= \frac{2}{\cos^2 x} = 2\sec^2 x,$$

which is the right-hand member. Since no new restrictions on x were introduced in the process, the original sentence is an identity when $\cos x \neq 0$. This restriction is sufficient to ensure that $\sin x \neq -1$ and $\sin x \neq 1$.

You can also prove that a given equation is an identity by showing that it is equivalent to a known identity.

EXAMPLE 2 Prove that $(\sec\alpha)(\sec\alpha - \cos\alpha) = \tan^2\alpha$, noting any restrictions on values for α.

SOLUTION Begin by noting that neither member is defined for $\cos\alpha = 0$; then write both expressions entirely in terms of $\sin\alpha$ and $\cos\alpha$.

$$\frac{1}{\cos\alpha}\left(\frac{1}{\cos\alpha} - \cos\alpha\right) = \frac{\sin^2\alpha}{\cos^2\alpha}$$

Apply the distributive property to the left-hand member to get

$$\frac{1}{\cos^2\alpha} - 1 = \frac{\sin^2\alpha}{\cos^2\alpha}.$$

Multiplying each member by $\cos^2\alpha$ then produces

$$1 - \cos^2\alpha = \sin^2\alpha$$

or $\sin^2\alpha + \cos^2\alpha = 1$, which is an identity for all values of α.

$\therefore$ for all values of α for which $\cos\alpha \neq 0$, the given sentence is an identity.

In proving trigonometric identities, it is helpful to observe the following items.

1. Use the eight fundamental trigonometric identities to simplify the more complicated member.
2. Use the transformations that produce equivalent equations.
3. Observe restrictions on variables and make sure that no new ones are introduced in your work.
4. If no other approach suggests itself, express all function values in terms of values of sine and cosine.
5. Introduce radicals only when absolutely necessary.

Oral Exercises

Give a reason for each of the steps in the following proof of the identity $2\cos^2 x - 1 = \cos^2 x - \sin^2 x$.

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin^2 x + 2\cos^2 x = 1 + \cos^2 x$
3. $2\cos^2 x = 1 + \cos^2 x - \sin^2 x$
4. $2\cos^2 x - 1 = \cos^2 x - \sin^2 x$
5. Are any restrictions on the variable necessary?

Written Exercises

Prove each identity.

A

1. $\sin\alpha(\csc\alpha - \sin\alpha) = \cos^2\alpha$
2. $\sec\alpha - \cos\alpha = \sin\alpha\tan\alpha$
3. $\tan\alpha + \cot\alpha = \sec\alpha\csc\alpha$
4. $\sin x(\csc x + \sin x\sec^2 x) = \sec^2 x$

Prove each identity.

5. $\sin x + \cos x \cot x = \csc x$

6. $\dfrac{1 + \cot^2 x}{1 + \tan^2 x} = \cot^2 x$

7. $\dfrac{(\sin \alpha - \cos \alpha)^2}{\cos \alpha} = \sec \alpha - 2 \sin \alpha$

8. $\dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x} = 2 \tan x$

9. $\dfrac{\cos \alpha}{\sec \alpha + 1} + \dfrac{\cos \alpha}{\sec \alpha - 1} = 2 \cot^2 \alpha$

10. $\csc \alpha(\sec \alpha - \tan \alpha \sin \alpha) = \cot \alpha$

11. $\dfrac{\sec \alpha + \csc \alpha}{1 + \tan \alpha} = \csc \alpha$

12. $\dfrac{1}{\sec \alpha - 1} + \dfrac{1}{\sec \alpha + 1} = 2 \cot \alpha \csc \alpha$

13. $\sec \alpha(\csc \alpha - \cot^2 \alpha \sin \alpha) = \tan \alpha$

14. $\dfrac{\sin \alpha}{1 + \cos \alpha} + \cot \alpha = \csc \alpha$

15. $(\tan x + \sin x)(1 - \cos x) = \sin^2 x \tan x$

16. $\dfrac{1 + \tan x}{\sin x + \cos x} = \sec x$

B 17. $\dfrac{\tan \alpha \csc \alpha}{\cos \alpha} - \dfrac{\sec \alpha - \cos \alpha}{\sin \alpha \tan \alpha} = \tan^2 \alpha$

18. $\dfrac{\sec \alpha}{\sec \alpha - 1} - \dfrac{\sec \alpha + 1}{\tan^2 \alpha} = 1$

19. $\sec^4 \alpha - \tan^4 \alpha = \sec^2 \alpha + \tan^2 \alpha$

20. $\dfrac{\sec \alpha}{1 + \csc \alpha} - \dfrac{\tan \alpha}{1 + \sin \alpha} = 0$

21. $\dfrac{1 - \sec x}{1 - \cos x} = -\sec x$

22. $\dfrac{\sin \alpha}{1 - \cos \alpha} = \dfrac{1 + \cos \alpha}{\sin \alpha}$

23. $\dfrac{\tan \alpha}{1 + \sec \alpha} = \dfrac{\sec \alpha - 1}{\tan \alpha}$

24. $(\sec x - \cos x)^2 = \tan^2 x - \sin^2 x$

25. $\dfrac{\sec x - 1}{\tan x} + \dfrac{\tan x}{\sec x + 1} = \dfrac{2 \sin x}{1 + \cos x}$

26. $\dfrac{\sin x}{\csc x - \cot x} + \dfrac{\sin x \cos x}{\csc x + \cot x} = \sin^2 x + 2 \cos x$

C 27. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = \left(\dfrac{1 - \sin A}{\cos A}\right)^2$

28. $\dfrac{\tan B + \sin B}{\tan B - \sin B} = \left(\dfrac{1 + \cos B}{\sin B}\right)^2$

Nicolas Bourbaki

Nicolas Bourbaki is the pen name for a group of European and American mathematicians, whose goal is "to present a view of the entire field of mathematical science as it exists." Since 1939, more than 40 volumes of Bourbaki's *Eléments de Mathématiques* have been published and there are more volumes to come. The composition of the Bourbaki group changes, with members retiring when they reach age 45 and being replaced by new ones. The *Eléments de Mathématiques* uses a formal axiomatic approach, attempting to integrate all of mathematics into a single framework.

Self-Test 1

Prove each identity.

1. $\sec\alpha(\cos\alpha + \sin\alpha\tan\alpha) = \sec^2\alpha$ *Obj. 1, p. 545*

2. $\dfrac{\sin^2\alpha}{1-\cos\alpha} = 1 + \cos\alpha$

3. $\dfrac{\cos\alpha}{\sec\alpha} + \dfrac{\sin\alpha}{\csc\alpha} = 1$

4. $\sec\alpha\csc\alpha - \tan\alpha = \cot\alpha$

Check your answers with those at the back of the book.

Functions: Sums and Differences

***OBJECTIVES** for Sections 15-3 through 15-6:*

1. Use reduction formulas to find function values in quadrants other than Quadrant I.

2. Use formulas of sums and differences of trigonometric function values to evaluate expressions involving such sums or differences.

3. Prove simple trigonometric identities involving double- and half-angle formulas.

15-3 The Cosine of a Sum or a Difference

If $m^R(\alpha_1) = x_1$ and $m^R(\alpha_2) = x_2$, then the **sum** of α_1 and α_2, denoted by $\alpha_1 + \alpha_2$, is an angle whose measure in radians is $x_1 + x_2$:

$$m^R(\alpha_1 + \alpha_2) = x_1 + x_2.$$

Similarly, the **difference** of α_1 and α_2, denoted by $\alpha_1 - \alpha_2$, has measure $x_1 - x_2$:

$$m^R(\alpha_1 - \alpha_2) = x_1 - x_2.$$

Angles $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2$, where all are in the first quadrant, are pictured in Figure 1.

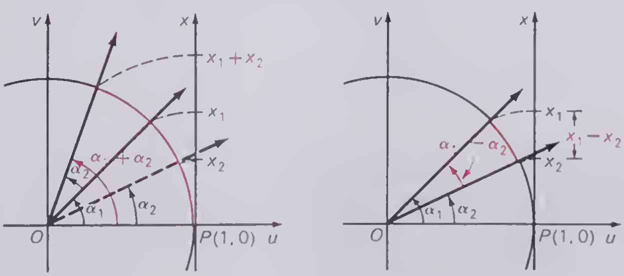


Figure 1

The negative of α_1, denoted by $-\alpha_1$, has measure $-x_1$:

$$m^{R}(-\alpha_1) = -x_1.$$

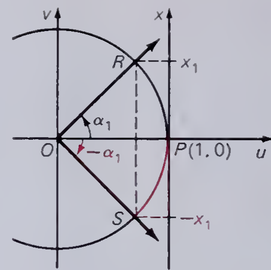


Figure 2

Angle α_1 (in the first quadrant) and angle $-\alpha_1$ (in the fourth quadrant) are pictured in Figure 2.

Figure 2 can be used to deduce two very important properties of sin $(-x)$ and cos $(-x)$ and of sin $(-\alpha)$ and cos $(-\alpha)$. Because of the symmetry of the unit circle with respect to the axes, the points R and S have *equal abscissas,* but *ordinates* which are the *negatives* of each other. Since a similar situation exists if α is in any of the other quadrants, you can assert that for each real number x,

$$\sin(-x) = -\sin x \qquad \text{and} \qquad \cos(-x) = \cos x.$$

In terms of angles, for each angle α you have

$$\sin(-\alpha) = -\sin\alpha \qquad \text{and} \qquad \cos(-\alpha) = \cos\alpha.$$

A function for which

$$f(-x) = -f(x)$$

for every x in the domain of f is called an **odd function,** while a function for which

$$f(-x) = f(x)$$

for every x in the domain of f is called an **even function** (see Section 8-1). Thus, sine is an *odd* function, and cosine is an *even* function.

Figure 3 shows $x_1 - x_2$ when x_1 corresponds to a second-quadrant angle and x_2 to a first-quadrant angle:

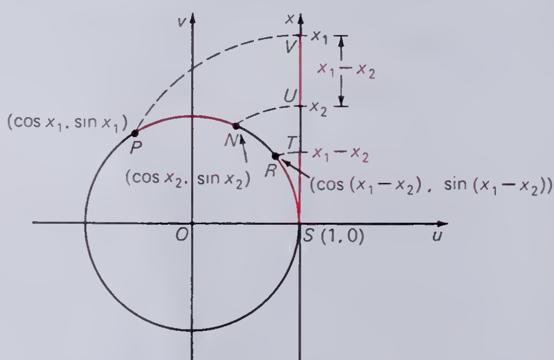


Figure 3

Observe that the segment $\overline{UV}$, whose endpoints are the graphs of x_2 and x_1 on the x number line, is congruent to the segment $\overline{ST}$ with endpoints corresponding to 0 and $x_1 - x_2$. Therefore, the arc of the unit circle from S to R is congruent to the arc from N to P, and consequently chord $\overline{SR}$ is congruent to chord $\overline{NP}$. Expressing this latter fact by means of the

distance formula (page 342), you have

$$\sqrt{[\cos (x_1 - x_2) - 1]^2 + [\sin (x_1 - x_2) - 0]^2} = \sqrt{(\cos x_1 - \cos x_2)^2 + (\sin x_1 - \sin x_2)^2}.$$

Squaring both members of this equation, you find that

$$\cos^2 (x_1 - x_2) - 2 \cos (x_1 - x_2) + 1 + \sin^2 (x_1 - x_2) = \cos^2 x_1 - 2 \cos x_1 \cos x_2 + \cos^2 x_2 + \sin^2 x_1 - 2 \sin x_1 \sin x_2 + \sin^2 x_2.$$

On rearranging terms, you have

$$\cos^2 (x_1 - x_2) + \sin^2 (x_1 - x_2) - 2 \cos (x_1 - x_2) + 1 = \cos^2 x_1 + \sin^2 x_1 - 2 \cos x_1 \cos x_2 + \cos^2 x_2 + \sin^2 x_2 - 2 \sin x_1 \sin x_2.$$

Because $\cos^2 x + \sin^2 x = 1$ for every real number x, you can replace each of the three pairs of terms shown in red in the preceding equation with "1" and simplify the result to obtain the following:

$$\cos (x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2$$

Since the development given is not limited by the particular values of x_1 and x_2 shown in Figure 3, the preceding equation is an identity. It is sometimes called the **formula for the cosine of a difference.** In terms of angles, this formula could be written:

$$\cos (\alpha_1 - \alpha_2) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2$$

EXAMPLE 1 Find $\cos \frac{\pi}{12}$.

SOLUTION Express $\frac{\pi}{12}$ as the difference of two numbers for which you know the sine and cosine. (Refer to the table on page 516.) Thus,

$$\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}.$$

Then, replacing x_1 and x_2 in the formula for the cosine of a difference with $\frac{\pi}{3}$ and $\frac{\pi}{4}$, respectively, you get

$$\cos \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}.$$

From the table, you have

$$\cos \frac{\pi}{3} = \frac{1}{2}, \quad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

(Solution continued on page 554.)

Therefore,

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{1}{4}(\sqrt{2} + \sqrt{6}). \quad \text{Answer.}$$

By replacing x_2 with $-x_2$ in the difference formula, you obtain

$$\cos[x_1 - (-x_2)] = \cos x_1 \cos(-x_2) + \sin x_1 \sin(-x_2).$$

Then, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ (page 552), you have the **formula for the cosine of a sum:**

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$$

In terms of angles, this formula could be written

$$\cos(\alpha_1 + \alpha_2) = \cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2.$$

You can develop a number of other valuable identities by using the formulas for the cosine of a difference and the cosine of a sum.

For example, if you let $x_1 = \pi$ and $x_2 = x$ in the difference formula, you have

$$\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$$
$$= (-1)\cos x + (0)\sin x = -\cos x.$$

Similarly, if $m(\alpha_1) = 180°$ and $\alpha_2 = \alpha$, you find from the difference formula that

$$\cos(180° - \alpha) = \cos 180° \cos\alpha + \sin 180° \sin\alpha$$
$$= (-1)\cos\alpha + (0)\sin\alpha = -\cos\alpha.$$

You can also show (Exercise 26, page 557) that

$$\cos(\pi + x) = -\cos x \quad \text{and} \quad \cos(180° + \alpha) = -\cos\alpha.$$

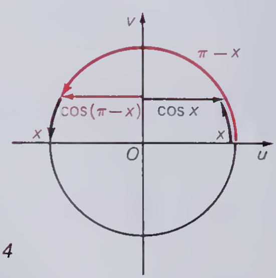


Figure 4

$\cos(\pi - x) = -\cos x$

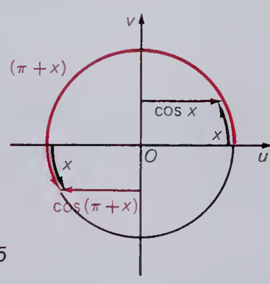


Figure 5

$\cos(\pi + x) = -\cos x$

These identities are valid for all real numbers x and all angles α. If $0 < x < \frac{\pi}{2}$, then $\pi - x$ is the measure of a unit-circle arc in Quadrant II

and $\pi + x$ is the measure of a unit-circle arc in Quadrant III. Thus, x becomes the measure of the *reference arc* for the arcs with measures $\pi - x$ and $\pi + x$. Compare Figures 4 and 5 with parts a and c of Figure 19, page 522, where the reference arc has measure x'.

Now if you let $x_1 = \frac{\pi}{2}$ and $x_2 = x$ in the difference formula, you have

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)\cos x + (1)\sin x = \sin x.$$

Notice that since the measures of the acute angles of a right triangle can be expressed as x and $\frac{\pi}{2} - x$ radians, the preceding formula is a generalization of the corresponding one given on page 536.

You can also show (Exercise 29, page 557) that

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x.$$

For $0 < x < \frac{\pi}{2}$, these results are pictured in Figures 6 and 7.

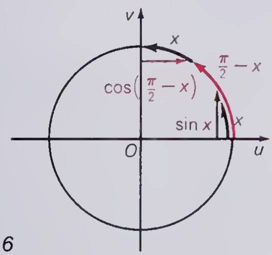


Figure 6

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

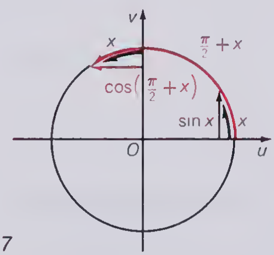


Figure 7

$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

By letting $x_1 = \frac{3\pi}{2}$ in the sum and difference formulas, you can find two more identities (Exercises 27 and 28, page 557). Because all these formulas can be used to "reduce" a given circular or trigonometric function value in any quadrant to a function value in Quadrant I, they are called **reduction formulas.** Here is a list of the most useful reduction formulas for cosine. From now on we shall work primarily with formulas stated in terms of x, where x is a real number, with the understanding that corresponding formulas hold in terms of angles.

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\cos(\pi + x) = -\cos x \qquad \cos(\pi - x) = -\cos x$$

$$\cos\left(\frac{3\pi}{2} + x\right) = \sin x \qquad \cos\left(\frac{3\pi}{2} - x\right) = -\sin x$$

EXAMPLE 2 Reduce $\cos \frac{13\pi}{9}$ to an equal function value for a number x such that $0 \le x \le \frac{\pi}{2}$.

SOLUTION Because $\pi < \frac{13\pi}{9} < \frac{3\pi}{2}$, you can use either

$$\frac{13\pi}{9} = \pi + \frac{4\pi}{9} \quad \text{or} \quad \frac{13\pi}{9} = \frac{3\pi}{2} - \frac{\pi}{18}.$$

Method I: Using $\frac{4\pi}{9}$, from $\cos(\pi + x) = -\cos x$ you have

$$\cos \frac{13\pi}{9} = \cos\left(\pi + \frac{4\pi}{9}\right) = -\cos \frac{4\pi}{9}. \quad \text{Answer.}$$

Method II: Using $\frac{\pi}{18}$, from $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$ you have

$$\cos \frac{13\pi}{9} = \cos\left(\frac{3\pi}{2} - \frac{\pi}{18}\right) = -\sin \frac{\pi}{18}. \quad \text{Answer.}$$

Note: $-\sin \frac{\pi}{18} = -\cos \frac{4\pi}{9}$ because $\frac{\pi}{18} + \frac{4\pi}{9} = \frac{\pi}{2}$.

Oral Exercises

Express each of the following in terms of a single cosine.

1. $\cos 45° \cos 30° - \sin 45° \sin 30°$
2. $\cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$
3. $\cos 18° \cos 35° + \sin 18° \sin 35°$
4. $-\cos \frac{\pi}{6} \cos \frac{\pi}{12} - \sin \frac{\pi}{6} \sin \frac{\pi}{12}$
5. $\cos \frac{2\pi}{9} \cos \frac{\pi}{9} - \sin \frac{\pi}{9} \sin \frac{2\pi}{9}$
6. $\sin 75° \sin 10° - \cos 75° \cos 10°$

1, 2, 7, 8, 9-12, 13-17 odd nearest tenth

Written Exercises

In Exercises 1–8 express each angle as a sum or difference of two of the angles listed in the table on page 516. Then apply a sum or difference formula to evaluate the given function. Express your answer in radical form.

A

1. $\cos 75°$
2. $\cos 105°$
3. $\cos 255°$
4. $\cos 195°$
5. $\cos 345°$
6. $\cos 285°$
7. $\cos 375°$
8. $\cos 165°$

Express the given function value as a function value for a real number x such that $0 \leq x \leq \frac{\pi}{2}$.

9. $\cos \frac{7\pi}{8}$
10. $\cos \frac{13\pi}{12}$
11. $\cos \frac{9\pi}{7}$
12. $\cos \left(-\frac{4\pi}{5}\right)$
13. $\cos \frac{11\pi}{8}$
14. $\cos \frac{13\pi}{16}$
15. $\cos 1.6\pi$
16. $\cos (-0.7\pi)$

9-23 odd

Express each quantity as a single cosine and evaluate the function.

17. $\cos \frac{5\pi}{12} \cos \frac{\pi}{12} + \sin \frac{5\pi}{12} \sin \frac{\pi}{12}$
18. $\cos \frac{\pi}{8} \cos \frac{5\pi}{8} - \sin \frac{\pi}{8} \sin \frac{5\pi}{8}$
19. $\cos \frac{8\pi}{9} \cos \frac{2\pi}{9} + \sin \frac{8\pi}{9} \sin \frac{2\pi}{9}$
20. $\cos 155° \cos 5° + \sin 155° \sin 5°$
21. $\cos 1.3\pi \cos 0.2\pi - \sin 1.3\pi \sin 0.2\pi$
22. $\cos 175° \cos 50° - \sin 175° \sin 50°$
23. $\cos (-25°) \cos 265° - \sin (-25°) \sin 265°$
24. $\cos 140° \cos (-15°) + \sin 140° \sin (-15°)$

25-31 odd

Use the formula for the cosine of a sum or a difference to prove each of the following identities.

25. $\cos \left(x - \frac{\pi}{2}\right) = \sin x$
26. $\cos (\pi + x) = -\cos x$
27. $\cos \left(\frac{3\pi}{2} + x\right) = \sin x$
28. $\cos \left(\frac{3\pi}{2} - x\right) = -\sin x$
29. $\cos \left(\frac{\pi}{2} + x\right) = -\sin x$
30. $\cos \left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3} \cos x + \sin x}{2}$
31. $\cos \left(\frac{\pi}{3} - x\right) = \frac{\cos x + \sqrt{3} \sin x}{2}$
32. $\cos \left(\frac{3\pi}{4} + x\right) = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

For each of the following identities draw a sketch, similar to that in Figure 7, that shows the geometric significance of the identity. Assume that $0 < x < \frac{\pi}{2}$.

B 33. $\cos \left(\frac{3\pi}{2} + x\right) = \sin x$
34. $\cos \left(\frac{3\pi}{2} - x\right) = -\sin x$
35. $\cos (x - \pi) = -\cos x$
36. $\cos \left(x - \frac{\pi}{2}\right) = \sin x$
37. $\cos \left(x - \frac{3\pi}{2}\right) = -\sin x$
38. $\cos \left(x + \frac{\pi}{2}\right) = -\cos \left(x - \frac{\pi}{2}\right)$

Find sin x, sin y, and use these values to find a value for cos $(x - y)$ under the given conditions.

39. $\cos x = \frac{3}{5}$; $0 < x < \frac{\pi}{2}$; $\cos y = \frac{5}{13}$; $0 < y < \frac{\pi}{2}$

40. $\cos x = -\frac{4}{5}$; $\frac{\pi}{2} < x < \pi$; $\cos y = \frac{12}{13}$; $0 < y < \frac{\pi}{2}$

41. $\cos x = -\frac{24}{25}$; $\pi < x < \frac{3\pi}{2}$; $\cos y = -\frac{3}{5}$; $\frac{\pi}{2} < y < \pi$

42. $\cos x = \frac{8}{17}$; $\frac{3\pi}{2} < x < 2\pi$; $\cos y = -\frac{4}{5}$; $\pi < y < \frac{3\pi}{2}$

Find sin x, sin y, and use these values to find a value for cos $(x + y)$ under the given conditions.

43. $\cos x = \frac{\sqrt{5}}{3}$; $0 < x < \frac{\pi}{2}$; $\cos y = \frac{1}{2}$; $\frac{3\pi}{2} < y < 2\pi$

44. $\cos x = -\frac{\sqrt{7}}{4}$; $\pi < x < \frac{3\pi}{2}$; $\cos y = \frac{1}{3}$; $0 < y < \frac{\pi}{2}$

45. $\cos x = -\frac{1}{4}$; $\frac{\pi}{2} < x < \pi$; $\cos y = \frac{2}{3}$; $\frac{3\pi}{2} < y < 2\pi$

C 46. $\cos x = a$; $0 < x < \frac{\pi}{2}$; $\cos y = \sqrt{1 - b^2}$; $0 < y < \frac{\pi}{2}$

15-4 The Sine and Tangent of a Sum or a Difference

To find formulas for the sine of a sum and of a difference, first recall (page 555) that

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

If you replace x with $\frac{\pi}{2} - x$, you obtain

$$\cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \sin\left(\frac{\pi}{2} - x\right), \quad \text{or} \quad \sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

To find a formula for $\sin (x_1 + x_2)$, use the above and the formula for the cosine of a difference to get:

$$\sin (x_1 + x_2) = \cos\left(\frac{\pi}{2} - (x_1 + x_2)\right) = \cos\left(\left(\frac{\pi}{2} - x_1\right) - x_2\right)$$
$$= \cos\left(\frac{\pi}{2} - x_1\right)\cos x_2 + \sin\left(\frac{\pi}{2} - x_1\right)\sin x_2$$

Since $\cos\left(\frac{\pi}{2} - x_1\right) = \sin x_1$ and $\sin\left(\frac{\pi}{2} - x_1\right) = \cos x_1$, you have the **formula for the sine of a sum:**

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2.$$

In terms of angles, you have

$$\sin(\alpha_1 + \alpha_2) = \sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2.$$

If, in the formula for the sine of a sum, you replace x_2 with $-x_2$, you obtain

$$\sin(x_1 + (-x_2)) = \sin x_1 \cos(-x_2) + \cos x_1 \sin(-x_2).$$

Then, because $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ (page 552), you have the **formula for the sine of a difference:**

$$\sin(x_1 - x_2) = \sin x_1 \cos x_2 - \cos x_1 \sin x_2$$

In terms of angles, you have:

$$\sin(\alpha_1 - \alpha_2) = \sin \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2$$

EXAMPLE 1 Find $\sin \frac{11\pi}{12}$.

SOLUTION $\frac{11\pi}{12}$ can be expressed as the sum of two numbers for which you know the sine and cosine, namely,

$$\frac{11\pi}{12} = \frac{\pi}{6} + \frac{3\pi}{4}.$$

Then, you can replace x_1 and x_2 in the sum formula with $\frac{\pi}{6}$ and $\frac{3\pi}{4}$, respectively, to obtain

$$\sin \frac{11\pi}{12} = \sin\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = \sin \frac{\pi}{6} \cos \frac{3\pi}{4} + \cos \frac{\pi}{6} \sin \frac{3\pi}{4}.$$

Since

$$\sin \frac{\pi}{6} = \frac{1}{2}, \ \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \ \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ and } \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2},$$

you have

$$\sin \frac{11\pi}{12} = \sin\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1). \quad \text{Answer.}$$

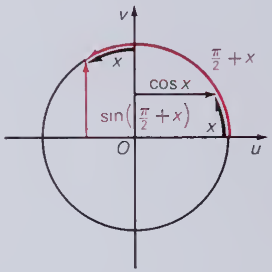
Following are the reduction formulas for sine corresponding to those for cosine given on page 555. The second of these was derived above. You can prove the others (Exercises 43–47, page 563).

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x \qquad \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin(\pi + x) = -\sin x \qquad \sin(\pi - x) = \sin x$$

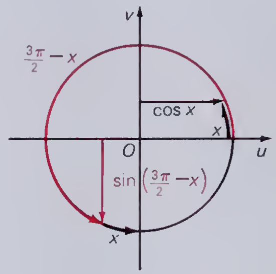
$$\sin\left(\frac{3\pi}{2} + x\right) = -\cos x \qquad \sin\left(\frac{3\pi}{2} - x\right) = -\cos x$$

Two of these are pictured in Figures 8 and 9.



$\sin\left(\frac{\pi}{2} + x\right) = \cos x$

Figure 8



$\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$

Figure 9

Sum, difference, and reduction formulas for the functions tangent, cotangent, secant, and cosecant can be obtained by using appropriate formulas established for sine and cosine, in conjunction with the definitions of the other functions. Although the formulas for sine and cosine hold for all values of the variable, there are certain restrictions placed on the domains of the variables in formulas for the other functions.

In practice, the only ones of these additional formulas that are of much concern to us are those for tangent, because to find values for cotangent, secant, or cosecant, you need only take the reciprocal of the corresponding values of tangent, cosine, or sine.

To start, notice that tangent, like sine, is an odd function (page 552), since

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x}.$$

Thus,

$$\tan(-x) = -\tan x, \quad \cos x \neq 0.$$

To obtain a difference formula for the tangent function, you can replace x with $(x_1 - x_2)$ in the definition of $\tan x$ to obtain

$$\tan(x_1 - x_2) = \frac{\sin(x_1 - x_2)}{\cos(x_1 - x_2)}, \quad \cos(x_1 - x_2) \neq 0.$$

From this, you find

$$\tan(x_1 - x_2) = \frac{\sin x_1 \cos x_2 - \cos x_1 \sin x_2}{\cos x_1 \cos x_2 + \sin x_1 \sin x_2}.$$

Then, if $\cos x_1 \cos x_2 \neq 0$, you can divide the numerator and denominator of the right-hand member of this identity by this product to obtain

$$\tan(x_1 - x_2) = \frac{\dfrac{\sin x_1 \cos x_2}{\cos x_1 \cos x_2} - \dfrac{\cos x_1 \sin x_2}{\cos x_1 \cos x_2}}{\dfrac{\cos x_1 \cos x_2}{\cos x_1 \cos x_2} + \dfrac{\sin x_1 \sin x_2}{\cos x_1 \cos x_2}}, \quad \cos x_1 \cos x_2 \neq 0.$$

From this and the definition of $\tan x$, you have the **formula for the tangent of a difference:**

$$\tan(x_1 - x_2) = \frac{\tan x_1 - \tan x_2}{1 + \tan x_1 \tan x_2}, \quad \begin{cases} \cos(x_1 - x_2) \neq 0 \\ \cos x_1 \neq 0,\ \cos x_2 \neq 0 \end{cases}$$

Using the fact that tangent is an odd function, you can replace x_2 with $-x_2$ in the formula for the tangent of a difference to deduce that

$$\tan[x_1 - (-x_2)] = \frac{\tan x_1 - \tan(-x_2)}{1 + \tan x_1 \tan(-x_2)},$$

from which you have the **formula for the tangent of a sum:**

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}, \quad \begin{cases} \cos(x_1 + x_2) \neq 0 \\ \cos x_1 \neq 0,\ \cos x_2 \neq 0 \end{cases}$$

You saw from the graph of tangent on page 532 that the period of tangent is π (or 180°) rather than 2π (or 360°). You can readily verify that π is a period by replacing x_1 with π and x_2 with x in the formula for $\tan(x_1 + x_2)$ to obtain

$$\tan(\pi + x) = \frac{\tan \pi + \tan x}{1 - \tan \pi \tan x} = \frac{0 + \tan x}{1 - (0)(\tan x)} = \tan x, \quad \cos x \neq 0.$$

This latter relationship can then be used to obtain a reduction formula for $\tan(\pi - x)$. If you replace x with $-x$, you have

$$\tan(\pi - x) = \tan(-x),$$

or

$$\tan(\pi - x) = -\tan x,\ \cos x \neq 0.$$

Notice that the formula for $\tan(x_1 + x_2)$ cannot be used to produce reduction formulas for tangent for values $\frac{\pi}{2} \pm x$ or $\frac{3\pi}{2} \pm x$, because $\tan \frac{\pi}{2}$ and $\tan \frac{3\pi}{2}$ are not defined. Such formulas, however, for tangent can be found directly from the definition of $\tan x$.

EXAMPLE 2 Prove that $\tan\left(\frac{\pi}{2} - x\right) = \cot x$, $\sin x \neq 0$.

SOLUTION By definition,

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x}, \quad \sin x \neq 0.$$

$$\therefore \tan\left(\frac{\pi}{2} - x\right) = \cot x, \quad \sin x \neq 0.$$

Following are reduction formulas for tangent. Three were derived above. You can prove the others (Exercises 51–53, page 563).

$$\tan\left(\frac{\pi}{2} + x\right) = -\cot x,\ \sin x \neq 0 \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x,\ \sin x \neq 0$$

$$\tan(\pi + x) = \tan x,\ \cos x \neq 0 \qquad \tan(\pi - x) = -\tan x,\ \cos x \neq 0$$

$$\tan\left(\frac{3\pi}{2} + x\right) = -\cot x,\ \sin x \neq 0 \qquad \tan\left(\frac{3\pi}{2} - x\right) = \cot x,\ \sin x \neq 0$$

Oral Exercises

State the requested value.

1. If $\sin x_1 = 0.4794$, then $\sin(-x_1) = ?$
2. If $\cos x_2 = -0.7648$, then $\cos(-x_2) = ?$
3. If $\tan x_3 = 1.557$, then $\tan(-x_3) = ?$
4. If $\sin x_4 = -0.8415$, then $\sin(\pi + x_4) = ?$
5. If $\cos x_5 = -0.2190$, then $\sin\left(\frac{\pi}{2} + x_5\right) = ?$

Written Exercises

Use the formulas for the sine and tangent of the sum or difference, and the chart on page 516, to evaluate the given function. Express your answers in radical form.

A

1. $\sin 75°$
2. $\sin 255°$
3. $\sin \frac{7\pi}{12}$
4. $\sin \frac{23\pi}{12}$
5. $\sin(-15°)$
6. $\sin 195°$
7. $\sin\left(-\frac{5\pi}{12}\right)$
8. $\sin \frac{17\pi}{12}$

Express the given function value as a function value of an angle α such that $0° \le m(\alpha) \le 90°$ or of a real number x such that $0 \le x \le \frac{\pi}{2}$.

9. $\sin \frac{7\pi}{8}$ **10.** $\sin \frac{13\pi}{16}$ **11.** $\sin 195°$ **12.** $\sin 320°$

13. $\sin\left(-\frac{7}{9}\pi\right)$ **14.** $\sin 1.7\pi$ **15.** $\sin 230°$ **16.** $\sin(-170°)$

17. $\tan \frac{13\pi}{20}$ **18.** $\tan \frac{11\pi}{8}$ **19.** $\tan 245°$ **20.** $\tan 325°$

21. $\tan \frac{12\pi}{7}$ **22.** $\tan \frac{13\pi}{9}$ **23.** $\tan(-130°)$ **24.** $\tan 290°$

25–32. Evaluate the tangents of the angles or real numbers given in Exercises 1–8 on page 562.

Use an identity to express each of the following as a function of a single angle and evaluate. Express your answer in radical form.

33. $\sin 75° \cos 15° - \cos 75° \sin 15°$

34. $\sin 80° \cos 160° + \cos 80° \sin 160°$

35. $\sin 255° \cos 75° + \cos 255° \sin 75°$

36. $\sin 345° \cos 120° - \cos 345° \sin 120°$

37. $2 \sin 105° \cos 105°$ **38.** $2 \sin 75° \cos 75°$

(*Hint for Exercises 37 and 38:* $2 \sin x \cos x = \sin x \cos x + \cos x \sin x$.)

39. $\dfrac{\tan 35° + \tan 85°}{1 - \tan 35° \tan 85°}$ **40.** $\dfrac{\tan 165° - \tan 30°}{1 + \tan 165° \tan 30°}$

41. $\dfrac{2 \tan 165°}{1 - \tan^2 165°}$ **42.** $\dfrac{2 \tan 112\frac{1}{2}°}{1 - \tan^2 112\frac{1}{2}°}$

Use the formulas for the sine and cosine of the sum or difference to prove each identity.

B **43.** $\sin\left(\frac{\pi}{2} + x\right) = \cos x$ **44.** $\sin(\pi + x) = -\sin x$

45. $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$ **46.** $\sin(\pi - x) = \sin x$

47. $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$ **48.** $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

49. $\sin(x - \pi) = -\sin x$ **50.** $\sin\left(x - \frac{3\pi}{2}\right) = \cos x$

(*Hint:* In Exercises 51–54, rewrite $\tan x$ in terms of $\sin x$ and $\cos x$.)

51. $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$ **52.** $\tan\left(\frac{3\pi}{2} + x\right) = -\cot x$

53. $\tan\left(\frac{3\pi}{2} - x\right) = \cot x$ **54.** $\tan\left(x - \frac{\pi}{2}\right) = -\cot x$

Use the definition of the function and the formulas for a sum or difference to find the given function value.

EXAMPLE $\sec 75° = \dfrac{1}{\cos 75°} = \dfrac{1}{\cos(45° + 30°)}$

$$= \frac{1}{\cos 45° \cos 30° - \sin 45° \sin 30°} = \sqrt{6} + \sqrt{2}$$

55. $\sec 165°$ **56.** $\csc 105°$ **57.** $\cot 15°$ **58.** $\csc 345°$

59. $\cot \dfrac{7\pi}{12}$ **60.** $\sec \dfrac{19\pi}{12}$ **61.** $\csc \dfrac{5\pi}{12}$ **62.** $\cot \dfrac{11\pi}{12}$

In Exercises 63–66, find $\sin(x + y)$, where x and y satisfy the given conditions.

63. $\sin x = \dfrac{3}{5}$; $0 < x < \dfrac{\pi}{2}$; $\sin y = \dfrac{5}{13}$; $0 < y < \dfrac{\pi}{2}$

64. $\sin x = \dfrac{12}{13}$; $\dfrac{\pi}{2} < x < \pi$; $\cos y = -\dfrac{4}{5}$; $\pi < y < \dfrac{3\pi}{2}$

65. $\cos x = -\dfrac{8}{17}$; $\dfrac{\pi}{2} < x < \pi$; $\sin y = -\dfrac{3}{5}$; $\dfrac{3\pi}{2} < y < 2\pi$

66. $\sin x = -\dfrac{\sqrt{3}}{2}$; $\pi < x < \dfrac{3\pi}{2}$; $\cos y = -\dfrac{1}{3}$; $\dfrac{\pi}{2} < y < \pi$

In Exercises 67–70, find $\sin(x - y)$, where x and y satisfy the given conditions.

67. $\sin x = \dfrac{7}{25}$; $0 < x < \dfrac{\pi}{2}$; $\cos y = \dfrac{3}{5}$; $0 < y < \dfrac{\pi}{2}$

68. $\sin x = \dfrac{\sqrt{5}}{3}$; $\dfrac{\pi}{2} < x < \pi$; $\sin y = \dfrac{\sqrt{3}}{2}$; $0 < y < \dfrac{\pi}{2}$

69. $\cos x = -\dfrac{\sqrt{7}}{4}$; $\pi < x < \dfrac{3\pi}{2}$; $\cos y = -\dfrac{4}{5}$; $\dfrac{\pi}{2} < y < \pi$

70. $\cos x = \dfrac{2\sqrt{2}}{3}$; $\dfrac{3\pi}{2} < x < 2\pi$; $\sin y = -\dfrac{2}{3}$; $\pi < y < \dfrac{3\pi}{2}$

Prove each of the following identities.

C **71.** $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$ **72.** $\sec(x + y) = \dfrac{\sec x \sec y}{1 - \tan x \tan y}$

73. $\csc(x + y) = \dfrac{\csc x \csc y}{\cot x + \cot y}$ **74.** $\dfrac{\sin(x - y)}{\sin y} + \dfrac{\cos(x - y)}{\cos y} = \sin x \sec y \csc y$

75. $\dfrac{\cos(x + y)}{\sin y} + \dfrac{\sin(x + y)}{\cos y} = \cos x \sec y \csc y$

15-5 Double-Angle and Half-Angle Formulas

All the identities derived in the preceding sections of this chapter result from the basic sum and difference identities:

$$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2$$
$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$$

We shall now derive still more identities.

If $x_1 = x_2 = x$, then

$$\sin(x + x) = \sin x \cos x + \sin x \cos x,$$

or

$$\sin 2x = 2 \sin x \cos x.$$

Similarly,

$$\cos(x + x) = \cos x \cos x - \sin x \sin x,$$

or

$$\cos 2x = \cos^2 x - \sin^2 x.$$

You can easily obtain two additional expressions for $\cos 2x$. Since $\sin^2 x + \cos^2 x = 1$, you have

$$\sin^2 x = 1 - \cos^2 x$$

and

$$\cos^2 x = 1 - \sin^2 x.$$

Therefore, on replacing $\sin^2 x$ with $(1 - \cos^2 x)$, in the identity for $\cos 2x$, you have

$$\cos 2x = \cos^2 x - (1 - \cos^2 x),$$

or

$$\cos 2x = 2\cos^2 x - 1.$$

On the other hand, if you replace $\cos^2 x$ with $(1 - \sin^2 x)$, you have

$$\cos 2x = (1 - \sin^2 x) - \sin^2 x,$$

or

$$\cos 2x = 1 - 2\sin^2 x.$$

Also, for all x for which $\tan x$ is defined and $|\tan x| \neq 1$,

$$\tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x},$$

or

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Corresponding identities can, of course, be written in terms of angles, and so these identities are often referred to as the **double-angle formulas** (or identities) for sine, cosine, and tangent.

Now, on replacing x with $\frac{x}{2}$ in $\cos 2x = 2\cos^2 x - 1$, you have

$$\cos x = 2\cos^2 \frac{x}{2} - 1,$$

or

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}.$$

Therefore,

$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}.$$

If you replace x with $\frac{x}{2}$ in $\cos 2x = 1 - 2\sin^2 x$, you can use a similar process to find that

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}.$$

In applying the formulas for $\sin \frac{x}{2}$ $\left(\text{or } \sin \frac{\alpha}{2}\right)$ and $\cos \frac{x}{2}$ $\left(\text{or } \cos \frac{\alpha}{2}\right)$, the choice of the positive or negative root depends on the quadrant in which $\frac{x}{2}$ $\left(\text{or } \frac{\alpha}{2}\right)$ lies. For example, if

$$0 < \frac{x}{2} < \frac{\pi}{2} \left(\text{or } 0° < m\left(\frac{\alpha}{2}\right) < 90°\right),$$

you would select the positive root for each, but if $\frac{\pi}{2} < \frac{x}{2} < \pi$ $\left(\text{or } 90° < m\left(\frac{\alpha}{2}\right) < 180°\right)$, you would select the positive root for sine and the negative root for cosine, and so on.

For tangent, if x is a value for which $\tan \frac{x}{2}$ is defined, you have

$$\tan^2 \frac{x}{2} = \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{\frac{1 - \cos x}{2}}{\frac{1 + \cos x}{2}} = \frac{1 - \cos x}{1 + \cos x},$$

from which

$$\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

Again, the choice of root is determined by the location of $\frac{x}{2}$ $\left(\text{or } \frac{\alpha}{2}\right)$.

A simpler formula for $\tan \frac{x}{2}$ $\left(\text{or } \tan \frac{\alpha}{2}\right)$ can be found by using the double-angle formulas as follows:

$$2\sin \frac{x}{2} \cos \frac{x}{2} = \sin x \qquad 2\cos^2 \frac{x}{2} = 1 + \cos x$$

Then

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}},$$

and so

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

Corresponding identities can be written in terms of angles, and so these identities are often called the **half-angle formulas** (or identities).

Oral Exercises

1. If $\sin x_1 = \frac{1}{\sqrt{3}}$ and $\cos x_1 = \frac{\sqrt{2}}{\sqrt{3}}$, find $\sin 2x_1$.
2. If $\cos x_2 = 0.8$ and $\sin x_2 = 0.6$, find $\cos 2x_2$.
3. If $\sin x_3 = -0.6$ and $\cos x_3 = -0.8$, find $\sin 2x_3$.
4. If $\tan x_4 = 0.2$, find $\tan 2x_4$.

Written Exercises

Use the half-angle formulas to find each function value.

EXAMPLE $\tan \frac{\pi}{8}$

SOLUTION Using $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$ with $x = \frac{\pi}{4}$, you have

$$\tan \frac{\left(\frac{\pi}{4}\right)}{2} = \frac{\sin \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}}.$$

Rationalizing the denominator yields

$$\tan \frac{\pi}{8} = \frac{\sqrt{2}(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1. \quad \text{Answer.}$$

A

1. $\cos 105°$
2. $\sin 112\frac{1}{2}°$
3. $\sin 165°$
4. $\tan 15°$
5. $\tan 157\frac{1}{2}°$
6. $\cos 165°$
7. $\sin \frac{\pi}{8}$
8. $\cos \frac{3\pi}{8}$
9. $\tan \frac{\pi}{12}$
10. $\sin \frac{15\pi}{8}$
11. $\cos \frac{5\pi}{12}$
12. $\tan \frac{13\pi}{12}$

Find sin 2α for the angle α in the given quadrant satisfying the given condition.

13. I; $\sin\alpha = \frac{3}{5}$

14. III; $\sin\alpha = -\frac{4}{5}$

15. II; $\cos\alpha = -\frac{12}{13}$

16. IV; $\cos\alpha = \dfrac{\sqrt{5}}{3}$

17. IV; $\sin\alpha = -\dfrac{3}{\sqrt{13}}$

18. I; $\tan\alpha = \dfrac{\sqrt{7}}{3}$

Find cos 2α for the angle α in the given quadrant satisfying the given condition.

19. I; $\cos\alpha = \dfrac{12}{13}$

20. II; $\sin\alpha = \dfrac{1}{\sqrt{5}}$

21. III; $\sin\alpha = -\dfrac{\sqrt{15}}{4}$

22. II; $\cos\alpha = -\dfrac{1}{\sqrt{10}}$

23. IV; $\cos\alpha = \dfrac{2}{3}$

24. I; $\tan\alpha = \dfrac{1}{5}$

Find sin $\dfrac{\alpha}{2}$ and cos $\dfrac{\alpha}{2}$ for the angle α in the given quadrant satisfying the given condition. Make a sketch of the angles α and $\dfrac{\alpha}{2}$ to decide the sign of the answer.

25. I; $\cos\alpha = \frac{7}{25}$

26. II; $\cos\alpha = -\frac{7}{25}$

27. I; $\sin\alpha = \frac{4}{5}$

28. II; $\sin\alpha = \frac{4}{5}$

29. III; $\cos\alpha = -\frac{15}{17}$

30. IV; $\sin\alpha = -\frac{8}{17}$

Use the half-angle and double-angle formulas to verify each of the following.

B 31. $\sin\left[2\left(\dfrac{x}{2}\right)\right] = \sin x$

32. $\cos\left[2\left(\dfrac{x}{2}\right)\right] = \cos x$

33. $\sin^2\dfrac{x}{2} + \cos^2\dfrac{x}{2} = 1$

34. $\sin^2 2x + \cos^2 2x = 1$

35. $1 + \tan^2\dfrac{x}{2} = \sec^2\dfrac{x}{2}$ $\left(\textit{Note: } \sec\dfrac{x}{2} = \dfrac{1}{\cos\dfrac{x}{2}}.\right)$

C 36. Show that if $\sin\alpha = \cos\beta$, where both α and β are in quadrant I, then $\sin 2\alpha = \sin 2\beta$. Make a sketch of the angles α, β, 2α, and 2β showing how this is possible.

15-6 More on Identities

The following is a list of useful identities to which you can refer as needed. These identities hold for all values of the variables for which the function values are defined. Corresponding identities hold in terms of angles under similar restrictions.

Summary of Useful Identities

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Reciprocal Identities

$$\sin x \csc x = 1$$
$$\cos x \sec x = 1$$
$$\tan x \cot x = 1$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Negatives

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Sum and Difference Identities

$$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2$$
$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$$
$$\tan(x_1 \pm x_2) = \frac{\tan x_1 \pm \tan x_2}{1 \mp \tan x_1 \tan x_2}$$

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Half-Angle Identities

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$
$$= \frac{\sin x}{1 + \cos x}$$

Written Exercises

Prove each identity.

A

1. $\dfrac{\cos 2\alpha}{\cos^2 \alpha} = 1 - \tan^2 \alpha$
2. $\cos^2 \alpha \tan \alpha = \dfrac{\sin 2\alpha}{2}$
3. $(\sin \alpha + \cos \alpha)^2 = 1 + \sin 2\alpha$
4. $\cos^2 \alpha(1 - \tan^2 \alpha) = \cos 2\alpha$
5. $(1 - \tan \alpha)^2 = \sec^2 \alpha(1 - \sin 2\alpha)$
6. $\sin^2 2\alpha = 4\sin^2 \alpha - 4\sin^4 \alpha$
7. $\cos \alpha - \sin \alpha \tan \alpha = \cos 2\alpha \sec \alpha$
8. $\dfrac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha$

Prove each identity.

9. $\dfrac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \cos 2\alpha$

10. $2 \cot 2\alpha = \cot \alpha - \tan \alpha$

$\left(\textit{Hint: } \cot 2\alpha = \dfrac{\cos 2\alpha}{\sin 2\alpha}.\right)$

11. $\cos^4 \alpha - \sin^4 \alpha = \cos 2\alpha$

12. $\dfrac{1}{\csc \alpha + \cot \alpha} = \tan \dfrac{\alpha}{2}$

B 13. $\cos^2 \dfrac{\alpha}{2} \tan \dfrac{\alpha}{2} = \dfrac{\sin \alpha}{2}$

14. $\cot \dfrac{\alpha}{2} = \dfrac{\sin \alpha}{1 - \cos \alpha}$

15. $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$

16. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$

17. $\cos 4\alpha = \cos^4 \alpha + \sin^4 \alpha - 6 \sin^2 \alpha \cos^2 \alpha$

18. $\sin 4\alpha = 4 \sin \alpha \cos \alpha - 8 \sin^3 \alpha \cos \alpha$

19. $\tan \dfrac{\alpha}{2} = \csc \alpha - \cot \alpha$

20. $\dfrac{\cos \alpha}{\cos \alpha - \sin \alpha} - \dfrac{\cos \alpha}{\cos \alpha + \sin \alpha} = \tan 2\alpha$

21. $\dfrac{\cos \alpha}{\cos \alpha + \sin \alpha} - \dfrac{\sin \alpha}{\cos \alpha - \sin \alpha} = 1 - \tan 2\alpha$

22. $\sin^2 \alpha \tan \dfrac{\alpha}{2} = \sin \alpha - \dfrac{\sin 2\alpha}{2}$

23. $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$

24. $\cos (\alpha + \beta) \cos (\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta$

The following are sometimes called the *sum and product identities*.

25. $\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta$

26. $\sin (\alpha + \beta) - \sin (\alpha - \beta) = 2 \cos \alpha \sin \beta$

27. $\cos (\alpha + \beta) + \cos (\alpha - \beta) = 2 \cos \alpha \cos \beta$

28. $\cos (\alpha + \beta) - \cos (\alpha - \beta) = -2 \sin \alpha \sin \beta$

C 29. $\sin A + \sin B = 2 \sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(*Hint:* Let $A = x_1 + x_2$ and $B = x_1 - x_2$ and use Exercise 25.)

30. $\sin A - \sin B = 2 \cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

31. $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

32. $\cos A - \cos B = -2 \sin \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

Self-Test 2

VOCABULARY reduction formula (p. 555) half-angle formula (p. 567)
double-angle formula (p. 565)

1. Reduce to a function value of an angle in quadrant I. *Obj. 1, p. 551*

a. $\sin 315°$ **b.** $\cos \dfrac{8\pi^R}{3}$

2. Use a sum formula to evaluate $\sin 345°$. *Obj. 2, p. 551*

3. Find $\cos 2\alpha$ if $\sin \alpha = \frac{2}{3}$ and α is in quadrant II.

4. Prove that $\csc 2\alpha = \dfrac{\csc \alpha}{2 \cos \alpha}$ is an identity with suitable restrictions on $\cos \alpha$. *Obj. 3, p. 551*

Check your answers with those at the back of the book.

Solving General Triangles

***OBJECTIVES** for Sections 15-7 and 15-8:*

1. Apply the law of cosines and the law of sines to solve general triangles.

2. Apply the law of cosines and the law of sines to solve simple practical problems.

15-7 The Law of Cosines

The Pythagorean Theorem enables you to find the length of one side of a *right* triangle when you know the lengths of the other two sides. To find a more general relationship between the lengths of the three sides of *any* triangle, look at Figure 10. In Figure 10a, $\angle BCA$ is in standard position, and so the coordinates of point A are $(b \cos C, b \sin C)$. By the distance formula then,

$$c = \sqrt{(b \cos C - a)^2 + (b \sin C - 0)^2}.$$

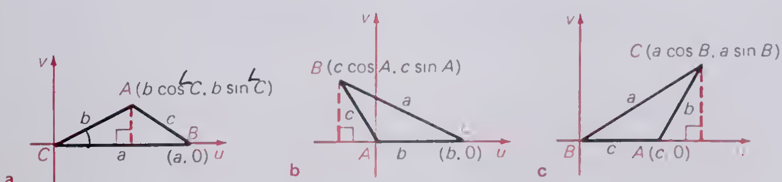


Figure 10

Squaring and simplifying the right-hand member, you have

$$\begin{aligned} c^2 &= b^2\cos^2 C - 2ab\cos C + a^2 + b^2\sin^2 C \\ &= a^2 + b^2(\cos^2 C + \sin^2 C) - 2ab\cos C, \end{aligned}$$

or, since $\cos^2 C + \sin^2 C = 1$,

$$c^2 = a^2 + b^2 - 2ab\cos C. \qquad (1)$$

By reorienting the axes so that angles A and B are, in turn, in standard position (See Figures 10b and 10c on page 571.), you can obtain the analogous relationships:

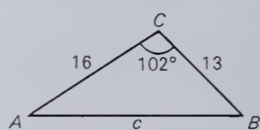
$$b^2 = a^2 + c^2 - 2ac\cos B \qquad (2)$$
$$a^2 = b^2 + c^2 - 2bc\cos A \qquad (3)$$

These three relationships together are the **law of cosines.**

Notice that if one of the angles of the triangle, say C, is a right angle, then $\cos C = 0$ and you have $c^2 = a^2 + b^2$, which is the Pythagorean Theorem. Thus, the Pythagorean Theorem is a "special case" of the law of cosines.

EXAMPLE Solve the triangle pictured, assuming the data to be exact.

SOLUTION $\cos C = \cos 102° = -\cos(180 - 102)° = -\cos 78°$. From Table 6 at the back of the book, $-\cos 78° \approx -0.2079$. Then, using the law of cosines, you have



$$\begin{aligned} c^2 &\approx (16)^2 + (13)^2 - 2(16)(13)(-0.2079) \\ &\approx 256 + 169 + 86.5 = 511.5. \end{aligned}$$
$$\therefore\ c \approx \sqrt{511.5}$$

From Table 3 you find that to the nearest unit, $c \approx 23$. To find $m(A)$, you can use

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

$0 = b^2 + c^2 - a^2$

$+2bc\cos A =$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

from which

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{256 + 511.5 - 169}{736} = \frac{598.5}{736} \approx 0.8132$$

$\therefore\ m(A) \approx 36°$.

Since in any triangle, $m(A) + m(B) + m(C) = 180°$, you have

$$36° + m(B) + 102° = 180° \text{ or } m(B) \approx 42°.$$

$\therefore\ c \approx 23,\ m(A) \approx 36°$, and $m(B) \approx 42°$. Answer.

Notice that solving (3) for $\cos A$ gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \qquad (4)$$

which, as illustrated in the Example above, can be used to find $m(A)$,

when a, b, and c are given. Similarly, solving (1) and (2) for $\cos C$ and $\cos B$ gives formulas for finding $m(C)$ and $m(B)$, respectively.

Written Exercises

Find the length of c to the nearest tenth in $\triangle ABC$.

A

1. $a = 6$, $b = 16$, $m(C) = 60°$
2. $a = 3\sqrt{2}$, $b = 9$, $m(C) = 45°$
3. $a = 13$, $b = 8\sqrt{3}$, $m(C) = 30°$
4. $a = 4$, $b = 3$, $m(C) = 60°$
5. $a = 5$, $b = 10$, $m(C) = 47°$
6. $a = 25$, $b = 10$, $m(C) = 70°$

In $\triangle ABC$ find $m(A)$ to the nearest $10'$.

7. $a = 2$, $b = 3$, $c = 4$
8. $a = 5$, $b = 10$, $c = 7$
9. $a = 5$, $b = 8$, $c = 12$
10. $a = 4$, $b = 9$, $c = 12$

Solve each triangle. Give measurements to the nearest unit of length and the nearest degree.

B

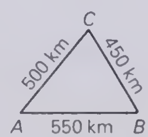
11. $a = 4$, $b = 10$, $m(C) = 59°$
12. $a = 25$, $c = 5$, $m(B) = 38°$
13. $b = 8$, $c = 9$, $m(A) = 60°$
14. $a = 15$, $b = 8$, $m(C) = 63°$
15. $a = 4$, $b = 7$, $c = 8$
16. $a = 12$, $b = 7$, $c = 9$
17. Show that one angle of a triangle with sides of lengths 5, 7, and 8 has the same measure as one angle of a triangle whose sides have lengths 3, 5, and 7.

Problems

In each problem, give lengths to the nearest tenth of a unit and angles to the nearest $10'$.

A

1. How far apart are the outer ends of the hour and minute hands of a clock at two o'clock if the hour hand is 10 cm long and the minute hand is 16 cm long?
2. Two ships leave from the same point at the same time on courses 32° apart at speeds of 15 km/h and 18 km/h, respectively. How far apart are the ships after 1 h?
3. Cities A, B, and C are located in the diagram at the right. If two airplanes take off from city A flying toward cities B and C, respectively, by what angle do their headings differ?

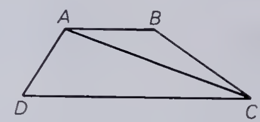


4. Two adjacent sides of a parallelogram form an acute angle and have lengths of 10 cm and 12 cm, respectively. If the shorter diagonal of the parallelogram has length 7 cm, what is the angle between the sides?

5. A rear windshield wiper of length 30 cm (distance from the pivot to the outer end of the blade) sweeps out an angle of 120°. How far apart are the two outermost corners of the swept-out area?

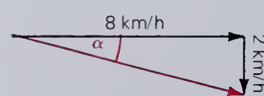
6. From a point near one end of a pond, the angle determined by a surveyor's transit and points A and B on opposite sides of the pond measures 28°. The transit is located 200 m from point A and 180 m from point B. How wide is the pond?

B 7. Points A and B and points C and D are the vertices of a trapezoid, in which the length of $\overline{AD} = 5$, the length of $\overline{CD} = 21$, and the length of $\overline{AB} = 10$. If $m\angle ADC = 60°$ and $m\angle DCA = 13°10'$, use the law of cosines to find the length of $\overline{BC}$.



8. Each of the two diagonals from one vertex of a regular pentagon has length 40 cm. What is the length of a side of the pentagon? (*Note:* The degree measure of each interior angle of a regular pentagon is 108.)

C 9. Noralie Cox, whose rowing speed in still water is 8 km/h, is rowing due east across a current heading south with a speed of 2 km/h. The diagram at right can be used to determine her speed and the angle by which her course diverges from an east-west line. Find the length of the hypotenuse to determine her speed of travel, and find the degree measure of $\angle\alpha$.



15-8 The Law of Sines

As noted in Section 15-7, by choosing the coordinate system appropriately, you can position any triangle ABC so that any one of the angles A,

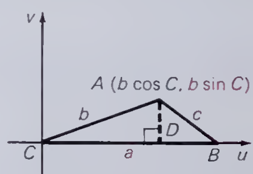


Figure 11

B, or C is in standard position. Because the area of a triangle is equal to one-half the product of the length of a base and the corresponding altitude, you can express the area of triangle ABC in Figure 11 by

$$\textbf{Area} = \tfrac{1}{2}ab \sin C.$$

Similarly, with angles A or B in standard position, we get

$$\textbf{Area} = \tfrac{1}{2}bc \sin A \qquad \text{and} \qquad \textbf{Area} = \tfrac{1}{2}ca \sin B.$$

Thus, in general, we have the following.

Theorem. The area of a triangle is equal to one-half the product of the lengths of two sides and the sine of their included angle.

EXAMPLE 1 Find the area of a triangle in which $a = 8$, $b = 14$, and $m(C) = 23°10'$.

SOLUTION From Table 6, $\sin 23°10' \approx 0.3934$, so that

$$\text{Area} = \tfrac{1}{2}ab \sin C \approx \tfrac{1}{2}(8)(14)(0.3934) \approx 22.03. \quad \text{Answer.}$$

Since each of the three expressions for the area of triangle ABC represents the same number, you know that

$$\tfrac{1}{2}bc \sin A = \tfrac{1}{2}ac \sin B = \tfrac{1}{2}ab \sin C.$$

Dividing each member of this compound sentence by $\frac{1}{2}abc$ produces

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This relationship is formalized in the following theorem.

Law of Sines

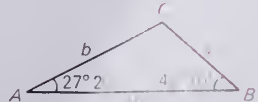
The sines of the angles of a triangle are proportional to the lengths of the opposite sides.

Notice that if one of the angles of the triangle, say C, is a right angle, then $\sin C = 1$, and the law of sines yields the familiar right-triangle relationships

$$\sin A = \frac{a}{c} \quad \text{and} \quad \sin B = \frac{b}{c}.$$

EXAMPLE 2 Solve the triangle pictured.

SOLUTION Since $m(A) + m(B) + m(C) = 180°$, you have



$$27°20' + 48°10' + m(C) = 180°$$

or $$m(C) = 104°30'.$$

From the law of sines:

$$\frac{\sin 27°20'}{a} = \frac{\sin 48°10'}{b} = \frac{\sin 104°30'}{40}$$

(Solution continued on page 576.)

Solving separately for a and b, you find:

$$\frac{\sin 27°20'}{a} = \frac{\sin 104°30'}{40}$$

$$a = \frac{40 \sin 27°20'}{\sin 104°30'}$$

$$\approx \frac{40(0.4592)}{0.9681} \approx 18.97$$

$$\frac{\sin 48°10'}{b} = \frac{\sin 104°30'}{40}$$

$$b = \frac{40 \sin 48°10'}{\sin 104°30'}$$

$$\approx \frac{40(0.7451)}{0.9681} \approx 30.79$$

$\therefore\ m(C) = 104°30'$, $a \approx 18.97$, and $b \approx 30.79$. Answer.

The law of sines gives you a way to solve any triangle if you know

(a) the measurements of two angles and the length of any one side, as illustrated in Example 2, or
(b) the lengths of two sides and the measurement of an angle opposite one of them—insofar as there is a solution.

In the latter case, the data may be ambiguous. Figures 12 and 13 illustrate the possibilities for a triangle when you are given a, b, and $m(A)$.

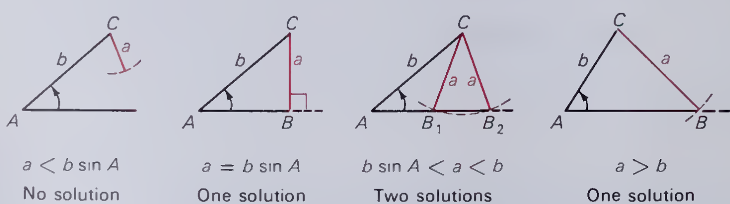


Figure 12

$m(A) < 90°$

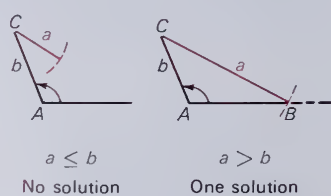
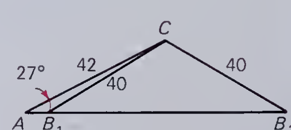


Figure 13

$90° \leq m(A) < 180°$

EXAMPLE 3 Solve the triangle for which $a = 40$, $b = 42$, and $m(A) = 27°$.



SOLUTION A sketch of the triangle, roughly to scale, suggests that there are two solutions. This can be verified by noting that $b \sin A \approx 42(0.4540) \approx 19.07$ and $19.07 < 40 < 42$ (see Figure 12, part 3).

To find $m(B)$, we use

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \sin B = \frac{b \sin A}{a} \approx \frac{19.07}{40} \approx 0.4768.$$

$\therefore$ either $m(B) \approx 28°28'$ or $m(B) \approx 180° - 28°28' = 151°32'$.

If $m(B) = 28°28'$, then $m(C) = 180° - 27° - 28° 28' = 124°32'$.
If $m(B) = 151°32'$, then $m(C) = 180° - 27° - 151°32' = 1°28'$.

From

$$\frac{\sin A}{a} = \frac{\sin C}{c},$$

$$\text{or} \quad c = \frac{a \sin C}{\sin A},$$

taking $m(C) = 124°32'$, you find that $c \approx \dfrac{40(0.8238)}{0.4540} \approx 72.58$, and

taking $m(C) = 1°28'$, you find that $c \approx \dfrac{40(0.0256)}{0.4540} \approx 2.26$.

$\therefore$ the two possible triangles have the following parts:

$m(B) \approx 28°28'$, $m(C) \approx 124°32'$, $c \approx 72.58$;

and $m(B) \approx 151°32'$, $m(C) \approx 1°28'$, $c \approx 2.26$. Answer.

Written Exercises

Solve the $\triangle ABC$ with the given parts. Give lengths to the nearest tenth of a unit and angles to the nearest 10′. If no solution exists, so state. If two solutions exist, give both.

A

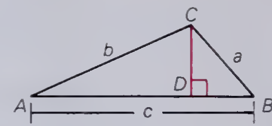
1. $m(A) = 30°$, $a = 20$, $m(B) = 36°$
2. $m(B) = 30°$, $m(C) = 63°$, $b = 150$
3. $a = 50$, $b = 32$, $m(A) = 120°$
4. $a = 40$, $c = 15$, $m(C) = 30°$ no Solution
5. $m(A) = 13°$, $m(B) = 130°$, $a = 90$
6. $m(A) = 30°$, $a = 18$, $c = 25$
7. $a = 95$, $m(B) = 30°40'$, $m(C) = 127°$
8. $a = 13$, $b = 20$, $m(A) = 57°40'$ no solution
9. $m(B) = 31°$, $a = 40$, $b = 50$
10. $m(C) = 27°$, $a = 26$, $c = 40$

For each triangle ABC with the given parts, find the area to the nearest square unit. If no triangle exists, so state. If two triangles exist, give the area of each.

11. $a = 6$, $b = 7$, $m(C) = 30°$
12. $b = 25$, $c = 15$, $m(A) = 41°50'$
13. $m(C) = 90°$, $b = 4$, $a = 3$
14. $m(B) = 75°10'$, $a = 20$, $c = 30$
15. $a = 40$, $b = 30$, $m(B) = 30°$
16. $a = 35$, $b = 18$, $m(B) = 20°$

(*Hint for Exercises 15 and 16:* Use the law of sines to find $m(C)$.)

In Exercises 17–19, use the diagram at right.



B 17. Find an expression for the length of $\overline{CD}$ in terms of b and a trigonometric function of $\angle A$.

18. Find an expression for the length of $\overline{CD}$ in terms of a and a trigonometric function of $\angle B$.

19. Use the results of Exercises 17 and 18 above to prove the first part of the law of sines:

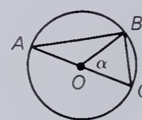
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

20. Use a diagram similar to the one above to prove $\frac{\sin B}{b} = \frac{\sin C}{c}$.

21. Use the fact that the area of a triangle can be given by Area $= \frac{1}{2}bc \sin A$ to prove the formula: Area $= \frac{b^2 \sin A \sin C}{2 \sin B}$.

22. Show that if a, $m(B)$, and $m(C)$ are given in $\triangle ABC$, then b can be found from the equation $\frac{\sin (B + C)}{a} = \frac{\sin B}{b}$.

23. In the diagram at the right, $\overline{AC}$ is a diameter of circle O. Show that for any angle α, the area of $\triangle AOB$ = the area of $\triangle BOC$.



Problems

Give lengths to the nearest tenth of a unit and angles to the nearest 10′.

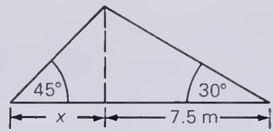
A 1. Find the area of a parallelogram with adjacent sides of length 10 cm and 25 cm and with an angle of measure 137° included between the sides.

2. Find the area of a regular 10-sided polygon inscribed in a circle of radius 50 cm.

3. The angle of elevation of a rocket measured from one tracking station is 79°, and 88° measured from a second station. The second station is 12.5 km from the first and lies on a line through the first station and the launch pad. How far was the rocket from the second station when the angles were observed?

4. Otis Washington starts out hiking from a camp at point A on a course 31° from the line through A and a town at point B 10 km away. He heads for a ridge at point C intending to change direction there and head for the town, but when he arrives at C he finds that the trail marker saying "12 km to town B" has fallen down. By what angle should he now turn in order to head straight for B?

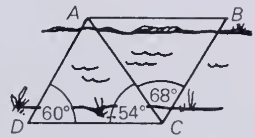
5. The pitches of the front and back roofs of a saltbox house are 45° and 30°, respectively. If the width of the back roof is 7.5 m, what is the width of the front roof?



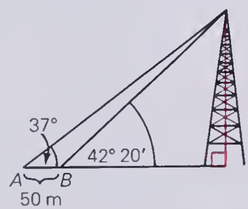
6. A clock pendulum swings through an angle of 20°. If the distance between the two extreme positions of the bob at the end of the shaft is 25 cm, how long is the shaft?

B 7. What is the acute angle between the diagonals of a rectangle if the area of the rectangle is 304 cm^2 and each diagonal has length 40 cm?

8. Sue Philips wants to know the distance between two points A and B on the opposite side of a river from her position. She finds that the length of $\overline{CD} = 91$ m, $m\angle ADC = 60°$, $m\angle ACD = 54°$, and $m\angle ACB = 68°$. What is the distance from A to B if A, B, C, and D are the vertices of a trapezoid?



C 9. Points A and B lie along a line with the base of a radio tower, and point B is 50 m closer to the tower than point A. The angle of elevation of the top of the tower measured from point A is 37°, and its angle of elevation measured from point B is 42°20′. Find the height of the tower.



Self-Test 3

VOCABULARY law of cosines (p. 572) law of sines (p. 575)

In $\triangle ABC$, find a to the nearest tenth of a unit.

1. $b = 7$, $c = 8$, $m(A) = 83°20'$ *Obj. 1, p. 571*
2. $m(A) = 27°$, $m(B) = 63°$, $b = 102$
3. Find the area of $\triangle ABC$ if $a = 10$, $b = 25$, $m(C) = 24°50'$.
4. Find the length of a side of a regular 9-sided polygon inscribed in a circle of radius 50 cm. *Obj. 2, p. 571*
5. A ship is sighted from two radar stations 125 km apart. The angle between the line segment joining the two stations and the radar beam of the first station is 43°, and between this segment and the beam from the second station is 107°. How far is the ship from the second station?

Check your answers with those at the back of the book.

Chapter Summary

1. You use the *fundamental trigonometric identities* together with properties of real numbers to establish additional identities.
2. The formula $\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2$ is the *formula for the cosine of a difference.* You can use this formula to derive *reduction formulas* and formulas for values of functions of sums and then for *double-* and *half-angle formulas.*
3. If you know the measures of three parts of a triangle, including the length of at least one side, you can find the measures of the remaining parts by using the *law of cosines,* $c^2 = a^2 + b^2 - 2ab \cos C$, or the *law of sines,* $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

   When given data for a triangle consist of the lengths of two sides and the measure of an angle opposite one of them, there may exist no, one, or two triangles with the given measurements.
4. The area of a triangle is equal to one-half the product of the lengths of two sides and the sine of their included angle.

Chapter Review

1. Rewrite $\dfrac{1}{\csc^2 \alpha}$ in terms of $\sin \alpha$. *15-1*

   a. $\dfrac{1}{\sin^2 \alpha}$ b. $\dfrac{1}{1 - \sin^2 \alpha}$ c. $\sin^2 \alpha$ d. $-\sin^2 \alpha$

2. Simplify $\dfrac{1}{\sec \alpha}(\tan \alpha + \cot \alpha)$ assuming $\sin \alpha \neq 0$, $\cos \alpha \neq 0$.

   a. $\csc \alpha$ b. 1 c. $\cos \alpha$ d. $\sec \alpha$

3. Which one of the following is equal to $\sin x \cos x + \sin^2 x \tan x$, assuming $\cos x \neq 0$? *15-2*

   a. $\cos x$ b. $\sec^2 x$ c. $\csc x$ d. $\tan x$

4. Express $\sin 105°$ as a function value of an angle α, $0° < m(\alpha) < 90°$. *15-3*

   a. $\cos 45°$ b. $\sin 75°$ c. $-\sin 15°$ d. $-\sin 45°$

5. Evaluate $\cos \dfrac{3\pi}{4} \cos \dfrac{\pi}{4} - \sin \dfrac{3\pi}{4} \sin \dfrac{\pi}{4}$. (First express it as the cosine of a single real number.)

   a. -1 b. 1 c. 0 d. $-\dfrac{\sqrt{2}}{2}$

6. If $\sin x = \frac{4}{5}$ and $\cos y = -\frac{12}{13}$, find $\sin(x - y)$. $\left(0 < x < \dfrac{\pi}{2}, \dfrac{\pi}{2} < y < \pi\right)$ *15-4*

   a. $-\frac{63}{65}$ b. $\frac{63}{65}$ c. $\frac{33}{65}$ d. $-\frac{33}{65}$

7. Which expression equals $\tan(x_1 + x_2)$?

a. $\dfrac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}$ b. $\dfrac{\tan x_1 - \tan x_2}{1 + \tan x_1 \tan x_2}$ c. $\dfrac{\tan x_1 - \tan x_2}{1 - \tan x_1 \tan x_2}$

X 8. Which expression equals $\cos \dfrac{x}{2}$? 15-5

a. $\pm\sqrt{\dfrac{1 + \cos x}{2}}$ b. $\pm\sqrt{\dfrac{1 - \cos x}{2}}$ c. $\pm\sqrt{\dfrac{1 - \sin x}{2}}$

9. Which expression equals $\tan 2x$?

a. $\dfrac{2 \tan x}{1 + \tan^2 x}$ b. $\dfrac{2 \tan x}{1 - \tan^2 x}$ c. $\dfrac{2 \sin x}{1 - \tan^2 x}$

In Review Items 10 and 11, answer (a) true or (b) false according to whether the given identity is true or false.

10. $\cos 2\alpha = 1 - 2 \sin^2 \alpha$ 11. $1 + \cot^2 x = \sec^2 x$ 15-6

12. Which is the correct expression of the law of cosines for $\triangle ABC$? 15-7

a. $b^2 = a^2 + c^2 - 2ab \cos C$ b. $b^2 = a^2 + c^2 - 2ac \cos B$

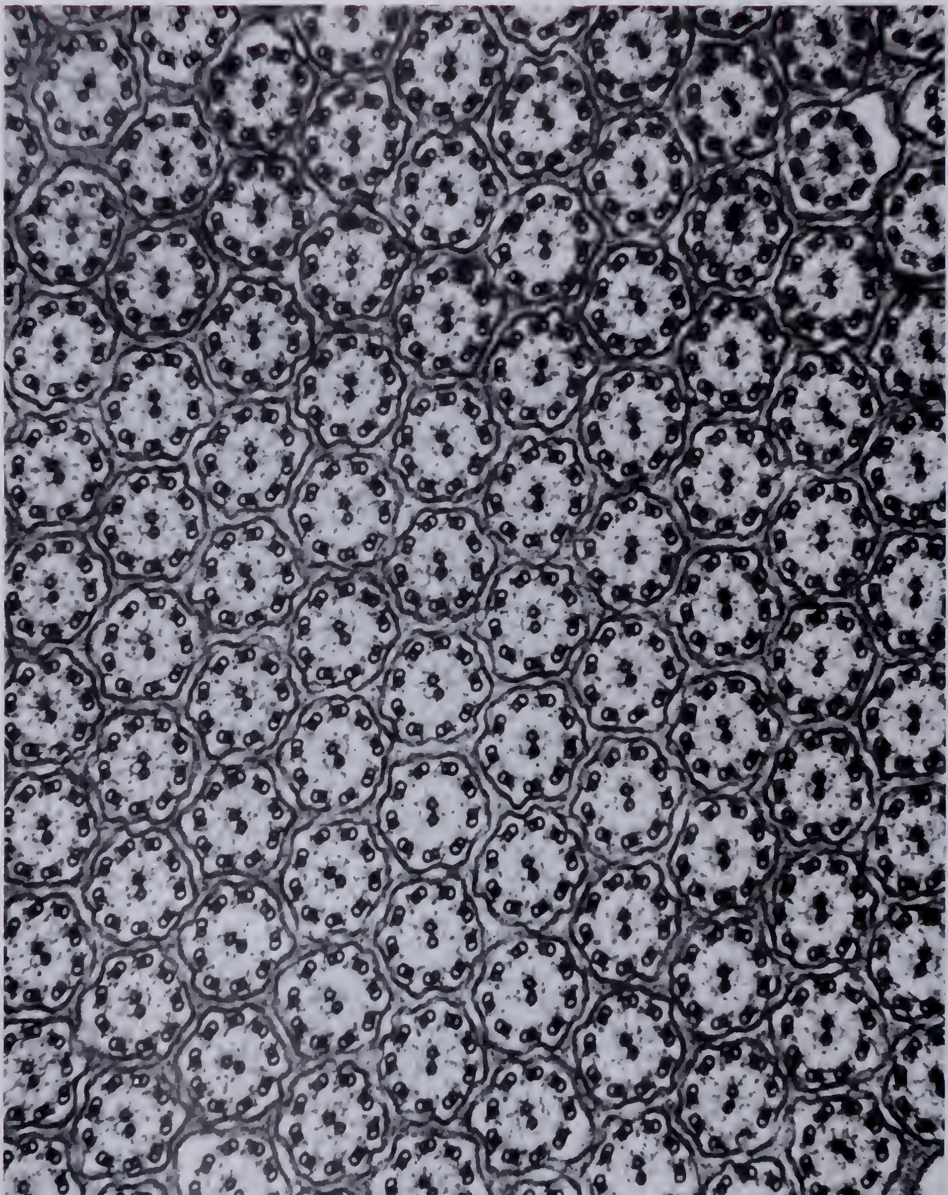
c. $b^2 = a^2 + c^2 - 2ab \cos A$ d. $b^2 = a^2 + b^2 - 2ac \cos B$

13. In $\triangle ABC$, if $\sin A = 0.8$, $\sin B = 0.2$, and $a = 4$, find b. 15-8

a. 0.8 b. 1.6 c. 1 d. 16

Chapter Test

1. Express $\cot^2 \alpha$ in terms of $\tan \alpha$. 15-1
2. Prove that $(1 - \sin \alpha) \sec \alpha = \dfrac{\cos \alpha}{1 + \sin \alpha}$ and state any necessary restrictions. 15-2
3. Express $\cos 105°$ as a function of a first-quadrant angle. 15-3
4. Use the formula for the sine of a sum to evaluate $\sin 75°$. 15-4
5. Use the formula for $\tan 2x$ to evaluate $\tan 2x_1$ when $\tan x_1 = 0.6$. 15-5
6. Given $\sin x \doteq 0.6$ and $\pi > x > \dfrac{\pi}{2}$, find $\sin \dfrac{x}{2}$, $2x$
7. Prove that $\dfrac{2}{\sin 2\alpha} = \sec \alpha \csc \alpha$ and state any necessary restrictions. 15-6
8. In triangle ABC, if $a = 15$, $b = 30$, and $m(C) = 60°$, find c. 15-7
9. In triangle ABC, if $m(A) = 45°$, $m(B) = 60°$, and $a = 30$, find b. 15-8



This photomicrograph shows a cross section of "cilia," which are small hair-like organelles found in both plants and animals. The black loops are small filaments which expand and contract to cause the cilia to move.

16
Inverses; Polar Coordinates

Inverse Functions

OBJECTIVES for Sections 16-1 and 16-2:
1. ***Find values of inverse trigonometric and circular functions.***
2. ***Solve simple trigonometric sentences.***

16-1 Inverse Functions; Principal Values

You find the inverse of the function specified by

$$\{(x, y)\colon\ y = \sin x\}$$

by interchanging the x and the y in the defining equation, thus obtaining the relation specified by

$$\{(x, y)\colon\ x = \sin y\}.$$

Both relations are graphed in Figure 1. The inverse relation is denoted

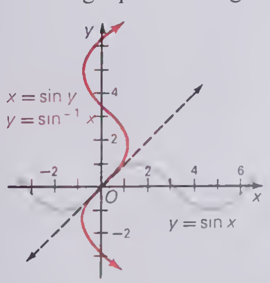


Figure 1

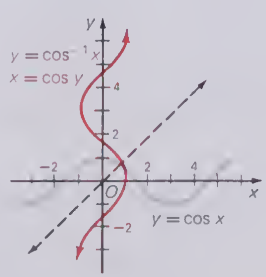


Figure 2

either by $\{(x, y)\colon y = \sin^{-1} x\}$ (read "y = the inverse sine of x") or by $\{(x, y)\colon y = \arcsin x\}$ (read "y = the arcsine of x"). Notice that the inverse relation $\{(x, y)\colon y = \sin^{-1} x\}$ is *not* a function.

Inverses of the other circular functions that you have studied are defined similarly:

$$\{(x, y)\colon y = \cos^{-1} x\} \qquad \{(x, y)\colon y = \tan^{-1} x\}$$
$$\{(x, y)\colon y = \cot^{-1} x\} \qquad \{(x, y)\colon y = \sec^{-1} x\}$$
$$\{(x, y)\colon y = \csc^{-1} x\}$$

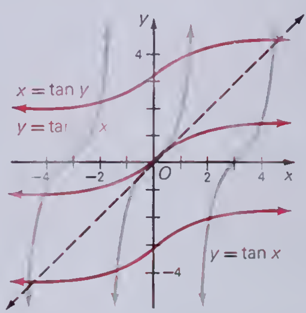


Figure 3

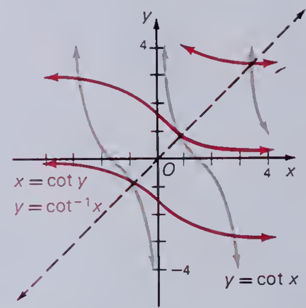


Figure 4

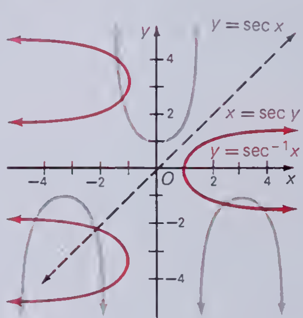


Figure 5

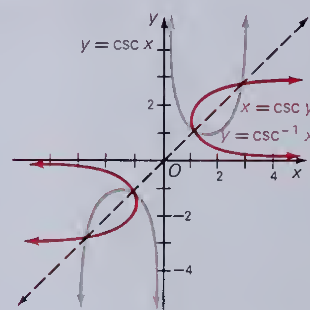


Figure 6

You can see from the graphs in Figures 1 through 6 that none of these inverse relations is a function.

By suitably restricting the ranges of the inverses of the circular and trigonometric functions, however, you can define for each such function any number of inverse functions. For example, as suggested by Figure 7, if the range is restricted to a suitable interval of length π, the resulting relation will then be a function.

According to custom, the **principal-value inverse function for sine** is defined for real numbers to be

$$\left\{(x, y)\colon x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$$

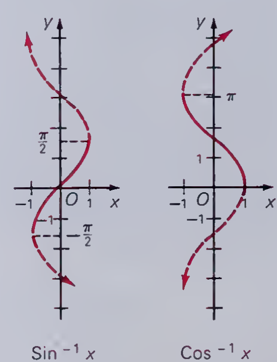


Figure 7

and is denoted by **Arcsin** or **Sin^{-1}**, with capital letters, and

read "principal arcsine of x" and "principal inverse sine of x," respectively. In terms of angles, you have:

$$\textbf{Arcsin} = \textbf{Sin}^{-1} = \{(x, \alpha)\colon x = \sin \alpha \text{ and } -90° \leq m(\alpha) \leq 90°\}.$$

The domain of Sin^{-1} is $\{x\colon |x| \leq 1\}$.

On the other hand, it is customary to define the **principal-value inverse function for cosine** to be

$$\textbf{Arccos} = \textbf{Cos}^{-1} = \{(x, y)\colon x = \cos y \text{ and } 0 \leq y \leq \pi\}.$$

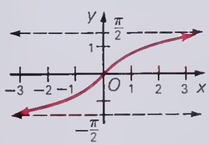
In terms of angles, you have

$$\textbf{Arccos} = \textbf{Cos}^{-1} = \{(x, \alpha)\colon x = \cos \alpha \text{ and } 0° \leq m(\alpha) \leq 180°\}.$$

The domain of Cos^{-1} is also $\{x\colon |x| \leq 1\}$.

The graphs and definitions of the remaining principal-value inverses for the circular functions are shown in Figures 8 through 11.

Figure 8

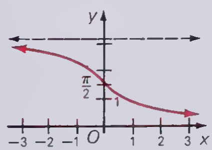


$\{(x, y)\colon x = \tan y\}$

Domain $= \mathcal{R}$

Range $= \left\{y\colon -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

Figure 9

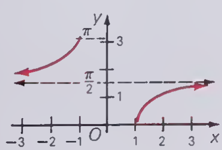


$\{(x, y)\colon x = \cot y\}$

Domain $= \mathcal{R}$

Range $= \{y\colon 0 < y < \pi\}$

Figure 10

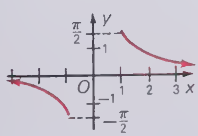


$\{(x, y)\colon x = \sec y\}$

Domain $= \{x\colon |x| \geq 1\}$

Range $= \left\{y\colon 0 \leq y \leq \pi, y \neq \frac{\pi}{2}\right\}$

Figure 11



$\{(x, y)\colon x = \csc y\}$

Domain $= \{x\colon |x| \geq 1\}$

Range $= \left\{y\colon -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0\right\}$

When $0, \frac{\pi}{2}, \pi$, etc., are replaced with $0°, 90°, 180°$, etc., and y with $m(\alpha)$, in the foregoing definitions, the results will define the principal-value inverse functions for the corresponding trigonometric functions.

EXAMPLE 1 a. Specify the members of $\{y: \cos y = -\frac{1}{2}, y \in \mathcal{R}\}$.
b. Find $\text{Cos}^{-1}(-\frac{1}{2})$ for both the inverse circular and the inverse trigonometric function.

SOLUTION a. First, find all values of y such that $\cos y = -\frac{1}{2}$ and $0 \leq y \leq 2\pi$. These are the values $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ because

$$\cos\frac{2\pi}{3} = -\frac{1}{2} \text{ and } \cos\frac{4\pi}{3} = -\frac{1}{2}.$$

$\therefore \{y: \cos y = -\frac{1}{2}, y \in \mathcal{R}\} =$

$$\left\{y: y = \frac{2\pi}{3} + 2k\pi\right\} \cup \left\{y: y = \frac{4\pi}{3} + 2k\pi\right\},$$

where $k \in$ {the integers}. Answer.

b. For Cos^{-1} the range is $0 \leq y \leq \pi$, or $0° \leq m(\alpha) \leq 180°$, and thus

$$\text{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \text{ or } \text{Cos}^{-1}\left(-\frac{1}{2}\right) = 120°. \quad \text{Answer.}$$

Throughout the remainder of this chapter, the variable k *will always have {the integers} as its replacement set.*

Sometimes you must use a table of function values to approximate elements in the range of an inverse relation.

EXAMPLE 2 a. Specify the members of $\{y: \sin y = \frac{7}{8}, y \in \mathcal{R}\}$ to the nearest hundredth.
b. Find $\text{Sin}^{-1}\frac{7}{8}$ for both the inverse circular and inverse trigonometric function.

SOLUTION a. Using $\frac{7}{8} = 0.8750$, you find from Table 7 that, for $0 \leq y \leq \frac{\pi}{2}$, $y \approx 1.07$.
Moreover, since in Quadrant II $\sin y$ is positive, you also have

$$y \approx \pi - 1.07 \approx 3.14 - 1.07 = 2.07.$$

$\therefore \{y: \sin y = \frac{7}{8}, y \in \mathcal{R}\} =$

$$\{y: y \approx 1.07 + 2k\pi\} \cup \{y: y \approx 2.07 + 2k\pi\}. \quad \text{Answer.}$$

b. For Sin^{-1}, the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, or $-90° \leq m(\alpha) \leq 90°$, and thus

$$\text{Sin}^{-1}\tfrac{7}{8} \approx 1.07 \text{ or } \text{Sin}^{-1}\tfrac{7}{8} \approx 61°3'. \quad \text{Answer.}$$

Thinking of principal-value inverses in terms of angles instead of numbers gives the easiest way to obtain values for such compositions of functions as

$$\sin(\text{Cos}^{-1}), \qquad \text{and} \qquad \text{Tan}^{-1}(\sin).$$

For example, Figure 12 pictures the angle function value $\alpha = \text{Cos}^{-1}\left(\frac{2}{3}\right)$. After using the Pythagorean theorem to compute the length of side $\overline{AB}$ of $\triangle AOB$ to be $\sqrt{5}$, you can read from the diagram such values as

$$\sin\left(\text{Cos}^{-1}\frac{2}{3}\right) = \frac{\sqrt{5}}{3}, \quad \tan\left(\text{Cos}^{-1}\frac{2}{3}\right) = \frac{\sqrt{5}}{2}, \quad \text{and}$$

$$\csc\left(\text{Cos}^{-1}\frac{2}{3}\right) = \frac{3}{\sqrt{5}}.$$

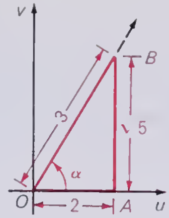


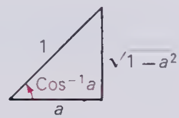
Figure 12

EXAMPLE 3 Simplify $\sin(\text{Cos}^{-1} a + \text{Sin}^{-1} b)$.

SOLUTION By the sum formula for $\sin(\alpha_1 + \alpha_2)$, you have

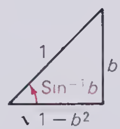
$$\begin{aligned}&\sin(\text{Cos}^{-1} a + \text{Sin}^{-1} b)\\ &\quad = [\sin(\text{Cos}^{-1} a)][\cos(\text{Sin}^{-1} b)] + [\cos(\text{Cos}^{-1} a)][\sin(\text{Sin}^{-1} b)].\end{aligned}$$

Now $\text{Cos}^{-1} a$ represents an angle in either Quadrant I or II, so that $\sin(\text{Cos}^{-1} a)$ is positive. Thus, you need only depict $\text{Cos}^{-1} a$ as an angle in a right triangle as shown. By inspection,



$$\sin(\text{Cos}^{-1} a) = \sqrt{1 - a^2}.$$

Similarly, $\text{Sin}^{-1} b$ lies in Quadrant I or Quadrant IV; in either case, $\cos(\text{Sin}^{-1} b)$ is positive, so that, from the sketch at the right, you see that



$$\cos(\text{Sin}^{-1} b) = \sqrt{1 - b^2}.$$

Then, because for every a in the domain of Sin^{-1}, $\sin(\text{Sin}^{-1} a) = a$ and for every b in the domain of Cos^{-1}, $\cos(\text{Cos}^{-1} b) = b$, you find

$$\begin{aligned}\sin(\text{Cos}^{-1} a + \text{Sin}^{-1} b) &= (\sqrt{1 - a^2})(\sqrt{1 - b^2}) + ab\\ &= \sqrt{(1 - a^2)(1 - b^2)} + ab. \quad \text{Answer.}\end{aligned}$$

Oral Exercises

State the value of each of the following for both the inverse circular function and the inverse angle function.

1. $\text{Sin}^{-1} 1$
2. $\text{Cos}^{-1} \frac{1}{2}$
3. $\text{Cos}^{-1}(-1)$
4. $\text{Sin}^{-1}(-1)$
5. $\text{Cos}^{-1} \dfrac{1}{\sqrt{2}}$
6. $\text{Sin}^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$
7. $\text{Cos}^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$
8. $\text{Tan}^{-1} 1$

Written Exercises

In each of the following, α denotes the degree measure of an angle. Specify the members of the given set using an expression in which k can have any integral value.

A

1. $\{\alpha\colon \sin\alpha = 0\}$
2. $\{\alpha\colon \cos\alpha = -1\}$
3. $\{\alpha\colon \sin\alpha = 1\}$
4. $\{\alpha\colon \tan\alpha = 1\}$
5. $\{\alpha\colon \csc\alpha = \sqrt{2}\}$
6. $\{\alpha\colon \sec\alpha = 2\}$

In each of the following, x denotes a real number. Specify the members of the given set using an expression in which k can have any integral value.

7. $\left\{x\colon \sin x = \frac{1}{\sqrt{2}}\right\}$
8. $\left\{x\colon \cos x = -\frac{1}{2}\right\}$
9. $\left\{x\colon \sin x = -\frac{1}{2}\right\}$
10. $\left\{x\colon \cos x = -\frac{1}{\sqrt{2}}\right\}$
11. $\{x\colon \tan x = -1\}$
12. $\left\{x\colon \tan x = \frac{1}{\sqrt{3}}\right\}$

State the value of each of the following for both the inverse angle and inverse circular functions. Use Tables 6 and 7 as needed.

13. $\text{Cos}^{-1}\frac{\sqrt{3}}{2}$
14. $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
15. $\text{Arcsin}\,(-1)$
16. $\text{Arccos}\,\frac{1}{\sqrt{2}}$
17. $\text{Tan}^{-1}(-\sqrt{3})$
18. $\text{Cot}^{-1}\,1$
19. $\text{Sin}^{-1}\,0.3584$
20. $\text{Cos}^{-1}\,0.4384$
21. $\text{Arcsin}\,(-0.9397)$
22. $\text{Cos}^{-1}(-0.7771)$
23. $\text{Tan}^{-1}\,0.2126$
24. $\text{Arctan}\,(-0.3640)$

Evaluate without using tables.

B

25. $\text{Sin}^{-1}\left(\cos\frac{\pi}{3}\right)$
26. $\text{Cos}^{-1}(\sin 180°)$
27. $\text{Sin}^{-1}(\sin 270°)$
28. $\text{Cos}^{-1}\left(\cos\frac{7\pi}{6}\right)$
29. $\text{Sin}^{-1}\left(\sin\frac{3\pi}{4}\right)$
30. $\text{Sin}^{-1}\left(\sin\frac{5\pi}{4}\right)$
31. $\text{Cos}^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right]$
32. $\text{Tan}^{-1}(\tan 45°)$
33. $\text{Tan}^{-1}\left(\tan\frac{2\pi}{3}\right)$

Use a diagram similar to Figure 12, page 587, to find each of the following values.

34. $\sin\left(\text{Cos}^{-1}\frac{3}{5}\right)$
35. $\cos\left(\text{Sin}^{-1}\frac{5}{13}\right)$
36. $\tan\left(\text{Sin}^{-1}\frac{24}{25}\right)$
37. $\sin\left(\text{Tan}^{-1}\frac{8}{15}\right)$
38. $\cos\left(\text{Tan}^{-1}\frac{\sqrt{7}}{4}\right)$
39. $\tan\left(\text{Sin}^{-1}\frac{1}{4}\right)$
40. $\sin\left[\text{Cos}^{-1}\left(-\frac{1}{3}\right)\right]$
41. $\cos\left[\text{Sin}^{-1}\left(-\frac{1}{5}\right)\right]$

Simplify.

42. $\cos\,(\mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Cos}^{-1}\frac{12}{13})$

43. $\sin\,(\mathrm{Sin}^{-1}\frac{3}{5} + \mathrm{Cos}^{-1}\frac{1}{2})$

44. $\sin\left(\mathrm{Cos}^{-1}\frac{1}{\sqrt{2}} + \mathrm{Tan}^{-1}\frac{3}{4}\right)$

45. $\cos\left(\mathrm{Sin}^{-1}\frac{12}{13} - \mathrm{Cos}^{-1}\frac{3}{5}\right)$

46. $\cos\,(2\,\mathrm{Sin}^{-1}\frac{5}{13})$

47. $\sin\,(2\,\mathrm{Cos}^{-1}\frac{1}{3})$

C 48. a. Is it true that if $0 \le x \le 1$, $\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x$?
b. Use part (a) above to show that if $0 \le x \le 1$, then $\cos\,(\mathrm{Sin}^{-1}(-x)) = \cos\,(\mathrm{Sin}^{-1}x)$.

49. a. Is it true for inverse circular functions that if $0 \le x \le 1$, $\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x$?
b. Use part (a) above to show that if $0 \le x \le 1$, then $\sin\,(\mathrm{Cos}^{-1}(-x)) = \sin\,(\mathrm{Cos}^{-1}x)$.

Prove the following for inverse circular functions over the indicated domains.

50. $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \frac{\pi}{2};\ 0 \le x \le 1$

51. $\mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}x = \frac{\pi}{2};\ x \ge 0$

16-2 Equations Involving Circular and Trigonometric Functions

The equation $\cos x = c$, where $c \in \mathcal{R}$ is a constant, either has an empty solution set (if $|c| > 1$), or else has a solution set containing infinitely many members. For example, the solution set over $\mathcal{R}$ of

$$\cos x = -\tfrac{1}{2}$$

was found in part a of Example 1 in Section 16-1 to be

$$\left\{x\colon x = \frac{2\pi}{3} + 2k\pi\right\} \cup \left\{x\colon x = \frac{4\pi}{3} + 2k\pi\right\}, \text{ where } k \in \{\text{the integers}\}.$$

Over the set of angles, the solution set of $\cos \alpha = -\frac{1}{2}$ is

$$\{\alpha\colon m(\alpha) = 120° + k \cdot 360°\} \cup \{\alpha\colon m(\alpha) = 240° + k \cdot 360°\},$$
$$\text{where } k \in \{\text{the integers}\}.$$

In either case, such a solution set is referred to as the **general solution** of the equation. The subset consisting of the solutions in a specified interval is called a **particular solution**. The particular solution of the equation $\cos x = -\frac{1}{2}$ in the interval $0 \le x < 2\pi$ is

$$\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}.$$

In the interval $0° \leq m(\alpha) < 360°$, the particular solution of $\cos \alpha = -\frac{1}{2}$ is

$$\{\alpha: m(\alpha) = 120°\} \cup \{\alpha: m(\alpha) = 240°\}.$$

In the latter case, we shall use the abbreviated notation $\{120°, 240°\}$. (Compare this with the abbreviated notation introduced in Section 14-4.)

EXAMPLE 1 a. Find the general solution over $\mathcal{R}$ of $3 \sin x + \cos x = 0$.

b. Find the particular solution in the interval $0 \leq x < 2\pi$.

SOLUTION a. To obtain an equation involving a single function value, you can add $-\cos x$ to each member to obtain:

$$3 \sin x = -\cos x.$$

Then, observing that $\cos x \neq 0$ (because if $\cos x = 0$, $\sin x \neq 0$), you can divide each member by $3 \cos x$ to obtain

$$\frac{\sin x}{\cos x} = -\frac{1}{3}.$$

Since $\dfrac{\sin x}{\cos x} = \tan x$, you have $\tan x = -\frac{1}{3} \approx -0.3333$. From Table 7 you see that $\tan (0.322) \approx \frac{1}{3}$. From Section 15-4 you know that $\tan(-x) = -\tan x$, so $\tan(-0.322) \approx -\frac{1}{3}$. Since -0.322 is in the range of Tan^{-1},

$$\text{Tan}^{-1}\left(-\tfrac{1}{3}\right) \approx -0.322.$$

Tangent has period π, so you have, as a general solution,

$$\{x: x \approx -0.322 + k\pi\}. \quad \text{Answer.}$$

b. Particular solutions in the interval $0 \leq x < 2\pi$ are obtained by letting $k = 1$ and $k = 2$. The particular solutions are $\{2.82, 5.96\}$. Answer.

EXAMPLE 2 a. Find the general solution over the set of angles of

$$\cos 2x + \sin x - 1 = 0.$$

b. Find the particular solution over the interval

$$0° \leq m(\alpha) < 360°.$$

SOLUTION a. You can begin by replacing $\cos 2x$ with $1 - 2 \sin^2 x$ to obtain $1 - 2 \sin^2 x + \sin x - 1 = 0$, from which $2 \sin^2 x - \sin x = 0$.

Factoring the left-hand member yields

$$\sin x\,(2 \sin x - 1) = 0,$$

which is equivalent to

$$\sin x = 0 \quad \text{or} \quad 2 \sin x - 1 = 0.$$

Since $\sin x = 0$ for $x = k \cdot 180°$, and $\sin x = \frac{1}{2}$ for $x = 30° + k \cdot 360°$

and $x = 150° + k \cdot 360°$, the general solution set is

$$\{x: x = k \cdot 180°\} \cup \{x: x = 30° + k \cdot 360°\}$$
$$\cup \{x: x = 150° + k \cdot 360°\} \text{ for } k \in \{\text{the integers}\}.$$ Answer.

b. Over the interval $0° \leq m(\alpha) < 360°$, the set of particular solutions is $\{0°, 30°, 150°, 180°\}$. Answer.

Written Exercises

Find (a) the general solution set of each equation for $m(\alpha)$ in degrees and (b) the particular solution set for $0° \leq m(\alpha) < 360°$.

A

1. $2 \sin \alpha + 1 = 0$

2. $2 \cos \alpha + \sqrt{3} = 0$

3. $\sqrt{2} \sin \alpha + 1 = 0$

4. $\sqrt{3} \sec \alpha - 2 = 0$

5. $\csc \alpha + 2 = 0$

6. $2 \cos^2 \alpha - 1 = 0$

7. $4 \sin^2 \alpha - 3 = 0$

8. $3 \tan^2 \alpha - 1 = 0$

Find (a) the general solution set of each equation for $x \in \mathcal{R}$ and (b) the particular solution set for $0 \leq x < 2\pi$. Be sure to check the proposed solution set.

9. $\sin x = -\cos x$

10. $\sin^2 x = 3 \cos^2 x$

11. $\cos x = \sin 2x$

12. $\sin x = -\sin 2x$

13. $\cos 2x = -1$

14. $2 \cos 2x = 1 - 2 \sin^2 x$

Find the particular solution set for $0° \leq m(\alpha) < 360°$ or for $0 \leq x < 2\pi$. Use Tables 6 and 7 as needed. If necessary, give your answer to the nearest 10′ or hundredth.

15. $2 \sin^2 \alpha - 1 = \sin \alpha$

16. $\tan \alpha = 2 \sin \alpha$

17. $2 + \cos \alpha = 2 \sin^2 \alpha$

18. $\cos x + \sec x = 2$

19. $\sin^2 2x = -\sin 2x$

20. $\cos 3\alpha = \sin 3\alpha$

21. $3 \sin^2 x - 2 \sin x = 1$

22. $\cos 2\alpha + \cos \alpha = 0$

23. $3 \csc^2 x - 7 \csc x + 4 = 0$

24. $\sec^2 x = 3 - \tan^2 x$

25. $(1 + \tan x)^2 = 2 \tan x + 2$

26. $2 \sin^2 \alpha - \cos \alpha = 1$

B

27. $\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2 = \frac{1}{2}$

28. $\tan^2 x - \sec x = 1$

29. $\cos 4\alpha = \cos 2\alpha$

30. $\sec^2 2\alpha = 2 \tan 2\alpha$

31. $(\sin x + \cos x)^2 = 2 \sin 2x$

32. $\cos 2x + 3 \cos x + 2 = 0$

33. $8 \cos^4 \alpha - 10 \cos^2 \alpha + 3 = 0$

34. $8 \sin^4 \alpha + 2 \sin^2 \alpha = 3$

C

35. $\sqrt{2 - 2 \cos \alpha} = 2 \cos \frac{\alpha}{2}$

36. $\cot \alpha - \tan \alpha = 2$

37. $\sqrt{3} \cos^2 \alpha - 2 \sin \alpha \cos \alpha = \sqrt{3} \sin^2 \alpha$

38. $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 2\sqrt{3}$

Self-Test 1

VOCABULARY principal-value inverse function for sine (p. 584)
principal-value inverse function for cosine (p. 585)
general solution of a trigonometric equation (p. 589)
particular solution of a trigonometric equation (p. 589)

1. Specify the members of $\left\{x\colon \sin x = -\frac{\sqrt{3}}{2},\ x \in \mathcal{R}\right\}$. *Obj. 1, p. 583*
2. State the value for the inverse angle function of:

   a. $\mathrm{Tan}^{-1}\sqrt{3}$ b. $\mathrm{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

3. Solve over $\mathcal{R}$: $\sec^2 x - 2 = 0$. *Obj. 2, p. 583*
4. Solve for $0° \le m(\alpha) < 360°$: $\cos 2\alpha + 1 = \cos \alpha$.

Check your answers with those at the back of the book.

ON THE CALCULATOR

Use of the calculator can make it a relatively simple matter to find a solution to an equation involving trigonometric functions. Note that the complete solution set may not be generated. Enter the constant first, then the inverse of the operations indicated on the left side of the equation. Begin with the operation farthest from the variable.

All of the examples on this page are to be done in the degree mode. Make sure your calculator is in the degree mode before you begin.

EXAMPLE Find a value for α such that $\frac{1}{3}\sin^2\alpha = 0.2$.

SOLUTION 0.2 [×] 3 [=] [√x] [arc] [sin] 50.76848°. Answer.

Exercises

Find a solution to each equation.

1. $4 \sin \alpha - 2.8 = 0.0284$
2. $3 \cos \alpha + 12 = 12.6$
3. $4 \sin \alpha^2 = 3$
4. $4 \sin^2 \alpha = 3$
5. $\frac{2}{3}\cos^2\alpha = 0.01$
6. $\cos^2(\frac{2}{3}\alpha) = 0.01$
7. $\tan \frac{1}{3}\alpha = 12$
8. $\sin(2\alpha - 72) = 0.63$
9. $3 \tan^2 \frac{\alpha}{2} = 15$

Polar Coordinates

***OBJECTIVES** for Sections 16-3 through 16-5:*

1. ***Express the coordinates of a point in Cartesian or polar form when it is given in the other form.***
2. ***Use DeMoivre's theorem to find powers and roots of complex numbers.***
3. ***Resolve a given vector into horizontal and vertical components, and find the sum of two given vectors.***

16-3 Polar Coordinates; Polar Graphs

In Section 3-2 you learned that each point in a plane can be paired uniquely with an ordered pair of real numbers (x, y) in a given Cartesian coordinate system for the plane. Another coordinate system can be developed as follows.

Each point in the plane lies on a ray with initial point at the origin. If r represents the distance from the origin to a point P, and if an angle having the nonnegative x-axis as its initial side and the ray $\overrightarrow{OP}$ as its terminal side is denoted by θ, then the ordered pair

$$(r, m(\theta))$$

clearly specifies the location of P in the plane (Figure 13). The components of the ordered pair $(r, m(\theta))$, usually written as simply (r, θ), are called a pair of **polar coordinates** of P. The nonnegative x-axis is then called the **polar axis,** and the origin is called the **pole.**

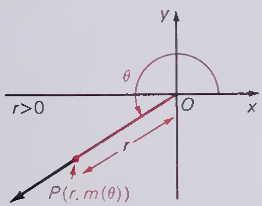


Figure 13

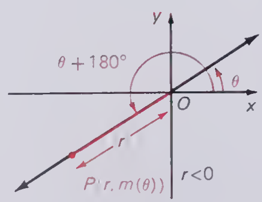


Figure 14

If r is negative, then we measure the distance $|r|$ along the extension of the terminal side of θ through the origin (Figure 14). For example, $(-3, 30°)$ is a pair of polar coordinates for the point P which also has coordinates $(3, 210°)$.

In this system, since θ is coterminal with infinitely many angles having the polar axis as initial side, the location of P can also be given, for example, by any of the ordered pairs

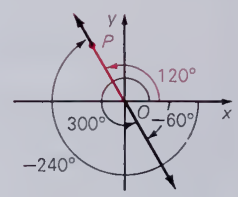
$$(r, \theta + 2k\pi) \qquad \text{or} \qquad (r, \theta + k \cdot 360°).$$

If $r = 0$, then *any* value might be assigned to $m(\theta)$. Thus the pole might, for example, be assigned polar coordinates $(0, 0°)$, $(0, 45°)$, or $(0, \frac{1}{2}\pi)$.

In general:

> In a system of polar coordinates, each ordered pair (r, θ) can be associated with one and only one point of the plane, but each point of the plane may be associated with any number of ordered pairs (r, θ).

EXAMPLE 1 List all other polar coordinates of $P(2, 120°)$ for which $-360° \le m(\theta) \le 360°$.



SOLUTION Sketch the graph of $P(2, 120°)$. By inspection, P has the additional coordinates

$$(-2, 300°), (-2, -60°), (2, -240°). \quad \text{Answer.}$$

When the polar axis of a polar coordinate system coincides with the nonnegative x-axis of a Cartesian coordinate system, the polar and Cartesian coordinates are related as shown in Figure 15.

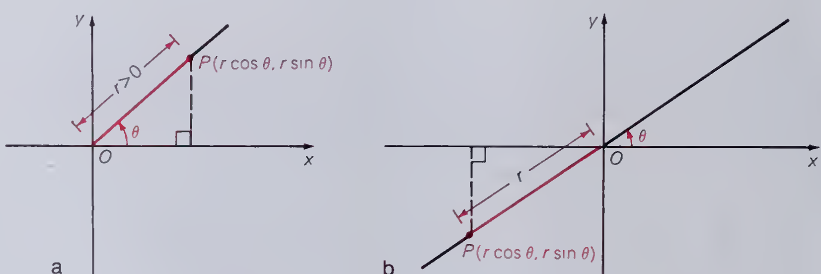


Figure 15 a b

If $r > 0$ (Figure 15a), then $(x, y) = (r\cos\theta, r\sin\theta)$.
If $r = 0$, $(x, y) = (r\cos\theta, r\sin\theta) = (0, 0)$.
If $r < 0$ (Figure 15b), then

$$\begin{aligned}(x, y) &= (|r|\cos(\theta + 180°), |r|\sin(\theta + 180°))\\ &= (-|r|\cos\theta, -|r|\sin\theta)\\ &= (r\cos\theta, r\sin\theta).\end{aligned}$$

In general:

> The polar and Cartesian coordinates of any point P are related by:
>
> $$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad (1)$$

These equations can be used to find Cartesian coordinates for a point whose polar coordinates are given.

EXAMPLE 2 Find the Cartesian coordinates of the point P with polar coordinates $(4, -120°)$.

SOLUTION Using Equations (*1*), you have:

$$x = r\cos\theta = 4\cos(-120°) = 4(-\tfrac{1}{2}) = -2$$

$$y = r\sin\theta = 4\sin(-120°) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$\therefore$ the required Cartesian coordinates are $(-2, -2\sqrt{3})$. Answer.

> Given the Cartesian coordinates (x, y) of a point other than the origin, a pair of polar coordinates of the point can be found from the equations:
>
> $$r = \pm\sqrt{x^2 + y^2}, \qquad \cos\theta = \frac{x}{\pm\sqrt{x^2 + y^2}}, \qquad (2)$$
>
> $$\sin\theta = \frac{y}{\pm\sqrt{x^2 + y^2}}$$

EXAMPLE 3 Find a pair of polar coordinates of the point P whose Cartesian coordinates are $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

SOLUTION From Equations (*2*), using $r = \sqrt{x^2 + y^2}$, you have

$$r = \sqrt{\left(-\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3.$$

Since $\cos\theta = \dfrac{-\frac{3\sqrt{3}}{2}}{3} = -\dfrac{\sqrt{3}}{2}$, and $\sin\theta = \dfrac{\frac{3}{2}}{3} = \dfrac{1}{2}$, you see by inspection that θ can be an angle measuring $150°$. Thus, a pair of polar coordinates of P is $(3, 150°)$. [By using $r = -\sqrt{x^2 + y^2} = -3$, you would find $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$, and so a second pair of coordinates is $(-3, 330°)$.] Answer.

Polar equations such as

$$r = 4\cos\theta, \qquad r = 2, \qquad r\sin\theta = 2, \qquad \text{and} \qquad \theta = 30°$$

have ordered pairs of the form (r, θ) as solutions. The graph of the set of all solutions of such an equation is called the **graph of the equation**.

EXAMPLE 4 Sketch the graph of $r = 4\cos\theta$.

SOLUTION The table below shows selected solutions of the equation for

$$0^\circ \le m(\theta) \le 360^\circ.$$

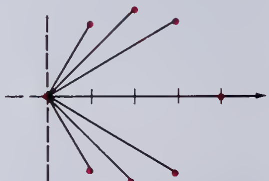
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
r	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0	-2	$-2\sqrt{2}$	$-2\sqrt{3}$	-4
θ	210°	225°	240°	270°	300°	315°	330°	360°
r	$-2\sqrt{3}$	$-2\sqrt{2}$	-2	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4

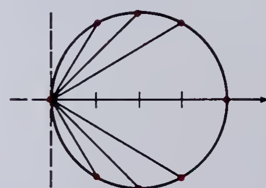
If you plot all these solutions in succession, you will twice plot the points in the figure at the left (below). That is, the points associated with

$$0^\circ \le m(\theta) \le 180^\circ$$

are the same as those associated with

$$180^\circ \le m(\theta) \le 360^\circ.$$





Connecting the points with a smooth curve yields the graph shown at the right (above). You can verify (Exercise 26, page 597) that the graph is a circle.

Oral Exercises

Give a set of polar coordinates with $-\pi < x \le \pi$ for the given point.

1. $(2, -30^\circ)$
2. $(-3, 210^\circ)$
3. $(-1, -135^\circ)$
4. $(4, 150^\circ)$

Written Exercises

Find a set of polar coordinates with $-180^\circ < m(\alpha) \le 180^\circ$ for the point whose Cartesian coordinates are given. Graph the point in the plane.

A
1. $(4, 4)$
2. $(-\sqrt{3}, 1)$
3. $(-3\sqrt{2}, -3\sqrt{2})$
4. $(5, -5\sqrt{3})$
5. $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$
6. $(\sqrt{2}, -\sqrt{6})$
7. $(-3, -4)$
8. $(-12, 5)$

Find the Cartesian coordinates of the given point. Graph the point in the plane.

9. $(3\sqrt{2}, 135°)$
10. $(5, 30°)$
11. $(-2, 60°)$
12. $(4, -120°)$
13. $\left(-1, \frac{\pi}{3}\right)$
14. $\left(-3, \frac{3\pi}{4}\right)$
15. $\left(\sqrt{3}, \frac{5\pi}{6}\right)$
16. $\left(-\sqrt{6}, -\frac{2\pi}{3}\right)$

Use Equations (*1*) on page 594 to transform the given equation into an equation in polar coordinates.

17. $x = 2$
18. $y = -3$
19. $x + y - 2 = 0$
20. $x^2 + y^2 = 4$
21. $x^2 - y^2 = 16$
22. $x^2 + y^2 - 9x = 0$

In Exercises 23–37, (a) sketch the graph of the given polar equation, and (b) transform the equation into an equation in Cartesian coordinates.

B

23. $r = 3$
24. $r = -4$
25. $\theta = 45°$
26. $r = 4\cos\theta$
27. $r = 2\sec\theta$
28. $r = -4\csc\theta$
29. $r = 2 - 2\cos\theta$
30. $r = 1 + \sin\theta$
31. $r = 2\sin 2\theta$
32. $r = \cos 2\theta$
33. $r = -2\cos 2\theta$
34. $r = 1 - 2\cos\theta$

C

35. $r = \theta$ (θ in radians)
36. $r^2 = 2\cos\theta$
37. $r^2 = \sin 2\theta$

ON THE CALCULATOR

You may use your calculator to find a solution to an equation involving circular functions of real numbers, as well as to an equation involving trigonometric functions. (Recall that the entire solution set may not be generated.) The solution technique is the same in either case. Just check your calculator manual for instructions about how to change from one mode to the other.

These exercises are all to be done in the radian mode.

EXAMPLE Find a solution to $5\cos x^2 = 1$.

SOLUTION 1 $\boxed{\div}$ 5 $\boxed{=}$ $\boxed{\text{arc}}$ $\boxed{\text{cos}}$ $\boxed{\sqrt{x}}$ 1.1702301 R. Answer.

Exercises

Find a solution to each equation.

1. $\sin\frac{x}{4} = 1$
2. $\cos(3x + 1) = 0.754$
3. $\tan^2\frac{x}{\pi} + 1 = 3.4$
4. $\frac{3}{5}\cos x = 0.4$
5. $\cos 3x + 1 = 1.5$
6. $\sin^2(x + \pi) = 0.25$
7. $\cos(2x - 1)^2 = 0.8$
8. $\tan(\frac{x}{2} + \pi) = 4$
9. $4\sin(2x) = 3.1$

16-4 Graphs of Complex Numbers; De Moivre's Theorem

Complex numbers may be graphed on a set of rectangular coordinates in the complex plane. (Such graphs are also called *Argand diagrams*.) Each complex number $x + iy$, where $x, y \in \mathcal{R}$, is associated with the point (x, y). We represent the complex number by a point or by an arrow from the origin to the associated point. In Figure 16, $-2 + i$ is represented by a point and $2 + 3i$ by an arrow. All of the real numbers appear on the horizontal axis in the form $x + 0i$, and the pure imaginary numbers appear on the vertical axis in the form $0 + iy$.

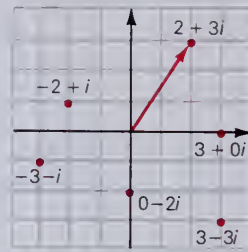


Figure 16

Complex numbers are frequently represented using polar coordinates. If $r \geq 0$, then $x + yi$ can be expressed as $r\cos\theta + (r\sin\theta)i$, or $r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{x^2 + y^2}, \qquad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \text{and} \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

The expression

$$r(\cos\theta + i\sin\theta)$$

is called the **polar form** or the **trigonometric form** for denoting the complex number $x + yi$. If

$$z = x + yi,$$

then $r = |z|$ is the **modulus** or the **absolute value** of z. An angle θ determined by the equations above is called an **amplitude** or an **argument** of z. Thus, $x + yi$ may be graphed as the point with rectangular coordinates (x, y) or polar coordinates (r, θ), $r \geq 0$, as shown in Figure 17.

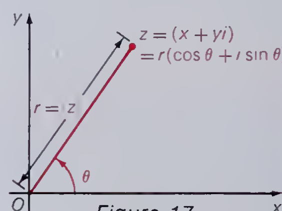


Figure 17

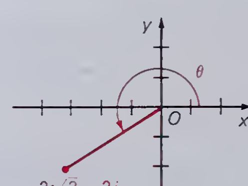
EXAMPLE 1 Express $-2\sqrt{3} - 2i$ in trigonometric form with $0° \leq m(\theta) < 360°$.

SOLUTION You have $r = |z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$
$= \sqrt{16} = 4.$

Then $\quad \cos\theta = -\dfrac{2\sqrt{3}}{4} = -\dfrac{\sqrt{3}}{2}$ and

$\sin\theta = -\dfrac{2}{4} = -\dfrac{1}{2}$, so that $m(\theta) = 210°$.

$\therefore\ -2\sqrt{3} - 2i = 4(\cos 210° + i\sin 210°)$.
Answer.



By expressing complex numbers in polar form, you can compute their products and quotients very readily by means of the following:

Theorem. If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then: (*1*) $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$,

(2) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (z_2 \neq 0).$

PROOF OF (*1*)

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2] \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

PROOF OF (2)

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 - i^2\sin\theta_1\sin\theta_2]}{r_2[\cos^2\theta - i^2\sin^2\theta_2]} \\ &= \frac{r_1[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{r_2[\cos^2\theta + \sin^2\theta]} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \end{aligned}$$

EXAMPLE 2 If $z_1 = 10(\cos 45° + i\sin 45°)$ and $z_2 = 2(\cos 15° + i\sin 15°)$, express in the form $a + bi$, **(a)** z_1z_2 and **(b)** $\dfrac{z_1}{z_2}$.

SOLUTION **a.** By Part (*1*) of the theorem,

$$\begin{aligned} z_1 \cdot z_2 &= 10\cdot 2[\cos(45° + 15°) + i\sin(45° + 15°)] \\ &= 20(\cos 60° + i\sin 60°) = 20\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 10 + 10i\sqrt{3}. \quad \text{Answer.} \end{aligned}$$

b. By Part (2) of the theorem,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{10}{2}[\cos(45° - 15°) + i\sin(45° - 15°)] \\ &= 5(\cos 30° + i\sin 30°) \\ &= 5\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{5\sqrt{3}}{2} + \frac{5}{2}i. \quad \text{Answer.} \end{aligned}$$

If $z = r(\cos\theta + i\sin\theta)$, you can see that successive applications of Part (*1*) of the theorem on page 599 yield

$$z^2 = z \cdot z = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$= r^2(\cos 2\theta + i\sin 2\theta),$$
$$z^3 = z^2 \cdot z = r^2(\cos 2\theta + i\sin 2\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$= r^3(\cos 3\theta + i\sin 3\theta),$$

and

$$z^4 = z^3 \cdot z = r^3(\cos 3\theta + i\sin 3\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$= r^4(\cos 4\theta + i\sin 4\theta).$$

Continuing this process suggests the general statement

$$z^n = z^{n-1}z = r^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]r(\cos\theta + i\sin\theta)$$
$$= r^n(\cos n\theta + i\sin n\theta).$$

This result was first published by the French mathematician De Moivre.

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and $n \in \{\text{the natural numbers}\}$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

A formal proof of this theorem requires mathematical induction, which is discussed in more advanced courses.

EXAMPLE 3 Express $(-1 + i)^4$ in the form $a + bi$.

SOLUTION Expressing $-1 + i$ in polar form, you have

$$r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2},\ \cos\theta = -\frac{1}{\sqrt{2}},\ \text{and}\ \sin\theta = \frac{1}{\sqrt{2}},$$

so that $$-1 + i = \sqrt{2}(\cos 135^\circ + i\sin 135^\circ).$$

Then, applying De Moivre's theorem, you see that

$$(-1 + i)^4 = [\sqrt{2}(\cos 135^\circ + i\sin 135^\circ)]^4$$
$$= (\sqrt{2})^4[\cos(4 \cdot 135^\circ) + i\sin(4 \cdot 135^\circ)]$$
$$= 4(\cos 540^\circ + i\sin 540^\circ)$$
$$= 4(\cos 180^\circ + i\sin 180^\circ) = 4(-1 + 0i) = -4.$$ Answer.

By defining $z^0 = 1$ and $z^{-n} = \frac{1}{z^n}$, it is possible to extend De Moivre's theorem to include all integral powers of nonzero complex numbers.

Theorem. If $z = r(\cos\theta + i\sin\theta) \neq 0 + 0i$ and $n \in$ {the integers}, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

The proof, which depends on the fact that

$$\frac{1}{z} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$$

(Exercise 38, page 604), is left for you (Exercise 43, page 604).

EXAMPLE 4 Express $\left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right)^{-4}$ in the form $a + bi$.

SOLUTION First, you express $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ in polar form:

$$r = |z| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{9} = 3,$$

$$\cos\theta = \frac{\frac{3\sqrt{3}}{2}}{3} = \frac{\sqrt{3}}{2}, \qquad \text{and} \qquad \sin\theta = \frac{\frac{3}{2}}{3} = \frac{1}{2}.$$

$$\therefore \frac{3\sqrt{3}}{2} + \frac{3}{2}i = 3(\cos 30° + i\sin 30°).$$

Then by the extended form of De Moivre's theorem,

$$\begin{aligned}[3(\cos 30° + i\sin 30°)]^{-4} &= 3^{-4}[\cos(-4\cdot 30°) + i\sin(-4\cdot 30°)]\\ &= \tfrac{1}{81}[\cos(-120°) + i\sin(-120°)]\\ &= \frac{1}{81}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{1}{162} - \frac{\sqrt{3}}{162}i.\end{aligned}$$

Answer.

To find roots of complex numbers using De Moivre's theorem, you can reason as in the following example.

EXAMPLE 5 Find all cube roots of 1 and express them in the form $a + bi$.

SOLUTION Expressing $1 = 1 + 0i$ in polar form, you have $r = 1$, $\cos\theta = \frac{1}{1} = 1$, and $\sin\theta = \frac{0}{1} = 0$. $\therefore 1 = 1(\cos 0° + i\sin 0°)$.

Now let $w = r(\cos\theta + i\sin\theta)$ represent a cube root of 1. Then

$$w^3 = r^3(\cos 3\theta + i\sin 3\theta) = 1(\cos 0° + i\sin 0°).$$

(Solution continued on page 602.)

Since two complex numbers are equal if and only if their moduli are equal and their arguments differ in degree measure by a multiple of 360°, you have

$$r^3 = 1 \qquad \text{and} \qquad 3\theta = 0 + k \cdot 360°.$$

$$\therefore\ r = \sqrt[3]{1} = 1 \text{ and } \theta = \frac{0 + k \cdot 360°}{3} = 0 + k \cdot 120°, \text{ and}$$

$$w = 1[\cos(0° + k \cdot 120°) + i \sin(0° + k \cdot 120°)].$$

Replacing k in turn with 0, 1, 2, 3, and so on, you find that:

If $k = 0$, then $w = 1(\cos 0° + i \sin 0°) = 1(1 + 0i) = 1.$

If $k = 1$, then $w = 1[\cos(0° + 120°) + i \sin(0° + 120°)]$
$= 1(\cos 120° + i \sin 120°)$
$= 1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$

If $k = 2$, then $w = 1[\cos(0° + 240°) + i \sin(0° + 240°)]$
$= 1(\cos 240° + i \sin 240°)$
$= 1\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$

If $k = 3$, then $w = 1[\cos(0° + 360°) + i \sin(0° + 360°)]$
$= 1(\cos 360° + i \sin 360°)$
$= 1(1 + 0i) = 1,$
which is the same as the value when $k = 0$.

Additional replacements for k will simply duplicate the three values of w already obtained. Therefore, the three cube roots of 1 are

$$1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \text{ and } -\frac{1}{2} - \frac{\sqrt{3}}{2}i. \quad \textsf{Answer.}$$

$$\left(\text{You can show that } 1^3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = 1.\right)$$

The three cube roots of 1 obtained in Example 5 above can be graphed as shown in Figure 18. The points are equally spaced around a circle with center at the origin and with radius 1.

The process used in Example 5 can be generalized as shown in the theorem at the top of page 603.

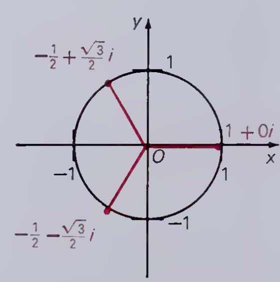


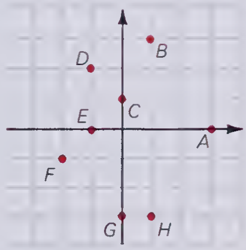
Figure 18

Theorem. The equation $z^n = r(\cos\theta + i\sin\theta)$, where $n \in$ {the natural numbers}, $r \in \mathcal{R}$, $r > 0$, has n roots:

$$z = \sqrt[n]{r}\left(\cos\frac{\theta + k(360°)}{n} + i\sin\frac{\theta + k(360°)}{n}\right)$$

for $k = 0, 1, 2, \ldots, n - 1$.

Oral Exercises



State the complex number represented by the given point.

1. A
2. B
3. C
4. D
5. E
6. F
7. G
8. H

Written Exercises

In Exercises 1–8, express the given complex number in polar form with $0° \le m(\theta) < 360°$.

A

1. $2 + 2i$
2. $3i$
3. -5
4. $-\sqrt{3} + i$
5. $\sqrt{2} - i\sqrt{2}$
6. $-2 - i2\sqrt{3}$
7. $-4 + 4i$
8. $\sqrt{6} - i\sqrt{2}$

Express the given complex number in the form $a + bi$.

9. $\sqrt{2}(\cos 45° + i\sin 45°)$
10. $4(\cos 60° + i\sin 60°)$
11. $\sqrt{3}(\cos 150° + i\sin 150°)$
12. $10(\cos 300° + i\sin 300°)$
13. $8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$
14. $\frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
15. $\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
16. $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

For the given values z_1 and z_2, (a) find z_1z_2 and (b) $\frac{z_1}{z_2}$. Express each answer in the form $a + bi$.

17. $z_1 = 6(\cos 30° + i\sin 30°)$, $z_2 = 2(\cos 330° + i\sin 330°)$
18. $z_1 = 4(\cos 225° + i\sin 225°)$, $z_2 = \cos 45° + i\sin 45°$
19. $z_1 = 8(\cos 135° + i\sin 135°)$, $z_2 = 4(\cos 105° + i\sin 105°)$
20. $z_1 = 3(\cos 285° + i\sin 285°)$, $z_2 = \frac{1}{2}(\cos 75° + i\sin 75°)$

Use De Moivre's theorem or its extension to express each of the following in the form $a + bi$.

21. $(1 + i\sqrt{3})^3$ **22.** $(1 + i\sqrt{3})^4$ **23.** $(-1 + i\sqrt{3})^5$ **24.** $(-\sqrt{2} - i\sqrt{2})^4$

25. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{10}$ **26.** $(-2 - 2i\sqrt{3})^3$ **27.** $(1 - i)^{-4}$ **28.** $(-\sqrt{3} + i)^{-3}$

29. $(0.2588 + 0.9659i)^3$
($Hint$: $15° = \text{Sin}^{-1}\, 0.2588 = \text{Cos}^{-1}\, 0.9659$)

30. $[2(-0.2588 + 0.9659i)]^2$

Find the required roots in polar form and graph the roots on a circle in the plane.

B **31.** the three cube roots of -1

32. the four fourth roots of 1

33. the four fourth roots of -16

34. the three cube roots of $-32 + 32i\sqrt{3}$

35. the three cube roots of $-4\sqrt{2} - 4i\sqrt{2}$

36. the five fifth roots of $-32i$

37. Let $z = a + bi = r(\cos\theta + i\sin\theta)$ and let $\bar{z} = a - bi$ denote the complex conjugate of z. Show that $\bar{z} = r[\cos(-\theta) + i\sin(-\theta)]$.

38. Show that if $z = r(\cos\theta + i\sin\theta)$, then $\frac{1}{z} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$.

39. Show that the product of the three cube roots of $1 + i$ equals $1 + i$.

40. Show that the product of the four fourth roots of -1 equals 1.

C **41.** Show that if $z = r(\cos\theta + i\sin\theta)$, then the product of the two square roots of z equals $-z$.

42. Show that if $z = r(\cos\theta + i\sin\theta)$, then the product of the three cube roots of z equals z.

43. Prove that if z^0 is defined to be 1 and z^{-n} is defined to be $\frac{1}{z^n}$, for $z \neq 0$, then De Moivre's theorem is valid for all integral exponents. $\left(Hint\text{: } \frac{1}{z^n} = \left(\frac{1}{z}\right)^n\text{; use this fact and the result of Exercise 38 above.}\right)$

16-5 Vectors

In order to describe a physical quantity such as velocity or force, you must give both its direction and its magnitude. In Figure 19 the arrow from O to P is a **directed line segment** which might represent, say, a velocity of r meters per second in a direction of θ with the x-axis. Any directed line segment in the plane (or space) is called a **vector** and is considered to point from one endpoint, called the **initial point,** to the other endpoint, called the **terminal point.**

A vector may be identified by means of a notation naming its endpoints, initial point first, such as $\overrightarrow{AB}$ (read "the vector AB"), or else by means of a lower-case letter in boldface type, such as

$\mathbf{v}$ (read "the vector $\mathbf{v}$")

(see Figure 20). The length of a vector is called its **norm** and is denoted

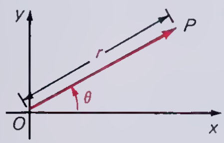


Figure 19

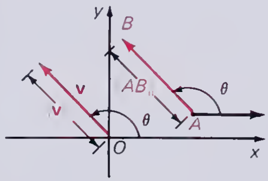


Figure 20

by $\|\overrightarrow{AB}\|$ or $\|\mathbf{v}\|$. In the plane, you ordinarily specify the direction of the vector by identifying the angle θ, where

$$-180° < m(\theta) \leq 180°,$$

that it determines with a ray directed parallel to, and in the direction of, the positive x-axis, as suggested by Figure 20. In particular, in Figure 20, $\mathbf{v}$ and $\overrightarrow{AB}$ have the same norm and the same direction. Such vectors are called **equivalent.**

For every vector $\mathbf{v}$ in the plane, each point in the plane is the initial point of a vector equivalent to $\mathbf{v}$. In particular, every vector $\mathbf{v}$ in the plane is equivalent to a vector in **standard position,** that is, a vector with initial point at the origin.

The sum, or **resultant,** $\overrightarrow{AB} + \overrightarrow{CD}$ of two vectors is defined as pictured in Figure 21. That is, if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are vectors, and $\overrightarrow{BE}$ is the vector equivalent to $\overrightarrow{CD}$ that has initial endpoint B, then

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AE}.$$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called **components** of $\overrightarrow{AE}$. Notice that the sum $\overrightarrow{AE}$ of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ can be pictured as the diagonal of a parallelogram with adjacent sides having the same lengths as $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and parallel to them, as illustrated in Figure 22. You can use this fact to help you find the norm and direction of a vector sum.

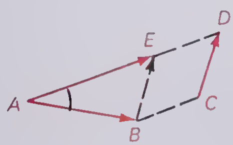


Figure 21

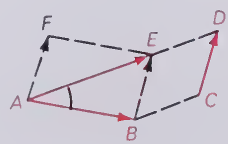
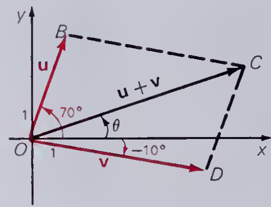


Figure 22

EXAMPLE 1 Given that $\|\mathbf{u}\| = 5$ and the direction angle of $\mathbf{u}$ measures $70°$, $\|\mathbf{v}\| = 8$ and the direction angle of $\mathbf{v}$ measures $-10°$, find an approximation to the nearest tenth of a unit for $\|\mathbf{u} + \mathbf{v}\|$, and to the nearest degree for the measure of the direction angle of $\mathbf{u} + \mathbf{v}$.

SOLUTION Make a sketch of the given vectors. By inspection, $m\angle DOB = 80°$ so that in the parallelogram $OBCD$,



$$m\angle ODC = 180° - 80° = 100°.$$

Then, to find $\|\mathbf{u} + \mathbf{v}\|$, you use the law of cosines:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos 100° \\ &\approx 5^2 + 8^2 - 2(5)(8)(-0.1736) \\ &= 25 + 64 + 13.888 \approx 102.89 \end{aligned}$$

$$\therefore\ \|\mathbf{u} + \mathbf{v}\| \approx \sqrt{102.89} \approx 10.1$$

To find the measure of the direction angle θ of $\mathbf{u} + \mathbf{v}$, use the law of sines first to find $m\angle COD$:

$$\frac{\sin \angle COD}{5} \approx \frac{\sin 100°}{10.1};\ \sin \angle COD \approx \frac{5 \sin 100°}{10.1} \approx \frac{5(0.9848)}{10.1}$$
$$\approx 0.4875$$

Then $m\angle COD \approx 29°$.

$\therefore\ m(\theta) \approx 29° - 10° = 19°$, and $\|\mathbf{u} + \mathbf{v}\| \approx 10.1$. Answer.

An alternative means of determining the norm and direction angle for the sum $\mathbf{u} + \mathbf{v}$ of two vectors is first to **resolve u and v** into sums of **horizontal** (parallel to x-axis) and **vertical** (parallel to y-axis) **components.** Figure 23 pictures a vector $\mathbf{u}$ together with its horizontal and

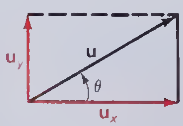


Figure 23

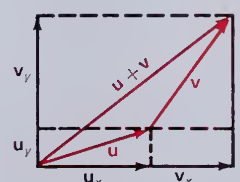


Figure 24

vertical components $\mathbf{u}_x$ and $\mathbf{u}_y$. Do you see that

$$\|\mathbf{u}_x\| = \|\mathbf{u}\|\,|\cos\theta|, \qquad \|\mathbf{u}_y\| = \|\mathbf{u}\|\,|\sin\theta|?$$

Now, examining Figure 24, you can see that the horizontal and vertical components of the sum $\mathbf{u} + \mathbf{v}$ of two vectors are the sums of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$:

$$(\mathbf{u} + \mathbf{v})_x = \mathbf{u}_x + \mathbf{v}_x \qquad \text{and} \qquad (\mathbf{u} + \mathbf{v})_y = \mathbf{u}_y + \mathbf{v}_y$$

so that

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{\|\mathbf{u}_x + \mathbf{v}_x\|^2 + \|\mathbf{u}_y + \mathbf{v}_y\|^2}.$$

Notice also that, for the direction angle θ of $\mathbf{u} + \mathbf{v}$, you have

$$|\cos \theta| = \frac{\|\mathbf{u}_x + \mathbf{v}_x\|}{\|\mathbf{u} + \mathbf{v}\|} \quad \text{and} \quad |\sin \theta| = \frac{\|\mathbf{u}_y + \mathbf{v}_y\|}{\|\mathbf{u} + \mathbf{v}\|},$$

or

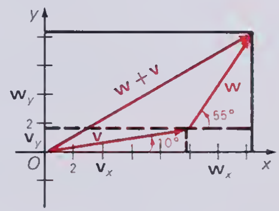
$$|\tan \theta| = \frac{\|\mathbf{u}_y + \mathbf{v}_y\|}{\|\mathbf{u}_x + \mathbf{v}_x\|}.$$

EXAMPLE 2 Given that $\|\mathbf{v}\| = 10$ and the direction angle of $\mathbf{v}$ measures $10°$, $\|\mathbf{w}\| = 8$, and the direction angle of $\mathbf{w}$ measures $55°$, find an approximation to the nearest tenth for $\|\mathbf{v} + \mathbf{w}\|$, and to the nearest degree for the measure of the direction angle of $\mathbf{v} + \mathbf{w}$.

SOLUTION Make a sketch showing the given vectors. Find the norms of the horizontal and vertical components of $\mathbf{v}$ and $\mathbf{w}$.

$$\|\mathbf{v}_x\| = 10 \cos 10° \approx 10(0.9848) \approx 9.8 \qquad \|\mathbf{w}_x\| = 8 \cos 55° \approx 8(0.5736) \approx 4.6$$

$$\|\mathbf{v}_y\| = 10 \sin 10° \approx 10(0.1736) \approx 1.7 \qquad \|\mathbf{w}_y\| = 8 \sin 55° \approx 8(0.8192) \approx 6.6$$



Then for the norm and direction of horizontal and vertical components of $\mathbf{v} + \mathbf{w}$, you find:

$$\|\mathbf{v}_x + \mathbf{w}_x\| \approx 9.8 + 4.6 = 14.4; \text{ direction } 0°$$
$$\|\mathbf{v}_y + \mathbf{w}_y\| \approx 1.7 + 6.6 = 8.3; \text{ direction } 90°$$

For θ, the direction angle of $\mathbf{v} + \mathbf{w}$, you find that

$$|\tan \theta| \approx \frac{8.3}{14.4} \approx 0.5764$$

from which you obtain $m(\theta) \approx 30°$.

Knowing $m(\theta)$, you can find the norm of $\mathbf{v} + \mathbf{w}$ by using

$$\cos \theta \approx \frac{14.4}{\|\mathbf{v} + \mathbf{w}\|},$$

so that

$$\|\mathbf{v} + \mathbf{w}\| \approx \frac{14.4}{\cos 30°} \approx \frac{14.4}{0.866} \approx 16.6.$$

Alternatively, you can compute $\|\mathbf{v} + \mathbf{w}\|$ as follows:

$$\|\mathbf{v} + \mathbf{w}\| = \sqrt{(14.4)^2 + (8.3)^2} \approx \sqrt{276} \approx 16.6$$

$\therefore \|\mathbf{v} + \mathbf{w}\| \approx 16.6$, $m(\theta) \approx 30°$. Answer.

Written Exercises

In Exercises 1–8, for vectors **u** and **v** with the given norms and directions, make a diagram similar to that of Example 1, page 606, and use the method of this example to find $\|\mathbf{u} + \mathbf{v}\|$ in simplest radical form.

A

1. **u**: 5, 130°; **v**: 8, 10°
2. **u**: 10, 90°; **v**: 6, 30°
3. **u**: 21, 120°; **v**: 5, 0°
4. **u**: 16, 80°; **v**: 5, 20°
5. **u**: 6, 45°; **v**: $\sqrt{2}$, 0°
6. **u**: 2, 75°; **v**: $3\sqrt{3}$, 45°
7. **u**: 12, 120°; **v**: $8\sqrt{2}$, −15°
8. **u**: $5\sqrt{3}$, 140°; **v**: 2, −10°

In Exercises 9–16, for vectors **u** and **v** with the given norms and directions, find $\|\mathbf{u}_x\|$, $\|\mathbf{u}_y\|$, $\|\mathbf{v}_x\|$, $\|\mathbf{v}_y\|$, $\|(\mathbf{u} + \mathbf{v})_x\|$, and $\|(\mathbf{u} + \mathbf{v})_y\|$ to the nearest tenth.

B

9. **u**: 32, 60°; **v**: 10, 0°
10. **u**: 8, 120°; **v**: 6, 30°
11. **u**: 20, 135°; **v**: 10, 60°
12. **u**: 10, 45°; **v**: 20, 120°
13. **u**: 12, 30°; **v**: 8, −45°
14. **u**: 16, −60°; **v**: 20, 60°
15. **u**: 10, 70°; **v**: 24, 40°
16. **u**: 8, 50°; **v**: 10, −30°

17–24. Find $m(\theta)$ to the nearest degree, where θ is the angle of $\mathbf{u} + \mathbf{v}$, and find $\|\mathbf{u} + \mathbf{v}\|$ to the nearest unit for the vectors **u** and **v** in Exercises 9–16 above.

Prove the following for vectors **u, v,** and **w.**

C

25. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
26. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

programming in BASIC

Exercises

1. Complete the program that will find, to the nearest tenth, the norms and direction angles of the horizontal and vertical components of a vector when its norm and direction angle are given. Refer to the discussion of Figure 23. (Recall that in using COS(A) and SIN(A) in BASIC, A is in radians.)

```
30  LET X = U*COS(A)
40  LET Y = U*SIN(A)
```

2. Complete the program that will find, to the nearest tenth, the norm of the sum of two vectors whose norms and direction angles are given. Refer to Example 1, page 606.

```
60  LET S = SQR(U↑2 + V↑2 - 2*U*V*COS(B))
```

Self-Test 2

VOCABULARY

polar coordinates (p. 593)
polar axis (p. 593)
the pole (p. 593)
modulus, or absolute value (p. 598)
argument, or amplitude (p. 598)
vector (p. 604)
equivalent vectors (p. 605)
norm (p. 605)
resultant (p. 605)
components of a vector (p. 605)

1. Find two sets of polar coordinates for $P(-2\sqrt{3}, 2)$ with $-180° < m(\theta) \leq 180°$. *Obj. 1, p. 593*
2. Find the Cartesian coordinates of the point with polar coordinates $(10, -60°)$.
3. Let $z_1 = 8(\cos 195° + i \sin 195°)$ and $z_2 = 2(\cos 75° + i \sin 75°)$. Find each of the following in the form $a + bi$. *Obj. 2, p. 593*

 a. z_1z_2 **b.** $\dfrac{z_1}{z_2}$ **c.** $(z_2)^4$

4. Find the three cube roots of $-32\sqrt{2} - i32\sqrt{2}$ in polar form.
5. For the vectors **u** and **v** such that $\|\mathbf{u}\| = 8$, direction of **u** is 120°, $\|\mathbf{v}\| = 10$, direction of **v** is 45°, find $\|\mathbf{u}_x\|$, $\|\mathbf{u}_y\|$, $\|\mathbf{v}_x\|$, and $\|\mathbf{v}_y\|$, all correct to the nearest tenth; $\|\mathbf{u} + \mathbf{v}\|$ to the nearest unit; and $m(\theta)$ for $\mathbf{u} + \mathbf{v}$ to the nearest degree. *Obj. 3, p. 593*

Check your answers with those at the back of the book.

Chapter Summary

1. The inverse of each of the circular or trigonometric functions discussed in this chapter is a relation that is not a function unless the range is restricted. For the *principal-value inverse functions* the ranges are:

 Sin^{-1}: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ Cos^{-1}: $0 \leq y \leq \pi$ Cot^{-1}: $0 < y < \pi$

 Csc^{-1}: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$ Sec^{-1}: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$ Tan^{-1}: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

2. Using trigonometric identities and the usual algebraic transformations, you can solve equations involving the circular or trigonometric functions.

3. Coordinates of points may be given in the Cartesian system as (x, y) or in the polar system as (r, θ), related as follows:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \pm\sqrt{x^2 + y^2} \qquad \cos\theta = \frac{x}{\pm\sqrt{x^2 + y^2}} \qquad \sin\theta = \frac{y}{\pm\sqrt{x^2 + y^2}}$$

4. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

5. De Moivre's theorem states that if $z = r(\cos\theta + i\sin\theta)$ and $n \in$ {the natural numbers}, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

6. The equation $z^n = r(\cos\theta + i\sin\theta)$, where $n \in$ {the natural numbers}, $r \in \mathcal{R}$, $r > 0$, has n solutions:

$$z = \sqrt[n]{r}\left(\cos\frac{\theta + k(360°)}{n} + i\sin\frac{\theta + k(360°)}{n}\right)$$

for $k = 0, 1, 2, \ldots, n - 1$.

7. A *vector* may be written in the form $\overrightarrow{AB}$ or $\mathbf{u}$; its norm, or length, is denoted by $\|\overrightarrow{AB}\|$ or by $\|\mathbf{u}\|$.

8. If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are any two vectors in the plane, and if $\overrightarrow{BE}$ is equivalent to $\overrightarrow{CD}$, then $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AE}$. Vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called *components* of $\overrightarrow{AE}$.

Chapter Review

1. Find $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ for the inverse angle function. 16-1

a. 30° b. 90° c. 60° d. 45°

2. Solve $2\cos x - 1 = 0$ for $0 \le x < 2\pi$. 16-2

a. $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ b. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ c. $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ d. $\left\{\frac{3\pi}{5}, \frac{2\pi}{5}\right\}$

3. Find the Cartesian coordinates of the point (3, 150°). 16-3

a. $\left(\frac{3\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right)$ b. $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ c. $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ d. $\left(\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right)$

4. Find a pair of polar coordinates of the point $(-2\sqrt{2}, 2\sqrt{2})$.

a. $(-2, 45°)$ b. $(2, 225°)$ c. $(4, 135°)$ d. $(-4, 45°)$

5. Express $1 + \sqrt{3}i$ in polar form. 16-4

a. $\sqrt{3}(\cos 30° + i \sin 30°)$
b. $2(\cos 60° + i \sin 60°)$
c. $4(\cos 150° + i \sin 150°)$
d. $2(\cos 150° + i \sin 150°)$

6. Find $[\frac{1}{2}(\cos 15° + i \sin 15°)]^2$.

a. $\frac{1}{8} + \frac{\sqrt{3}}{8}i$ b. $\frac{\sqrt{3}}{8} + \frac{1}{8}i$ c. $\frac{1}{4} - \frac{\sqrt{3}}{4}i$ d. $\frac{1}{4} + \frac{\sqrt{3}}{4}i$

Review Items 7 and 8 refer to the vector **u** with initial point (0, 0) and terminal point (3, 4).

7. Find $\|\mathbf{u}\|$. 16-5

a. 3 b. 4 c. 5 d. 25

8. Find $\|\mathbf{u}_x\|$.

a. 3 b. 4 c. 5 d. 9

Chapter Test

1. State the value of $\text{Sin}^{-1}(\cos 120°)$. 16-1
2. Solve $4 \sin^2 x - 3 = 0$ over $\mathcal{R}$. 16-2
3. Transform $r = -2 \sin \theta$ into an equation with Cartesian coordinates. 16-3
4. Express $2\sqrt{3} - 2i$ in polar form. 16-4
5. Find all the cube roots of $1 + i\sqrt{3}$ in polar form.

In Test Items 6 and 7 let $z_1 = 2(\cos 30° + i \sin 30°)$ and $z_2 = 3(\cos 60° + i \sin 60°)$.

6. Compute $z_1 z_2$ in polar form.
7. Compute $\frac{z_1}{z_2}$ in polar form.

Test Items 8–10 refer to **u** such that $\|\mathbf{u}\| = 3$ and direction of **u** is 60°; **v** such that $\|\mathbf{v}\| = 5$ and direction of **v** is 105°. Answer to the nearest 0.1 unit.

8. Compute $\|\mathbf{v}_x\|$ and $\|\mathbf{u}_y\|$. 16-5
9. Compute $\|\mathbf{u} + \mathbf{v}\|$.
10. Compute $m(\theta)$ for $\mathbf{u} + \mathbf{v}$.

Cumulative Review (Chapters 13–16)

1. Simplify: $\begin{bmatrix} -3 & 5 & 0 \\ 7 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 3 \\ 2 & -4 & 3 \end{bmatrix}$

   a. $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 6 & 0 \end{bmatrix}$ b. $\begin{bmatrix} -7 & 8 & -3 \\ 5 & 6 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 & 3 \\ 9 & -2 & 6 \end{bmatrix}$

2. Solve for X: $3X - \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}$

   a. $\begin{bmatrix} \frac{1}{3} & \frac{5}{5} \\ -1 & -\frac{1}{3} \end{bmatrix}$ b. $\begin{bmatrix} 1 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix}$ c. $\begin{bmatrix} 1 & \frac{1}{3} \\ 1 & -3 \end{bmatrix}$ d. $\begin{bmatrix} 9 & 3 \\ 9 & -27 \end{bmatrix}$

3. Simplify: $\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

   a. $\begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 6 \\ -2 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ d. $\begin{bmatrix} -6 & 6 \\ -2 & 1 \end{bmatrix}$

4. Find the multiplicative inverse of the matrix $\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$.

   a. $\begin{bmatrix} -\frac{2}{7} & -\frac{3}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$ b. $\begin{bmatrix} \frac{2}{11} & \frac{3}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{bmatrix}$ c. $\begin{bmatrix} \frac{1}{11} & -\frac{3}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix}$ d. $\begin{bmatrix} -\frac{1}{7} & \frac{3}{7} \\ -\frac{3}{7} & -\frac{2}{7} \end{bmatrix}$

5. Use a matrix equation to solve the system:

$$\begin{aligned} 2x - 5y &= 6 \\ x - 3y &= 2 \end{aligned}$$

   a. $\{(8, 2)\}$ b. $\{(26, 8)\}$ c. $\{(32, 10)\}$ d. $\{(-4, -2)\}$

6. Find the coordinates of the image P' of the point $P(-1, 3)$ under the translation of the plane given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

   a. $(-1, 8)$ b. $(-3, 8)$ c. $(-3, 2)$ d. $(-1, 2)$

7. Find the coordinates of the image Q' of the point $Q(8, 3)$ under the transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

   a. $(3, -2)$ b. $(2, -3)$ c. $(-2, -3)$ d. $(3, 3)$

8. Convert 75° to radian measure.

   a. $\frac{25\pi}{36}^R$ b. $\frac{\pi}{4}^R$ c. $\frac{5\pi}{12}^R$ d. $\frac{3\pi}{7}^R$

9. Find $\cos \alpha$ if $\sin \alpha = \frac{2}{3}$ and the terminal side of angle α is in quadrant II.

   a. $-\frac{1}{3}$ b. $-\frac{\sqrt{3}}{3}$ c. $-\frac{5}{9}$ d. $-\frac{\sqrt{5}}{3}$

10. Use the table on page 516 to find cos 495°.

a. $\frac{1}{\sqrt{2}}$ b. $-\frac{1}{\sqrt{2}}$ c. $\frac{1}{2}$ d. $-\frac{\sqrt{3}}{2}$

11. Find $\cot \alpha$ if the terminal side of $\angle \alpha$ is in quadrant IV and $\cos \alpha = \frac{5}{13}$.

a. $-\frac{5}{12}$ b. $-\frac{5}{13}$ c. $\frac{12}{13}$ d. $-\frac{13}{12}$

12. In $\triangle ABC$, $m(C) = 90°$, $m(A) = 30°$, and $a = 4$. Find c.

a. 8 b. 2 c. $\sqrt{3}$ d. $\frac{\sqrt{3}}{2}$

13. Simplify: $\frac{\sin x - \cos x}{\cos x} + 1$

a. $\sin x$ b. $\tan x$ c. $\cos x$ d. $1 + \sin x$

II quad I quad.

14. If $\cos x = -\frac{3}{5}$ where $\frac{\pi}{2} < x < \pi$ and $\sin y = \frac{5}{13}$ where $0 < y < \frac{\pi}{2}$, find $\cos(x + y)$.

a. $-\frac{16}{65}$ b. $-\frac{56}{65}$ c. $\frac{33}{65}$ d. $\frac{63}{65}$

15. Simplify $\frac{\sin 2\alpha}{2 \sin^2 \alpha}$.

a. 1 b. $\frac{1}{2}$ c. $\cot \alpha$ d. $\tan \alpha$

16. In $\triangle ABC$, $m(B) = 42°30'$, $m(C) = 76°20'$, and $b = 5.2$. Find a.

a. 6.7 b. 4.7 c. 4.0 d. 5.8

17. Which is the correct expression of the Law of Cosines for $\triangle ABC$?

a. $c^2 = a^2 + b^2 - 2ac \cos C$ b. $c^2 = a^2 + b^2 - 2ab \cos B$
c. $c^2 = a^2 + b^2 - 2ac \cos A$ d. $c^2 = a^2 + b^2 - 2ab \cos C$

18. $\text{Sin}^{-1}\left(\cos \frac{2\pi}{3}\right) = \underline{\ ?\ }$.

a. $\frac{7\pi}{6}$ b. $\frac{\pi}{6}$ c. $\frac{5\pi}{6}$ d. $-\frac{\pi}{6}$

19. Solve $2 \sin^2 x - 3 \sin x + 1 = 0$ for $0 < x < 2\pi$.

a. $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$ b. $\left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}\right\}$ c. $\left\{0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$

20. Express $-1 - i\sqrt{3}$ in polar form.

a. $2(\cos 60° + i \sin 60°)$ b. $2(\cos 240° + i \sin 240°)$
c. $2(\cos 120° + i \sin 120°)$ d. $2(\cos 300° + i \sin 300°)$

21. Find $\|\mathbf{v}\|$, if $\mathbf{v}$ is the vector with initial point (0, 0) and terminal point (6, 8).

a. 6 b. 8 c. 10 d. 12

Comprehensive Test *(Chapters 1–16)*

1. Solve over $\mathcal{R}$: $\frac{1}{4}(-x) = 8$

   a. $\{-32\}$ b. $\{-2\}$ c. $\{2\}$ d. $\{12\}$

2. Simplify $-\frac{1}{4}[4 - (3 + 5)]$.

   a. -1 b. 1 c. -3 d. 3

3. Simplify $-2(a + 2b - 3) + (a - 3b + 1)$

   a. $a + 7b - 2$ b. $-a + 7b + 7$ c. $-a - 7b + 7$

4. Solve over $\mathcal{R}$: $6a - (2a + 7) = 9$

   a. $\{\frac{1}{4}\}$ b. $\{2\}$ c. $\{4\}$ d. $\{\frac{1}{2}\}$

5. Solve over $\mathcal{R}$: $|x + 6| < 4$.

   a. $\{x: x < -10\}$ b. $\{x: -10 < x < -2\}$

   c. $\{x: x < -10\} \cap \{x: x > -2\}$

6. If $f: x \rightarrow 3x$ and $g: x \rightarrow 2x^2 - 1$, then $g(f(-1)) = \underline{\ ?\ }$.

   a. 3 b. 17 c. -19 d. -9

7. Find the slope of the line through the points $(-1, 2)$ and $(5, 4)$.

   a. 3 b. 1 c. $\frac{1}{3}$ d. $\frac{2}{3}$

8. If p varies directly as q and $p = 8$ when $q = 3$, what does p equal when $q = 6$?

   a. 2 b. 48 c. 3 d. 16

9. Solve the system
$$\begin{aligned} 2x - 5y &= 6 \\ x - 3y &= 2 \end{aligned}$$

   a. $\{(8, 2)\}$ b. $\{(26, 8)\}$ c. $\{(32, 10)\}$ d. $\{(-4, -2)\}$

10. If $\begin{vmatrix} 3k & 6 \\ k & 3 \end{vmatrix} = -12$, then $k = \underline{\ ?\ }$.

    a. 4 b. -4 c. $\frac{1}{4}$ d. $-\frac{1}{4}$

11. Which point lies in the xz-plane?

    a. $(1, 2, 3)$ b. $(0, 1, 3)$ c. $(1, 0, 3)$ d. $(1, 3, 0)$

Review Items 12 and 13 refer to the following system of equations.

$$\begin{aligned} 2x + y - z &= 3 \\ 4x - y + 4z &= 0 \\ -3y + 2z &= 6 \end{aligned}$$

12. Which determinant is D_z?

    a. $\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 4 \\ 0 & 6 & 2 \end{vmatrix}$ b. $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ 0 & -3 & 6 \end{vmatrix}$ c. $\begin{vmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 6 & -3 & 2 \end{vmatrix}$

13. Evaluate D_y.

a. 24 b. -96 c. 48 d. -3

14. Factor $2x^2 - 4x - 30$ completely.

a. $(2x + 6)(x - 5)$ b. $(2x - 10)(x + 3)$ c. $2(x - 5)(x + 3)$

15. Solve $2x^2 - 72x = 0$ by factoring.

a. $\{2, 36\}$ b. $\{2, -6, 6\}$ c. $\{-6, 6\}$ d. $\{0, 36\}$

16. Express $\dfrac{x^2 - x - 2}{(x - 2)^2} \div \dfrac{x^3 + x^2}{3x^2 - 12}$ in lowest terms.

a. $\dfrac{3x + 6}{x^2}$ b. $\dfrac{x^2(x + 1)^2}{3(x + 2)(x - 2)^2}$ c. $\dfrac{3x + 6}{x^3 - 2x^2}$

17. Find the sum of the arithmetic series $\sum_{n=0}^{6} 2(n - 3)$.

a. 6 b. 0 c. 12 d. -6

18. Find the sum of the geometric series $\sum_{i=1}^{4} 9(\frac{1}{3})^{i-1}$

a. $\frac{40}{3}$ b. 9 c. $-\frac{10}{9}$ d. $\frac{80}{3}$

19. Find the sum of the infinite geometric series $\sum_{n=1}^{\infty} 72(\frac{1}{3})^{n-1}$

a. 108 b. 48 c. 72 d. 216

20. Simplify $\dfrac{6}{\sqrt{3} - 1}$.

a. $\frac{3}{2}(\sqrt{3} + 1)$ b. $3\sqrt{3} + 3$ c. $\frac{3}{2}(\sqrt{3} - 1)$ d. $3\sqrt{3} - 3$

21. Solve $x^2 - 6x + 3 = 0$ over $\mathcal{R}$.

a. $\{3 + \sqrt{6}, -3 - \sqrt{6}\}$ b. $\{3 + 2\sqrt{6}, 3 - 2\sqrt{6}\}$

c. $\{3 + \sqrt{6}, 3 - \sqrt{6}\}$

22. Simplify $10\sqrt{-\frac{2}{5}}$

a. $-2\sqrt{10}$ b. $2i\sqrt{10}$ c. $-2i$ d. $-5i\sqrt{10}$

23. Find the vertex of the graph of $y = x^2 - 4x + 5$.

a. $(2, 5)$ b. $(-2, 4)$ c. $(-2, 5)$ d. $(2, 1)$

24. Solve $x^3 - 3x^2 + 3x + 7 = 0$ over $\mathcal{C}$ given that $2 - i\sqrt{3}$ is a root.

a. $\{-2 - i\sqrt{3}, 2 + i\sqrt{3}, -1\}$ b. $\{-2 - i\sqrt{3}, 2 - i\sqrt{3}, -1\}$

c. $\{2 - i\sqrt{3}, 2 + i\sqrt{3}, -1\}$

25. Find the distance between $P(-3, 6)$ and $Q(2, -4)$.

a. $\sqrt{101}$ b. $5\sqrt{5}$ c. $\sqrt{29}$ d. $\sqrt{5}$

26. Which point is the center of the circle with equation $x^2 + 2x + y^2 - 6y = 15$?

a. $(-1, 3)$ b. $(1, -3)$ c. $(-1, -3)$ d. $(1, 3)$

27. Write an equation of an ellipse with foci at $(0, 3)$ and $(0, -3)$ whose x-intercepts are $(4, 0)$ and $(-4, 0)$.

a. $\frac{x^2}{16} + \frac{y^2}{25} = 1$ b. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ c. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

28. Give the equations of the asymptotes to the hyperbola $\frac{x^2}{36} - \frac{y^2}{25} = 1$.

a. $y = \frac{25}{36}x,\ y = -\frac{25}{36}x$ b. $y = \frac{6}{5}x,\ y = -\frac{6}{5}x$

c. $y = \frac{5}{6}x,\ y = -\frac{5}{6}x$

29. Simplify $\left(\frac{a^3}{8}\right)^{-\frac{2}{3}}$

a. $\frac{a^2}{4}$ b. $-\frac{a^2}{4}$ c. $-\frac{4}{a^2}$ d. $\frac{4}{a^2}$

30. Solve $\log_{10} x = 2$.

a. $\{20\}$ b. $\{100\}$ c. $\{5\}$ d. $\{\sqrt{10}\}$

31. For what value of x does $3(5^x) = 27$?

a. $\frac{\log 15}{\log 27}$ b. $\frac{\log 9}{\log 5}$ c. $\frac{\log 5}{\log 9}$ d. $\frac{\log 27}{\log 15}$

32. How many different permutations exist of the letters in COFFEE?

a. 720 b. 45 c. 180 d. 90

33. In how many ways can a committee of 3 be chosen from a group of 8 people?

a. 56 b. 336 c. 120 d. 60

34. Find the second term in the binomial expansion of $(2k - 3)^5$.

a. $-48k^3$ b. $-240k^4$ c. $-56k^3$ d. $-240k^3$

35. If $A = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -4 \\ 1 & -1 \end{bmatrix}$, then simplify $2A - B$.

a. $\begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 4 \\ 5 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 4 & 8 \\ 1 & 3 \end{bmatrix}$ d. $\begin{bmatrix} -4 & -8 \\ -1 & -3 \end{bmatrix}$

36. Find the coordinates of the image Q' of the point $Q(3, 4)$ under the transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

a. $(-3, 12)$ b. $(5, -4)$ c. $(5, 4)$ d. $(3, 12)$

37. Convert $\frac{3\pi}{5}^R$ to degree measure.

a. $72°$ b. $216°$ c. $108°$ d. $36°$

38. Find $\sin\alpha$ if $\cos\alpha = -\frac{3}{5}$ and the terminal side of angle α is in quadrant II.

a. $\frac{4}{5}$ b. $\frac{3}{4}$ c. $-\frac{3}{4}$ d. $-\frac{4}{5}$

39. If $\sin x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$.

a. $-\frac{5}{12}$ b. $-\frac{12}{5}$ c. $\frac{12}{5}$ d. $\frac{5}{12}$

40. Simplify $\cos\alpha \cot\alpha + \sin\alpha$.

a. $\sec\alpha$ b. $\csc\alpha$ c. 1 d. $2\sin\alpha$

41. Find $\sin 2\alpha$ if $\cos\alpha = \frac{3}{5}$ and the terminal side of α is in quadrant IV.

a. $-\frac{24}{25}$ b. $\frac{24}{25}$ c. $-\frac{7}{25}$ d. $\frac{7}{25}$

42. Use the Law of Cosines to find b, if $a = 6.1$, $c = 3.8$ and $m(B) = 52°$.

a. 4.8 b. 4.7 c. 4.9 d. 4.6

43. Find $\tan(\text{Cos}^{-1} - \frac{1}{2})$.

a. $-\sqrt{3}$ b. $\sqrt{3}$ c. $\frac{\sqrt{3}}{3}$ d. $-\frac{\sqrt{3}}{3}$

44. Solve: $4\sin^2 x - 1 = 0$, $0 \le x \le 2\pi$.

a. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ b. $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ c. $\left\{\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}\right\}$

45. Find polar coordinates for the point $(-\sqrt{3}, 1)$.

a. $(-4, 210°)$ b. $(-2, 210°)$ c. $(2, 150°)$ d. $(4, 150°)$

Extra Practice

For use after Chapter 1

Specify the solution set of the given equation over $\mathcal{R}$. If the solution set is $\emptyset$, so state.

1. $3t - 2 = 16$
2. $2x + x = 12 - x$
3. $3y - 2 = 3y + 5$
4. $6 - 5z = 21$
5. $5k - k = 4k$
6. $b + 2 = 6b + 2$

Graph the given set. Select a suitable unit of measure.

7. $\{-3, 4, 0, -2, 1\}$
8. $\{-\frac{1}{2}, \frac{3}{4}, \frac{5}{2}, 0, -\frac{3}{2}\}$
9. $\{5, 15, -20, 25, 30, -5\}$

Simplify each expression.

10. $-28 + 39 - 12$
11. $3.2 - (5 + 7.8)$
12. $-15 + 36 - (-29 - 54)$
13. $-(7 + 16) - 3(8 - 12)$
14. $-72(\frac{1}{8} - \frac{1}{9} + \frac{1}{3})$
15. $36(\frac{1}{2})(-\frac{1}{3})(-\frac{1}{4})$
16. $29(-\frac{1}{4})(0)(\frac{8}{15})$
17. $\frac{3}{2}(6 + 5) - \frac{5}{2}(-8 + 5)$
18. $(2.6)(5)[4 + (-4)]$
19. $-3[7(2 - 5) + 6] + 4$
20. $-3[7(2 - 5 + 6) + 4]$
21. $-3[7 \cdot 2 - (5 + 6) + 4]$
22. $6 + 8 \div \frac{2}{3}$
23. $(6 + 8) \div \frac{2}{3}$
24. $12 \cdot \frac{3}{4} \div \frac{9}{2}$
25. $[-5 + 6 + 29] \cdot [\frac{1}{2} - \frac{1}{3} + \frac{1}{5}]$
26. $[\frac{1}{4}(-29 - 15) + 4] \div [6 - \frac{1}{8}(45 + 19)]$

For use after Chapter 2

Simplify.

1. $(-3x^3 + x^2 - 8) - (2x^2 + 3x - 15)$
2. $(p^4 - 3p^2q^2 - q^4) + (3p^4 - 2p^2q^2 - q^4)$
3. $(y^3 - 3y^2) - (y^2 - 9) + (-y^3 - 7)$
4. $3(t^2 - 4t) - 2(-2t^2 + 3t - 5)$

Solve each equation.

5. $4a - 3(7 - 6a) = 12$
6. $\frac{2}{3}n - 6 = 2(n - 11)$
7. $5[r - 4(r + 3) - 6] = 10(r - 4)$
8. $\dfrac{3d}{5} - \dfrac{7d}{5} = 16$
9. $\frac{1}{3}(4k - 7) - \frac{4}{3}(8 - 2k) = 5$
10. $8[p - \frac{3}{2}(7 - p) - 4] = 16$
11. The measure of one base angle of an isosceles triangle is 6° more than three times the measure of the vertex angle. Find the measures of the angles of the triangle.
12. One train can travel the same distance in 3 h that another train, going 8 km/h faster than the first, can travel in 2.5 h. What are the speeds of the two trains?

Solve each sentence over $\mathcal{R}$ and graph its solution set.

13. $5 - 3x \leq 17$
14. $2k - 9 > 3(k - 4)$
15. $-7 - \frac{1}{3}z > 1 - z$

16. $-6 < \frac{2}{3}(x - 1) < 2$

17. $-3|n - 2| \geq 9$

18. $\frac{|4p - 1|}{2} < 6$

For use after Chapter 3

If $g: x \to -2x^2 + 3x + 5$, find each of the following:

1. $g(0)$
2. $g(1)$
3. $g(-1)$
4. $g(\frac{1}{2})$
5. $g(a)$

Graph each relation over the domain $\{-2, -1, 0, 1, 2, 3\}$ and tell whether the relation is a function. If it is, give the range of the function.

6. $\{(x, y): y = 3x - 2\}$
7. $\{(x, y): y^2 = x^2\}$
8. $\{(x, y): y = 2|x|\}$

Graph each equation over $\mathcal{R}$.

9. $y = \frac{1}{2}x - 3$
10. $y = 8 - 2x$
11. $2x - 3y = 6$
12. $\frac{1}{2}x + 2y = 4$
13. $-2x + 3y = 4$
14. $x - 3y = -12$
15. $-\frac{1}{2}x + \frac{1}{6}y = 2$
16. $3(x - 2) = 2(y + 1)$
17. $2y = -8$

Graph each inequality as a shaded region on a coordinate plane.

18. $x < -2y$
19. $-x + 3y \leq 3$
20. $x - \frac{1}{2}y > -5$

Find the slope of the line passing through the two given points.

21. (3, 5) and (4, 9)
22. (−2, 5) and (3, −10)
23. (0, 4) and (−3, 7)
24. (−1, 3) and (−5, −5)
25. (6, 0) and (2, −2)
26. (10, 3) and (6, 13)
27. (8, 2) and (−2, 17)
28. $(\frac{1}{2}, -3)$ and $(-\frac{3}{2}, -1)$
29. $(0, -\frac{1}{3})$ and $(\frac{1}{3}, 2)$

Find an equation of the line with slope m and passing through point P.

30. $m = -2$; $P = (-3, 4)$
31. $m = 4$; $P = (0, 5)$
32. $m = 3$; $P = (2, -1)$
33. $m = -\frac{1}{2}$; $P = (-2, -7)$
34. $m = 0$; $P = (4, -6)$
35. $m = \frac{2}{3}$; $P = (12, -1)$

36–44. Find the equation of the line through each pair of points in Exercises 21–29 above.

In each of the following y varies directly as x, and a pair of values for x and y is given. Find the y-value for the second x-value.

45. (4, 6); (10, _?_)
46. (15, 6); (3, _?_)
47. (4, 7); (6, _?_)
48. (8, 5); (3, _?_)
49. (9, 4); (12, _?_)
50. $(\frac{1}{2}, 3)$; (5, _?_)

51. Florence Burton receives \$7.71 interest on the \$1500 she has on deposit for one month in a savings bank. How much interest would she have received on \$2000?

52. Archie Pakor took 2.5 h to type a 35-page report. How long would it take him to type a 49-page report?

For use after Chapter 4

Graph each pair of equations on one set of axes. If the system is consistent, estimate the solution set from the graph. If it is inconsistent, so state.

1. $x + y = 5$
   $4x - y = 0$
2. $2x + y = 1$
   $2x - y = -5$
3. $2x - 3y = 7$
   $3y - 2x = 6$
4. $x - y = 8$
   $3x + 4y = -4$
5. $x - 3y = 2$
   $6y - 2x = -4$
6. $5x + 2y = 10$
   $-5x - 2y = -20$

Solve each system for x and y by the linear combination or the substitution method.

7. $5x = 8 - y$
   $6x + 2y = 4$
8. $2x - 3y = 9$
   $5x + 6 = -2y$
9. $x = 7y - 4$
   $4x - 9y = 3$

Solve each system by Cramer's Rule.

10. $x - 4y = -2$
    $-2x - 6y = 1$
11. $4x - y = 6$
    $2x + 4y = 6$
12. $7x - 3y = 6$
    $5x + 7y = 6$
13. $4x + 3y = -1$
    $5x + y = 7$
14. $5x + 4y = 22$
    $3x + y = 9$
15. $5x + 3y = -9$
    $10x + 6y = 8$
16. If a boat can travel 3 km upstream in 0.5 h and 15 km downstream in 1.5 h, find the rate of the boat and the rate of the current.
17. The cost of a shipment of 24 snow shovels, made up of two different models, costing \$5 or \$8, is \$150. Find the number of shovels at each price in the shipment.
18. Three boxes of lawn seed and 4 bags of fertilizer cost \$23.50. Two boxes of lawn seed and 3 bags of fertilizer cost \$17.00. How much does one box of lawn seed cost?

In a coordinate plane, graph the solution set of each system.

19. $y > 2x - 3$
    $x > 2$
20. $x + 2y < 0$
    $2x - y > 6$
21. $y - 3 \leq x$
    $-x + 2y \geq 3$

For use after Chapter 5

Sketch the graph of each equation and give three pairs of equations that respectively define the trace of the graph in each coordinate plane. If there is no trace in one or more coordinate planes, so state.

1. $2x + 3y + 4z = 12$
2. $3z = 15$
3. $6x + 9y - 4z = -36$

Solve each system. If it is inconsistent, so state.

4. $\begin{aligned} 3x - 2y &= 12 \\ y + 4z &= 1 \\ 2x + y + z &= 2 \end{aligned}$

5. $\begin{aligned} 5x - y - 2z &= 5 \\ 4x + 6y + 2z &= 4 \\ 2x + 3y + z &= 5 \end{aligned}$

6. $\begin{aligned} x - 3y + 4z &= 10 \\ 2x - y - z &= 7 \\ x - 4y &= 1 \end{aligned}$

7. $\begin{aligned} 4x + y - z &= 0 \\ x - 5z &= 9 \\ 3x + 6y + z &= 7 \end{aligned}$

8. $\begin{aligned} 2x - 3y + z &= -3 \\ 3x + 3y - 2z &= 2 \\ -4x + 6y - z &= 6 \end{aligned}$

9. $\begin{aligned} 2x + y - 2z &= -2 \\ -x - 3y - 2z &= 5 \\ -4x - 2y + 3z &= 2 \end{aligned}$

Evaluate the given determinant.

10. $\begin{vmatrix} 3 & -1 & 2 \\ 4 & 0 & -1 \\ 8 & -2 & 5 \end{vmatrix}$

11. $\begin{vmatrix} -3 & 2 & -2 \\ -1 & 4 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

12. $\begin{vmatrix} 2 & 3 & -1 \\ 2 & 1 & 5 \\ -4 & -6 & 2 \end{vmatrix}$

13. $\begin{vmatrix} -2 & 1 & 5 \\ 2 & 0 & 4 \\ 0 & -1 & 3 \end{vmatrix}$

14. $\begin{vmatrix} 5 & -2 & 4 \\ 3 & 0 & 0 \\ -7 & 2 & -1 \end{vmatrix}$

15. $\begin{vmatrix} 4 & -1 & 0 \\ 2 & -2 & 3 \\ -7 & 5 & 6 \end{vmatrix}$

Solve using Cramer's Rule.

16. $\begin{aligned} x + y &= 2 \\ 2x - z &= 1 \\ 2y - 3z &= -1 \end{aligned}$

17. $\begin{aligned} 3x - 2y + 5z &= 6 \\ 4x - 4y + 3z &= 0 \\ 5x - 4y + z &= -5 \end{aligned}$

18. $\begin{aligned} 2x + y - z &= 3 \\ 3x + 4y + z &= 6 \\ 2x - 3y + z &= 1 \end{aligned}$

19. A collection of \$1-, \$20-, and \$50-bills with a total value of \$1200 consists of 90 bills. There are twice as many \$1-bills as \$20-bills and \$50-bills combined. How many of each are there?

Evaluate the given determinant using expansion by minors.

20. $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -4 & 1 \end{vmatrix}$

21. $\begin{vmatrix} 0 & 5 & -2 \\ -1 & 0 & 0 \\ 0 & 3 & 4 \end{vmatrix}$

22. $\begin{vmatrix} 8 & -5 & 1 & 2 \\ 0 & -3 & 0 & 7 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

For use after Chapter 6

Write each expression without negative exponents in simplest form. Assume that no variable is equal to zero.

1. $(6x^4y^{-2})(5x^{-3}y)$
2. $(\frac{3}{2}a^{-4})(-8a^6)$
3. $(2^{-3}r^4s)(16r^{-3}s^3)$
4. $(3^5ab^3c^{-2})(-3^{-2}a^{-3}bc^{-1})$
5. $(2x^2y^2)^{-3}(4x^3y)^2$
6. $(-5p^{-3}q^4)^2(-5pq^4)^{-3}$
7. $\dfrac{6m^4n^{-2}}{-3m^{-1}n^3}$
8. $\dfrac{3a^3b^{-2}}{9a^5b^{-3}}$
9. $\dfrac{(-2a^3d^{-2})^{-2}}{a^{-5}d}$
10. $\dfrac{(x+y)^2}{x^{-1}+y^{-1}}$

Write each product in simple form.

11. $(x^2 - 3)^2$
12. $(xy + 8)^2$
13. $(n^3 - m^2)^2$
14. $(7 - 3b^2c)(7 + 3b^2c)$
15. $(1 - b^5)(1 + b^5)$
16. $(4r^2 + 5s^3)^2$
17. $(t^4 - 3)^2$
18. $(2w^4 - 7)(2w^4 + 7)$

Factor the given polynomial completely.

19. $36n^2 - 60n + 25$
20. $8 - 125x^3$
21. $3a^2 - 8a - 35$
22. $d^3 + 27x^3y^3$
23. $4x^2 - yx - 18y^2$
24. $16x^4 + 56x^2 + 49$
25. $64r^3 - s^6$
26. $6t^2 - 25t + 14$
27. $16 - 49b^2c^4$
28. $8p^2 - 6pq - 27q^2$
29. $8a^3b^3 + c^3$
30. $7 - 17k^2 - 3k^4$

Determine the solution set of the given equation.

31. $x^2 - 16x + 39 = 0$
32. $3x^2 - 48 = 0$
33. $2y^2 + y - 15 = 0$
34. $7x^2 = 28x$
35. $9a^2 - 30a + 25 = 0$
36. $6n^2 - 11n = 35$
37. $4 - 5r - 6r^2 = 0$
38. $4x^2 + 5 = 21x$
39. $8m^2 = 26m + 45$

40. Two adjacent sides of a rectangular lot of area 60 m² are to be fenced with 19 m of fencing. What should the dimensions of the lot be?
41. One leg of a right triangle with hypotenuse 15 cm is 3 cm longer than the other leg. Find the lengths of the two legs.
42. A margin of uniform width is to be left around the edges of a printed page that measures 16 cm by 20 cm and is to contain 192 cm² of printing. How wide should the margin be?

Solve each inequality and graph its solution set.

43. $x^2 - 3x - 18 \geq 0$
44. $x^2 - 10x + 21 < 0$
45. $2y^2 - 50 > 0$
46. $80 < 5k^2$
47. $x^2 - 14x + 49 > 0$
48. $1 - 6x + 9x^2 \leq 0$

Simplify the given expression.

49. $(x - y)^2(x^2 - y^2)^{-1}$
50. $(z^3 + 8)(z + 2)^{-2}$
51. $(2a^2 - 3a + 1)(4a^2 - 1)^{-1}$

Transform the given rational expression into a sum by dividing.

52. $\dfrac{3c^3 - 5c^2 + c - 6}{c - 2}$
53. $\dfrac{2x^3 - 5x^2 + 13x - 13}{2x - 3}$
54. $\dfrac{125y^3 - 8}{5y - 2}$
55. $\dfrac{3x^3 - 19x^2 + 3}{3x - 1}$
56. $\dfrac{2t^4 + 5t^3 - 4t^2 + 9}{t + 3}$
57. $\dfrac{4x^4 - 19x^3 + 5x + 1}{4x - 3}$

Express each product or quotient in lowest terms.

58. $\dfrac{x^2 - 3x + 2}{x^2 - 4} \cdot \dfrac{(x + 1)^2}{4x^2 - 4}$
59. $\dfrac{a^3 - b^3}{a^2 - b^2} \cdot \dfrac{(a + b)^2}{a^3 + b^3}$
60. $\dfrac{2y^2 - 5y - 3}{4(y - 3)^2} \div \dfrac{2y^2 + 7y + 3}{2(y^2 - 9)}$
61. $\dfrac{z^3 - 16z^2}{2z^2 - 4z + 2} \div \dfrac{z^3 - 8z^2 + 16z}{z^2 - 5z + 4}$

Simplify the given rational expression.

62. $\dfrac{x+y}{x^2-4y^2} - \dfrac{x}{x^2-4xy+4y^2}$

63. $\dfrac{x+1}{x^2-3x+2} - \dfrac{x-2}{x^2-2x+1}$

64. $\dfrac{a}{a-b} - \dfrac{b}{a+b} - \dfrac{a^2+b^2}{a^2-b^2}$

65. $\dfrac{r^2+t^2}{2(r+t)^3} - \dfrac{r-t}{2(r+t)^2}$

Determine the solution set over $\mathcal{R}$.

66. $\dfrac{x}{x+1} - \dfrac{2x+5}{(x+1)(x+3)} = 3$

67. $\dfrac{r-2}{r-1} = \dfrac{r^2-3}{r^2-1} - \dfrac{r-2}{r+1}$

68. $\dfrac{y-3}{(y-2)^2} + 5 = \dfrac{-3y}{y-2}$

69. $\dfrac{z-1}{z^2-5z+6} - \dfrac{z-2}{z^2-4z+3} = \dfrac{z+1}{z^2-3z+2}$

70. Three machines can plow a field in 4 h, 6 h, and 12 h, respectively, each working alone. How long would it take all three working together?

71. Herb rows 8 km upstream and 8 km downstream in 5 h 20 min. If the speed of the current is 2 km/h, how fast does Herb row?

For use after Chapter 7

Find the specified term of each arithmetic sequence.

1. $5, 2, -1, \ldots; a_{14}$
2. $-7, -\frac{5}{2}, 2, \ldots; a_{17}$
3. $0.8, 0.45, 0.1, \ldots; a_{15}$
4. $-0.4, 2.6, 5.6, \ldots; a_{16}$
5. $\frac{5}{6}, \frac{3}{2}, \frac{13}{6}, \ldots; a_{19}$
6. $-\frac{7}{8}, -\frac{1}{2}, -\frac{1}{8}, \ldots; a_{18}$

Insert the stated number of arithmetic means between the given numbers.

7. Three between -7 and 9
8. Four between $\frac{5}{2}$ and 8
9. Four between 78 and -68
10. Three between $-\frac{1}{5}$ and $-\frac{1}{80}$

Find d and a_1 for the arithmetic sequence with the specified values.

11. $a_3 = 9, a_{11} = -15$
12. $a_{12} = -27, a_8 = -21$
13. $a_4 = \frac{1}{2}, a_{13} = 2$
14. $a_7 = -4.6, a_{15} = 0.6$

Find the sum of each of the following arithmetic series.

15. $\sum_{n=1}^{9} (3n-5)$

16. $\sum_{k=1}^{10} (1-7k)$

17. $\sum_{i=1}^{14} (\frac{1}{3}i + 4)$

18. A series of 9 terms, beginning $-13 - 5 + 3 \cdots$

19. A series of 12 terms beginning $\frac{7}{2} + 2 + \frac{1}{2} \cdots$

20. The multiples of 3 between 0 and 100

Find the sum of the arithmetic series.

21. The even numbers between 101 and 199

22. The series $-2 + 1 + 4 + \cdots + 73$

Find the requested term of the given geometric sequence.

23. $\frac{5}{8}, \frac{5}{4}, \frac{5}{2}, \ldots; a_8$

24. $250, 50, 10, \ldots; a_6$

25. $9, -6, 4, \ldots; a_7$

26. $0.003, 0.3, 30, \ldots; a_6$

27. $-\frac{2}{9}, \frac{2}{3}, -2, \ldots; a_7$

28. $111\frac{1}{9}, 11\frac{1}{9}, 1\frac{1}{9}, \ldots; a_7$

Insert the stated number of geometric means between the given pairs of numbers.

29. Two between $\frac{2}{9}$ and 6

30. Two between -24 and $\frac{3}{8}$

31. Two between $\frac{2}{9}$ and $\frac{3}{4}$

32. Three between -125 and $-\frac{16}{5}$

Find the sum of each geometric series.

33. $\sum_{n=1}^{8} \frac{1}{4}(2)^{n-1}$

34. $\sum_{k=1}^{6} \frac{9}{2}(-\frac{1}{3})^{k-1}$

35. $\sum_{i=1}^{7} -\frac{3}{2}(-5)^{i-1}$

36. The first 5 positive integral powers of $\frac{1}{2}$

37. The first 6 positive integral powers of -2

38. The first 7 terms of the series $\frac{5}{2} + 5 + 10 + \cdots$

39. The first 6 terms of the series $160 - 40 + 10 - \cdots$

Write the first 4 terms of each of the following sequences and guess the limit of the sequence, or state that the sequence is not convergent.

40. $a_n = \dfrac{n+1}{2n}$

41. $a_n = 2 - \dfrac{1}{2n}$

42. $a_n = \dfrac{n^3}{1000}$

43. $a_n = (-1)^n \dfrac{3n^2}{n^2 - 2}$

Find the sum of the infinite geometric series. If the series is not convergent, so state.

44. $\frac{1}{8} + \frac{3}{4} + \frac{9}{2} + \cdots$

45. $500 - 100 + 20 - \cdots$

46. $\sum_{i=1}^{\infty} \frac{3}{4}(\frac{2}{3})^{i-1}$

Convert the given nonterminating decimal to a fraction by rewriting as an infinite geometric series and finding the sum of the series.

47. $0.30303\cdots$

48. $0.545454\cdots$

49. $0.037037037\cdots$

50. $0.801801801\cdots$

For use after Chapter 8

Find the value of a for which the given point lies on the graph of the given equation.

1. $(-\frac{1}{2}, 6)$; $y = ax^2$

2. $(2, -3)$; $y = ax^3$

3. $(3, 24)$; $y = ax^3$

4. $(3, 54)$; $y = ax^4$

5. $(\frac{3}{4}, -\frac{1}{2})$; $y = ax^2$

6. $(-\frac{3}{2}, -\frac{9}{8})$; $y = ax^3$

Solve over the set of rational numbers.

7. $25x^2 = 49$
8. $0.027x^3 + 0.008 = 0$
9. $81 + x^4 = 0$
10. $1 - 2.25x^2 = 0$
11. $2x^3 - 0.25 = 0$
12. $27x^4 - 5\frac{1}{3} = 0$

Find the rational roots of the given equation by trying each of the possible roots as a value for x.

13. $x^3 - 8x + 3 = 0$
14. $x^3 + 2x^2 - x - 2 = 0$
15. $2x^3 + 6x^2 + 2x - 4 = 0$
16. $x^4 - x^3 - 7x^2 + x + 6 = 0$

Express as a fraction in lowest terms.

17. $0.\overline{78}$
18. $0.4\overline{09}$
19. $2.\overline{027}$
20. $0.19\overline{3}$

Simplify.

21. $\sqrt[3]{(-8)^4}$
22. $\dfrac{3\sqrt{50}}{5}$
23. $\sqrt{\frac{27}{125}} \cdot \sqrt{18}$
24. $\sqrt[3]{\frac{135}{4}}$
25. $-4\sqrt{75x^3} + 3\sqrt{147x^3}$
26. $2\sqrt{6}(3\sqrt{12} - 5\sqrt{27})$
27. $(2\sqrt{7} - \sqrt{5})(2\sqrt{7} + \sqrt{5})$
28. $(3\sqrt{5} - 4)^2$
29. $\dfrac{4}{2\sqrt{3} - 5}$
30. $\dfrac{\sqrt{2} - 7}{\sqrt{5} + 3}$

Solve over $\mathcal{R}$.

31. $\sqrt{x - 4} + x = 6$
32. $\sqrt{6 + x} - 3x = 8$
33. $\sqrt{x} + \sqrt{x + 4} = 4$
34. $\sqrt{x + 4} - \sqrt{x} = \sqrt{3}$
35. $\sqrt{x - 2} - \sqrt{x + 3} = 1$
36. $\sqrt{2 - x} + \sqrt{8 + x} = 4$
37. $\sqrt{2x + 1} - \sqrt{x - 3} = 2$
38. $\sqrt{3 - 2x} = \sqrt{3 + x} + 3$

Solve by completing the square.

39. $x^2 - 6x + 4 = 0$
40. $x^2 + 10x = -13$
41. $2x^2 = 6x - 1$
42. $3x^2 - 2x = 7$
43. $5x^2 + 10x + 1 = 0$
44. $2x^2 = x + 7$

Solve using the quadratic formula.

45. $x^2 - 7x = 5$
46. $2x^2 - 4x + 1 = 0$
47. $5x^2 - 4x = 3$
48. $4x^2 - 7x - 2 = 0$
49. $x^2 - \frac{2}{3}x = 2$
50. $3x^2 - \frac{1}{2}x = 1$
51. The length of a rectangle is 2 cm greater than its width and its area is 25 cm^2. Find the dimensions of the rectangle.
52. What is the length of a side of the square that must be cut out of a rectangle measuring 6 cm by 4 cm, if the remaining area is to equal that of a rectangle of width 3 cm and length equal to a side of the cutout square?

For use after Chapter 9

Express each of the following in the form $a + bi$, or as a real number a, or an imaginary number bi, or 0.

1. $\sqrt{-80}$
2. $(-3i)(7i)$
3. $\sqrt{-12}\sqrt{-3}$
4. $\frac{-18i}{-6}$
5. $\frac{6}{3i}$
6. $-2\sqrt{-50}$
7. $\frac{-1}{2i^3}$
8. $\frac{9}{i^6}$
9. $(3 - 5i) + (-7 + 2i)$
10. $\left(-6 - \frac{i}{2}\right) - \left(4 + \frac{3}{2}i\right)$
11. $(5 - 2i) - \frac{2i}{3}$
12. $\frac{5}{2} - \left(-3 + \frac{i}{4}\right)$
13. $(2 + 7i)(-3 + 4i)$
14. $(\sqrt{2} - i\sqrt{2})^2$
15. $(-3 + 5i)(-3 - 5i)$
16. $(4 - 3i)^2$
17. $\frac{3}{4 - i}$
18. $\frac{5i}{2 + i}$
19. $\frac{1 + 6i}{2 - 3i}$
20. $\frac{7 - i}{3 + 4i}$

Solve over $\mathcal{C}$, the set of complex numbers.

21. $x^2 - 4x + 5 = 0$
22. $x^2 + 8x + 25 = 0$
23. $x^2 + 6x + 12 = 0$
24. $2x^2 - 3x + 7 = 0$
25. $-3x^2 + 4x - 4 = 0$
26. $5x^2 + 2x\sqrt{6} + 3 = 0$
27. $3x^2 - x\sqrt{15} + 8 = 0$
28. $x^2 3\sqrt{2} - 4x + \sqrt{2} = 0$

Write a quadratic equation with the given solution set.

29. $\{2, -7\}$
30. $\left\{\frac{i}{3}, -\frac{i}{3}\right\}$
31. $\{5 - 2\sqrt{3}, 5 + 2\sqrt{3}\}$
32. $\{3 + 2i, 3 - 2i\}$
33. $\{3i, -3i\}$
34. $\{3 + i\sqrt{5}, 3 - i\sqrt{5}\}$

Give the equation of the axis of symmetry and the coordinates of the vertex of the graph of each function.

35. $f(x) = x^2 - 8x + 18$
36. $f(x) = x^2 + 5x$
37. $f(x) = -x^2 + 6x$
38. $f(x) = 2x^2 - 20x + 47$
39. $f(x) = 3x^2 - 8$
40. $f(x) = \frac{1}{2}x^2 - 3x - 2$

Solve by sketching the graph of the related quadratic function.

41. $x^2 - 16 \geq 0$
42. $5x - x^2 \geq 0$
43. $2x^2 + 3x < 0$
44. $x^2 - x \leq 12$
45. $2x^2 - 3x > 2$
46. $5 - 2x^2 \leq 3x$

Use synthetic substitution to find the requested values of the given polynomial.

$P(x) = x^4 - 2x^3 - x^2 - 8x - 12$: 47. $P(2)$ 48. $P(3)$ 49. $P(-2)$ 50. $P(2i)$

For each polynomial $P(x)$, find the remainder when the polynomial is divided by the given binomial.

51. $P(x) = x^4 - 3x^3 - 8x^2 + x - 6$; $x + 2$

52. $P(x) = x^3 - 5x - 10$; $x - 3$

Find the other roots of the equation, given the root at the right.

53. $x^4 - x^3 - 5x^2 - x - 6 = 0$; i

54. $x^4 + x^3 - 2x^2 + 2x + 4 = 0$; $1 - i$

For use after Chapter 10

Find the distance between each pair of points and, also, the midpoint of the line segment joining them.

1. $(-2, 5), (4, -3)$
2. $(1, -2), (5, 1)$
3. $(-8, -3), (4, 2)$
4. $(7, 12), (0, -12)$
5. $(2\sqrt{5}, -5), (4\sqrt{5}, -7)$
6. $(-3, \sqrt{2}), (-4, 3\sqrt{2})$

Find an equation of the line perpendicular to the given line and passing through the given point.

7. $x + 2y = 3$; $(-3, 5)$
8. $2x - 3y = 5$; $(4, -1)$
9. $3x = 4y - 7$; $(-6, -5)$
10. $2x + 5y = 0$; $(3, -2)$
11. $5x = 3y + 8$; $(-2, \frac{4}{5})$
12. $4y + 3x = 12$; $(4, -6)$

Sketch the graph of the circle defined by each equation.

13. $x^2 + y^2 - 6x = 0$
14. $x^2 + y^2 - 4x + 6y - 3 = 0$
15. $x^2 + y^2 + 2x + 8y + 16 = 0$
16. $2x^2 + 2y^2 + 2x - 10y + 5 = 0$

Sketch the graph of the parabola defined by each equation.

17. $y = x^2 - 6x + 4$
18. $y = -2x^2 - 8x - 7$
19. $x = -\frac{1}{2}y^2 - y + \frac{7}{2}$

Find an equation of the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ for the parabola with focus F and directrix the line with the given equation.

20. $F(4, 0)$; $x = -2$
21. $F(0, 3)$; $y = 1$
22. $F(0, 4)$; $y = -2$

Sketch the graph of the ellipse defined by each equation, and give the coordinates of the foci.

23. $100x^2 + 49y^2 = 4900$
24. $9x^2 + 144y^2 = 36$
25. $100x^2 + 25y^2 = 225$

Find an equation for the ellipse with center at the origin and with the given characteristics.

26. Major axis of length 10; x-intercepts 4 and -4.
27. Foci at $(3, 0)$ and $(-3, 0)$; sum of focal radii 8.

Sketch the hyperbola defined by each equation, by first drawing the asymptotes of the graph. Give the coordinates of the foci.

28. $5y^2 - 5x^2 = 45$ **29.** $4x^2 - 25y^2 = 225$ **30.** $100y^2 - 625x^2 = 900$

Find an equation for the hyperbola with center at the origin and with the given characteristics.

31. Foci at $(4, 0)$ and $(-4, 0)$; length of conjugate axis 6.

32. Intercepts at $(3, 0)$ and $(-3, 0)$; foci at $(4, 0)$ and $(-4, 0)$.

In the given variation y varies inversely as x. Determine x_2 or y_2.

33. $(6, \frac{1}{3}), (-10, y_2)$ **34.** $(-3, \frac{1}{2}), (12, y_2)$ **35.** $(\frac{1}{2}, \frac{8}{3}), (x_2, 6)$

Solve each system by substitution.

36. $\begin{aligned} x^2 + y^2 &= 10 \\ 16x^2 + y^2 &= 25 \end{aligned}$ **37.** $\begin{aligned} 100x^2 + 225y^2 &= 22{,}500 \\ x^2 + y^2 &= 145 \end{aligned}$ **38.** $\begin{aligned} 16x^2 - y^2 &= 256 \\ x^2 + y^2 &= 169 \end{aligned}$

39. $\begin{aligned} 16x^2 + 25y^2 &= 625 \\ x^2 - y^2 &= 16 \end{aligned}$ **40.** $\begin{aligned} 25x^2 + 9y^2 &= 625 \\ y^2 - x^2 &= 9 \end{aligned}$ **41.** $\begin{aligned} 16 &= y^2 - x^2 \\ y &= x^2 - 4 \end{aligned}$

For use after Chapter 11

Write in simplified radical form.

1. $\sqrt[6]{8}$ **2.** $\sqrt[9]{\frac{125}{27}}$ **3.** $\sqrt{(\frac{4}{9})^3}$ **4.** $\sqrt[4]{36x^2}$

5. $\sqrt[3]{4} \cdot \sqrt[6]{32}$ **6.** $\dfrac{\sqrt[12]{81}}{\sqrt{3}}$ **7.** $\dfrac{\sqrt{2}}{\sqrt[6]{2}}$ **8.** $\sqrt[3]{4} \cdot \sqrt[12]{16}$

Solve over $\mathcal{R}$.

9. $4^{x+3} = 16^x$ **10.** $3^{2x} = 27^{x-2}$ **11.** $4^{2x-7} = 2^{x+1}$

12. $4^{2x+1} = 8^{x-1}$ **13.** $3^{2x+2} = 81^{x-1}$ **14.** $125^{x-6} = 25^{2x-3}$

15. $(\frac{1}{3})^{x+1} = 9^{x+5}$ **16.** $(\frac{1}{4})^{x-1} = 8^{-2x-2}$ **17.** $(\frac{1}{16})^{x+1} = (\frac{1}{32})^{1-2x}$

Solve by converting each statement to exponential form and solving the resulting equation.

18. $\log_x 8 = \frac{3}{2}$ **19.** $\log_5 x = -3$ **20.** $\log_{27} x = -\frac{2}{3}$

21. $\log_x \sqrt{2} = \frac{3}{4}$ **22.** $\log_{\frac{1}{25}} 125 = x$ **23.** $\log_{\sqrt{3}} x = -4$

24. $\log_x \sqrt{3} = -\frac{1}{6}$ **25.** $\log_2 \sqrt{8} = 2x$ **26.** $\log_5 \dfrac{1}{125} = \dfrac{x}{2}$

Solve over $\mathcal{R}$.

27. $\log x + \log 2 = 1$ **28.** $\log x - \log 4 = \log 9 - \log x$

29. $\log 2x^2 - \log 18 = 2$ **30.** $\log 3x^3 - \log x^2 = \log 12$

31. $2 \log x + \log 5 = \log 45$

32. $2 \log 4 - 3 \log x = \log 54$

33. $\frac{1}{2} \log x - \log 5 = -2$

34. $4 \log 2 + \frac{1}{3} \log x = 3 \log 2$

Use interpolation to find the logarithm of the given number.

35. 2.217 36. 14,230 37. 0.04962 38. 0.001634

Use interpolation to find the antilog of the given number.

39. 0.1685 40. 2.3897 41. 9.9296 − 10 42. 7.2388 − 10

Compute each of the following using logarithms.

43. $\dfrac{(671)(0.0038)}{984}$

44. $\dfrac{(97{,}600)(0.0704)}{(653)(0.142)}$

45. $\dfrac{7350}{(0.00463)(81)}$

46. $\dfrac{\sqrt[5]{0.354}}{\sqrt[6]{0.0418}}$

47. $\dfrac{\sqrt[7]{35{,}200}(0.897)^3}{714}$

48. $\dfrac{\sqrt[3]{(0.00213)(8.68)}}{\sqrt[4]{0.497}}$

49. To the nearest hundred dollars, how much would \$3000, invested at 12% compounded quarterly, grow to in 11 yr?

Solve over $\mathcal{R}$. Express each solution to 3 significant figures.

50. $2.3^x = 148$ 51. $x^{1.8} = 76300$ 52. $(2x)^{\frac{5}{6}} = 200$ 53. $(3.09^{2.4})x = 75$

54. A town's population (N) increases according to the formula

$$N = N_0(1.02)^{2t}$$

where N is the town's population t yr after it was N_0. If the population is now 20,400, how long ago was it 10,000?

For use after Chapter 12

1. How many sequences of 3 numbers can be generated by rolling a die three times in succession?
2. How many 4-digit numbers are there whose first and third digits are both odd and which are divisible by 5?
3. How many even 4-digit numbers are there whose second and third digits are odd?
4. In how many different orders can 7 actors in a play be listed in the program?
5. In how many different ways can 5 differently colored stones be set symmetrically in a ring around another stone?
6. How many rearrangements of the letters of the word COSINE begin and end with a vowel?

How many distinguishable permutations are there of each word?

7. MILLION 8. TOMORROW 9. BOOKKEEPER 10. PHILOSOPHICAL

11. How many pairs of co-chairpersons can be chosen from among a group of 12?

12. How many ways are there of getting exactly 4 heads out of 10 flips of a coin?

13. How many different sums of money can be made by taking 3 coins from a change purse containing one penny, one nickel, one dime, one quarter, and one half-dollar?

14. How many ways can 5 students be chosen for a study group out of a class of 15 if one particular student must be included and another one excluded?

How many ways are there of choosing 2 red and 3 green marbles from a bag containing the given number of each type?

15. 5 red, 4 green

16. 4 red, 6 green

17. 6 red, 5 green

18. In how many ways can 4 boys and 4 girls be chosen from a group of 7 boys and 8 girls?

19. How many choices of 3 shades of each color can be chosen from a paint box containing 4 shades of red, 6 of yellow, and 7 of blue?

20. In how many ways can 5 paintings be lined up on one wall of an art gallery and 4 other paintings be lined up on the opposite wall?

Expand each binomial and write the result in simplified form.

21. $(b + 2)^5$

22. $(ab - 1)^7$

23. $(x^2 - \frac{1}{2})^6$

24. $(2c + d^2)^6$

Given the eighth row of Pascal's triangle: 1, 7, 21, 35, 35, 21, 7, 1

25. Find the ninth row.

26. Find the tenth row.

List a sample space for the experiment: a coin is flipped 4 times.

27. Event: exactly 1 head

28. Event: exactly 2 heads

A change purse contains 4 pennies, 5 dimes, and 7 quarters. If two coins are drawn out at random, what is the probability of the following events?

29. Both are pennies.

30. Both are dimes.

31. Neither is a dime.

32. Neither is a quarter.

33. One is a dime and one is a penny.

A coin is flipped 5 times in succession. What is the probability of getting:

34. at least 4 tails?

35. at least 3 tails?

36. at least 3 heads or at least 3 tails?

For use after Chapter 13

Find the values of the variables for which the given statement is true.

1. $[w + x \quad x - 1 \quad x + y] = [5 \quad 2 \quad 2]$

2. $[w - x \quad 1 \quad y - 3] = [-1 \quad x + y \quad 3]$

3. $\begin{bmatrix} w + y & x + y \\ x + 2 & w - z \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} w - 2x \\ 2w + x \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

5. $[w \quad x \quad y] + [x \quad y \quad w] = [2 \quad 3 \quad 7]$

6. $\begin{bmatrix} w & x \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ -z & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}$

Solve for the matrix X.

7. $X + \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & -6 \end{bmatrix}$

8. $\begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix} - X = \begin{bmatrix} -1 & 3 \\ -5 & 8 \end{bmatrix}$

9. $X - [3 \quad -2 \quad 5] = [4 \quad 3 \quad -1]$

10. $[-7 \quad 3 \quad -9] + X = [-5 \quad 2 \quad 0]$

Find the 2×2 matrix X that satisfies the given equation for

$$J = \begin{bmatrix} 17 & -12 \\ -5 & 16 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}.$$

11. $X = 2J$
12. $X = -3K$
13. $X = J - 2K$
14. $\frac{1}{4}X = K$
15. $X + J = 2K$
16. $X - 4K = J$
17. $3X = 2J + K$
18. $5X = J - K$

Find the given product for

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix},\ \text{and } C = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & 4 \end{bmatrix}.$$

19. AB
20. BA
21. AC
22. BC
23. A^2
24. $(A + B)^2$

For the same A, B, and C as above compute each of the following pairs and compare the answers for each pair to see whether they are equal.

25. $A(BC)$ and $(AB)C$
26. $(A + B)C$ and $AC + BC$
27. $(AB)C$ and $(BA)C$

Find the inverse of the given matrix.

28. $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$
29. $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$
30. $\begin{bmatrix} 11 & 3 \\ -7 & -2 \end{bmatrix}$
31. $\begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix}$
32. $\begin{bmatrix} -7 & 5 \\ -5 & 4 \end{bmatrix}$

Solve each matrix equation for X, for

$$D = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix},\ E = \begin{bmatrix} -2 & -3 \\ 4 & 7 \end{bmatrix},\ \text{and } F = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$

33. $DX = E$
34. $FX = E$
35. $EX = D$
36. $EX = F$

For use after Chapter 14

Find the distance a point on the rim of the wheel of given radius will travel in the given number of revolutions or by rotation through an angle with the given measure. Use $\pi \approx \frac{22}{7}$.

1. 35 cm; 2.7 revolutions
2. 154 cm; 135°

Given $m^\circ(\alpha)$, find $m^R(\alpha)$.

3. 150°
4. −210°
5. 420°
6. −315°
7. −740°

Given $m^R(\alpha)$, find $m^\circ(\alpha)$.

8. $\frac{2\pi}{3}^R$
9. $\frac{7\pi}{4}^R$
10. $-\frac{5\pi}{3}^R$
11. $-\frac{9\pi}{4}^R$
12. $\frac{16\pi}{3}^R$

Find the given function values without tables.

13. $\sin 390^\circ$
14. $\cos 720^\circ$
15. $\cos 495^\circ$
16. $\sin \frac{11\pi}{3}^R$
17. $\cos \left(\frac{-5\pi}{2}^R\right)$
18. $\sin 8\pi^R$
19. $\cos(-690^\circ)$
20. $\sin(-600^\circ)$
21. $\cos \frac{17\pi}{6}^R$
22. $\sin \frac{15\pi}{4}^R$

Use Tables 6 and 7 to find a four-significant-digit approximation of the given function value.

23. $\cos 54^\circ 32'$
24. $\sin 32^\circ 17'$
25. $\cos 0.872^R$

In Exercises 26–28, give the maximum and minimum values of the function and state the amplitude.

26. $y = 3 \sin x + 2$
27. $y = -2 \cos x + 1$
28. $y = \frac{1}{2} \sin x - 3$

Sketch the graph of each function over the interval $-2\pi \le x \le 2\pi$.

29. $y = \frac{3}{2} \cos x$
30. $y = 2 \sin x - 1$
31. $y = -3 \cos x + 2$

Find the values of the other 5 trigonometric functions of the angle α in the given quadrant and having the given function value.

32. IV; $\cos \alpha = \frac{4}{5}$
33. III; $\sec \alpha = -2$
34. II; $\csc \alpha = 3$

Solve the right triangle with the given parts. In each triangle $\angle C$ is a right angle. State lengths of sides to three significant digits and angle measures to the nearest 10′.

35. $m(B) = 25^\circ 30'$; $c = 70$
36. $a = 43$, $c = 50$

For use after Chapter 15

Prove each identity.

1. $\dfrac{1 + \cos \alpha}{\cos \alpha} = 1 + \sec \alpha$

2. $\dfrac{\sec^2 x}{1 + \cot^2 x} = \tan^2 x$

3. $\sec x(\sec x - \cos x) = \tan^2 x$

4. $\sin \theta + \cos \theta \cot \theta = \csc \theta$

5. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

6. $(\tan B + \cot B)^2 = \sec^2 B + \csc^2 B$

7. $\tan y + \cot y = \dfrac{1}{\cos y \sin y}$

8. $\csc x - \cot x = \dfrac{\sin x}{1 + \cos x}$

9. $1 - \dfrac{\cot^2 A - 1}{\cot^2 A + 1} = 2 \sin^2 A$

10. $\sin \alpha = 1 - \dfrac{\cot \alpha \cos \alpha}{1 + \csc \alpha}$

Given $\alpha = 135°$, $\beta = 60°$, $\gamma = 210°$, evaluate each of the following.

11. $\sin (\alpha + \beta)$

12. $\cos (\alpha - \beta)$

13. $\cos (\alpha + \gamma)$

14. $\sin (\gamma - \alpha)$

15. $\sin (\alpha - \beta)$

16. $\tan (\alpha + \beta)$

17. $\tan (\beta - \alpha)$

18. $\tan (\alpha + \gamma)$

Prove each identity.

19. $\dfrac{\sin 2x}{2 \sin^2 x} = \cot x$

20. $\dfrac{\tan x + \cot x}{\csc 2x} = 2$

21. $\tan \left(\dfrac{x}{2}\right) = \dfrac{1 - \cos x}{\sin x}$

22. $\csc 2x + \cot 2x = \cot x$

If angle x is in the second quadrant and $\sin x = \frac{4}{5}$, evaluate the given function.

23. $\sin 2x$

24. $\sin \left(\dfrac{x}{2}\right)$

25. $\cos 2x$

26. $\cos \left(\dfrac{x}{2}\right)$

27. $\tan 2x$

28. $\tan \left(\dfrac{x}{2}\right)$

Evaluate the given function using the half-angle formulas.

29. $\sin \dfrac{\pi}{12}$

30. $\tan \dfrac{\pi}{8}$

31. $\cos \left(-\dfrac{\pi}{8}\right)$

32. $\sin \left(-\dfrac{3\pi}{8}\right)$

33. $\tan \dfrac{5\pi}{12}$

34. $\cos \dfrac{7\pi}{8}$

35. In $\triangle ABC$, if $b = 5\sqrt{2}$, $c = 9$, and $m(A) = 45°$, find a.

36. In $\triangle ABC$, if $m(B) = 42°$, $m(C) = 54°$, and $b = 80$, find the length of a to the nearest tenth.

For use after Chapter 16

Evaluate without using tables.

1. $\text{Sin}^{-1}(0)$
2. $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
3. $\text{Sin}^{-1}\left(\frac{\sqrt{2}}{2}\right)$
4. $\text{Sin}^{-1}(\sin 180°)$
5. $\text{Sin}^{-1}\left(\cos\frac{\pi}{4}\right)$
6. $\text{Cos}^{-1}\left(\sin\frac{5\pi}{6}\right)$

State the value of each of the following for both inverse angle and inverse circular functions to the nearest degree and nearest hundredth, respectively.

7. $\text{Sin}^{-1}(0.7923)$
8. $\text{Cos}^{-1}(0.3581)$
9. $\text{Sin}^{-1}(0.2387)$
10. $\text{Tan}^{-1}(1.423)$

Solve for x, $0 \leq x < 2\pi$.

11. $-\cos x = \sin x$
12. $\sqrt{3}\sec x - 2 = 0$
13. $\cos 2x + \sin x = 0$
14. $\sin 2x = 2\cos x$

Give the general solution for $x \in \mathcal{R}$.

15. $2\sin x\cos x = \sqrt{3}\cos x$
16. $3\tan^2 x + \sec^2 x = 5$
17. $2\sin^2 x - 3\cos x = 3$
18. $2\cos^2 2x = 1 - \cos 2x$

Find a set of polar coordinates for the given point with $-180° < m(\alpha) \leq 180°$. Answer to the nearest 10′.

19. $(-\sqrt{2}, -\sqrt{2})$
20. $(3, -3\sqrt{3})$
21. $(7, 24)$

Find the Cartesian coordinates of the given point.

22. $(5, 135°)$
23. $(3, -60°)$
24. $\left(\sqrt{2}, \frac{7\pi}{6}\right)$

Given $z_1 = 4(\cos 60° + i\sin 60°)$ and $z_2 = 20(\cos 210° + i\sin 210°)$, evaluate the following. Express each answer in the form $a + bi$.

25. $z_1 z_2$
26. $\frac{z_2}{z_1}$
27. z_2^2
28. z_1^4

Use De Moivre's theorem to solve each of the following. Leave each solution in polar form.

29. $z^3 = 8$
30. $z^3 = -i$
31. $z^4 = 16i$

32. For the vectors **u** and **v** such that $\|\mathbf{u}\| = 10$ and the direction of **u** is 60° and $\|\mathbf{v}\| = 12$ and the direction of **v** is 45°, find $\|(\mathbf{u}+\mathbf{v})_x\|$ and $\|(\mathbf{u}+\mathbf{v})_y\|$ to the nearest tenth and $\|\mathbf{u}+\mathbf{v}\|$ to the nearest unit.

Appendix

Table 1 Formulas

Circle	$A = \pi r^2, C = 2\pi r$	Cube	$V = s^3$
Parallelogram	$A = bh$	Rectangular Box	$V = lwh$
Right Triangle	$A = \frac{1}{2}bh, c^2 = a^2 + b^2$	Cylinder	$V = \pi r^2 h$
Square	$A = s^2$	Pyramid	$V = \frac{1}{3}Bh$
Trapezoid	$A = \frac{1}{2}h(b + b')$	Cone	$V = \frac{1}{3}\pi r^2 h$
Triangle	$A = \frac{1}{2}bh$	Sphere	$V = \frac{4}{3}\pi r^3$
Sphere	$A = 4\pi r^2$		

Table 2 Metric Units of Measure

Base Units

Length: meter (m)
Mass: kilogram (kg)*

Time

second (s), minute (min)
day (d), month (mo), year (yr)

Temperature

degree Celsius (°C)
degree Kelvin (°K)

Capacity

liter (L)
$1 \text{ L} = 1000 \text{ cm}^3$

Force

Newton (N)

Pressure

Pascal (Pa)

Prefixes

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10	deka	da	10^{-18}	atto	a

A prefix multiplies a unit by the factor given in the table.

Examples gigameter: $1 \text{ Gm} = 10^9 \text{ m} = 1{,}000{,}000{,}000 \text{ m}$
milligram: $1 \text{ mg} = 10^{-3} \text{ g} = 0.001 \text{ g}$*

Compound units may be formed by division or multiplication.

Examples kilometers per hour: km/h square centimeters: cm^2 cubic meters: m^3

*Although the kilogram is defined as the base unit, the gram (g) is used with the prefixes to name other units of mass.

Table 3 Squares and Square Roots

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1.0	1.00	1.000	3.162	**5.5**	30.25	2.345	7.416
1.1	1.21	1.049	3.317	**5.6**	31.36	2.366	7.483
1.2	1.44	1.095	3.464	**5.7**	32.49	2.387	7.550
1.3	1.69	1.140	3.606	**5.8**	33.64	2.408	7.616
1.4	1.96	1.183	3.742	**5.9**	34.81	2.429	7.681
1.5	2.25	1.225	3.873	**6.0**	36.00	2.449	7.746
1.6	2.56	1.265	4.000	**6.1**	37.21	2.470	7.810
1.7	2.89	1.304	4.123	**6.2**	38.44	2.490	7.874
1.8	3.24	1.342	4.243	**6.3**	39.69	2.510	7.937
1.9	3.61	1.378	4.359	**6.4**	40.96	2.530	8.000
2.0	4.00	1.414	4.472	**6.5**	42.25	2.550	8.062
2.1	4.41	1.449	4.583	**6.6**	43.56	2.569	8.124
2.2	4.84	1.483	4.690	**6.7**	44.89	2.588	8.185
2.3	5.29	1.517	4.796	**6.8**	46.24	2.608	8.246
2.4	5.76	1.549	4.899	**6.9**	47.61	2.627	8.307
2.5	6.25	1.581	5.000	**7.0**	49.00	2.646	8.367
2.6	6.76	1.612	5.099	**7.1**	50.41	2.665	8.426
2.7	7.29	1.643	5.196	**7.2**	51.84	2.683	8.485
2.8	7.84	1.673	5.292	**7.3**	53.29	2.702	8.544
2.9	8.41	1.703	5.385	**7.4**	54.76	2.720	8.602
3.0	9.00	1.732	5.477	**7.5**	56.25	2.739	8.660
3.1	9.61	1.761	5.568	**7.6**	57.76	2.757	8.718
3.2	10.24	1.789	5.657	**7.7**	59.29	2.775	8.775
3.3	10.89	1.817	5.745	**7.8**	60.84	2.793	8.832
3.4	11.56	1.844	5.831	**7.9**	62.41	2.811	8.888
3.5	12.25	1.871	5.916	**8.0**	64.00	2.828	8.944
3.6	12.96	1.897	6.000	**8.1**	65.61	2.846	9.000
3.7	13.69	1.924	6.083	**8.2**	67.24	2.864	9.055
3.8	14.44	1.949	6.164	**8.3**	68.89	2.881	9.110
3.9	15.21	1.975	6.245	**8.4**	70.56	2.898	9.165
4.0	16.00	2.000	6.325	**8.5**	72.25	2.915	9.220
4.1	16.81	2.025	6.403	**8.6**	73.96	2.933	9.274
4.2	17.64	2.049	6.481	**8.7**	75.69	2.950	9.327
4.3	18.49	2.074	6.557	**8.8**	77.44	2.966	9.381
4.4	19.36	2.098	6.633	**8.9**	79.21	2.983	9.434
4.5	20.25	2.121	6.708	**9.0**	81.00	3.000	9.487
4.6	21.16	2.145	6.782	**9.1**	82.81	3.017	9.539
4.7	22.09	2.168	6.856	**9.2**	84.64	3.033	9.592
4.8	23.04	2.191	6.928	**9.3**	86.49	3.050	9.644
4.9	24.01	2.214	7.000	**9.4**	88.36	3.066	9.695
5.0	25.00	2.236	7.071	**9.5**	90.25	3.082	9.747
5.1	26.01	2.258	7.141	**9.6**	92.16	3.098	9.798
5.2	27.04	2.280	7.211	**9.7**	94.09	3.114	9.849
5.3	28.09	2.302	7.280	**9.8**	96.04	3.130	9.899
5.4	29.16	2.324	7.348	**9.9**	98.01	3.146	9.950
5.5	30.25	2.345	7.416	**10**	100.00	3.162	10.000

Table 4 Cubes and Cube Roots

N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
1.0	1.000	1.000	2.154	4.642	**5.5**	166.375	1.765	3.803	8.193
1.1	1.331	1.032	2.224	4.791	**5.6**	175.616	1.776	3.826	8.243
1.2	1.728	1.063	2.289	4.932	**5.7**	185.193	1.786	3.849	8.291
1.3	2.197	1.091	2.351	5.066	**5.8**	195.112	1.797	3.871	8.340
1.4	2.744	1.119	2.410	5.192	**5.9**	205.379	1.807	3.893	8.387
1.5	3.375	1.145	2.466	5.313	**6.0**	216.000	1.817	3.915	8.434
1.6	4.096	1.170	2.520	5.429	**6.1**	226.981	1.827	3.936	8.481
1.7	4.913	1.193	2.571	5.540	**6.2**	238.328	1.837	3.958	8.527
1.8	5.832	1.216	2.621	5.646	**6.3**	250.047	1.847	3.979	8.573
1.9	6.859	1.239	2.668	5.749	**6.4**	262.144	1.857	4.000	8.618
2.0	8.000	1.260	2.714	5.848	**6.5**	274.625	1.866	4.021	8.662
2.1	9.261	1.281	2.759	5.944	**6.6**	287.496	1.876	4.041	8.707
2.2	10.648	1.301	2.802	6.037	**6.7**	300.763	1.885	4.062	8.750
2.3	12.167	1.320	2.844	6.127	**6.8**	314.432	1.895	4.082	8.794
2.4	13.824	1.339	2.884	6.214	**6.9**	328.509	1.904	4.102	8.837
2.5	15.625	1.357	2.924	6.300	**7.0**	343.000	1.913	4.121	8.879
2.6	17.576	1.375	2.962	6.383	**7.1**	357.911	1.922	4.141	8.921
2.7	19.683	1.392	3.000	6.463	**7.2**	373.248	1.931	4.160	8.963
2.8	21.952	1.409	3.037	6.542	**7.3**	389.017	1.940	4.179	9.004
2.9	24.389	1.426	3.072	6.619	**7.4**	405.224	1.949	4.198	9.045
3.0	27.000	1.442	3.107	6.694	**7.5**	421.875	1.957	4.217	9.086
3.1	29.791	1.458	3.141	6.768	**7.6**	438.976	1.966	4.236	9.126
3.2	32.768	1.474	3.175	6.840	**7.7**	456.533	1.975	4.254	9.166
3.3	35.937	1.489	3.208	6.910	**7.8**	474.552	1.983	4.273	9.205
3.4	39.304	1.504	3.240	6.980	**7.9**	493.039	1.992	4.291	9.244
3.5	42.875	1.518	3.271	7.047	**8.0**	512.000	2.000	4.309	9.283
3.6	46.656	1.533	3.302	7.114	**8.1**	531.441	2.008	4.327	9.322
3.7	50.653	1.547	3.332	7.179	**8.2**	551.368	2.017	4.344	9.360
3.8	54.872	1.560	3.362	7.243	**8.3**	571.787	2.025	4.362	9.398
3.9	59.319	1.574	3.391	7.306	**8.4**	592.704	2.033	4.380	9.435
4.0	64.000	1.587	3.420	7.368	**8.5**	614.125	2.041	4.397	9.473
4.1	68.921	1.601	3.448	7.429	**8.6**	636.056	2.049	4.414	9.510
4.2	74.088	1.613	3.476	7.489	**8.7**	658.503	2.057	4.431	9.546
4.3	79.507	1.626	3.503	7.548	**8.8**	681.472	2.065	4.448	9.583
4.4	85.184	1.639	3.530	7.606	**8.9**	704.969	2.072	4.465	9.619
4.5	91.125	1.651	3.557	7.663	**9.0**	729.000	2.080	4.481	9.655
4.6	97.336	1.663	3.583	7.719	**9.1**	753.571	2.088	4.498	9.691
4.7	103.823	1.675	3.609	7.775	**9.2**	778.688	2.095	4.514	9.726
4.8	110.592	1.687	3.634	7.830	**9.3**	804.357	2.103	4.531	9.761
4.9	117.649	1.698	3.659	7.884	**9.4**	830.584	2.110	4.547	9.796
5.0	125.000	1.710	3.684	7.937	**9.5**	857.375	2.118	4.563	9.830
5.1	132.651	1.721	3.708	7.990	**9.6**	884.736	2.125	4.579	9.865
5.2	140.608	1.732	3.733	8.041	**9.7**	912.673	2.133	4.595	9.899
5.3	148.877	1.744	3.756	8.093	**9.8**	941.192	2.140	4.610	9.933
5.4	157.464	1.754	3.780	8.143	**9.9**	970.299	2.147	4.626	9.967
5.5	166.375	1.765	3.803	8.193	**10**	1000.000	2.154	4.642	10.000

Table 5 Common Logarithms of Numbers*

x	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

*Mantissas, decimal points omitted. Characteristics are found by inspection.

Table 5 Common Logarithms of Numbers

x	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table 6 Values of Trigonometric Functions for Angles in Degrees

$m(\alpha)$ Degrees	$m(\alpha)$ Radians	$\sin \alpha$	$\csc \alpha$	$\tan \alpha$	$\cot \alpha$	$\sec \alpha$	$\cos \alpha$		
0° 00′	.0000	.0000	Undefined	.0000	Undefined	1.000	1.0000	1.5708	90° 00′
10′	.0029	.0029	343.8	.0029	343.8	1.000	1.0000	1.5679	50′
20′	.0058	.0058	171.9	.0058	171.9	1.000	1.0000	1.5650	40′
30′	.0087	.0087	114.6	.0087	114.6	1.000	1.0000	1.5621	30′
40′	.0116	.0116	85.95	.0116	85.94	1.000	.9999	1.5592	20′
50′	.0145	.0145	68.76	.0145	68.75	1.000	.9999	1.5563	10′
1° 00′	.0175	.0175	57.30	.0175	57.29	1.000	.9998	1.5533	89° 00′
10′	.0204	.0204	49.11	.0204	49.10	1.000	.9998	1.5504	50′
20′	.0233	.0233	42.98	.0233	42.96	1.000	.9997	1.5475	40′
30′	.0262	.0262	38.20	.0262	38.19	1.000	.9997	1.5446	30′
40′	.0291	.0291	34.38	.0291	34.37	1.000	.9996	1.5417	20′
50′	.0320	.0320	31.26	.0320	31.24	1.001	.9995	1.5388	10′
2° 00′	.0349	.0349	28.65	.0349	28.64	1.001	.9994	1.5359	88° 00′
10′	.0378	.0378	26.45	.0378	26.43	1.001	.9993	1.5330	50′
20′	.0407	.0407	24.56	.0407	24.54	1.001	.9992	1.5301	40′
30′	.0436	.0436	22.93	.0437	22.90	1.001	.9990	1.5272	30′
40′	.0465	.0465	21.49	.0466	21.47	1.001	.9989	1.5243	20′
50′	.0495	.0494	20.23	.0495	20.21	1.001	.9988	1.5213	10′
3° 00′	.0524	.0523	19.11	.0524	19.08	1.001	.9986	1.5184	87° 00′
10′	.0553	.0552	18.10	.0553	18.07	1.002	.9985	1.5155	50′
20′	.0582	.0581	17.20	.0582	17.17	1.002	.9983	1.5126	40′
30′	.0611	.0610	16.38	.0612	16.35	1.002	.9981	1.5097	30′
40′	.0640	.0640	15.64	.0641	15.60	1.002	.9980	1.5068	20′
50′	.0669	.0669	14.96	.0670	14.92	1.002	.9978	1.5039	10′
4° 00′	.0698	.0698	14.34	.0699	14.30	1.002	.9976	1.5010	86° 00′
10′	.0727	.0727	13.76	.0729	13.73	1.003	.9974	1.4981	50′
20′	.0756	.0756	13.23	.0758	13.20	1.003	.9971	1.4952	40′
30′	.0785	.0785	12.75	.0787	12.71	1.003	.9969	1.4923	30′
40′	.0814	.0814	12.29	.0816	12.25	1.003	.9967	1.4893	20′
50′	.0844	.0843	11.87	.0846	11.83	1.004	.9964	1.4864	10′
5° 00′	.0873	.0872	11.47	.0875	11.43	1.004	.9962	1.4835	85° 00′
10′	.0902	.0901	11.10	.0904	11.06	1.004	.9959	1.4806	50′
20′	.0931	.0929	10.76	.0934	10.71	1.004	.9957	1.4777	40′
30′	.0960	.0958	10.43	.0963	10.39	1.005	.9954	1.4748	30′
40′	.0989	.0987	10.13	.0992	10.08	1.005	.9951	1.4719	20′
50′	.1018	.1016	9.839	.1022	9.788	1.005	.9948	1.4690	10′
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
10′	.1076	.1074	9.309	.1080	9.255	1.006	.9942	1.4632	50′
20′	.1105	.1103	9.065	.1110	9.010	1.006	.9939	1.4603	40′
30′	.1134	.1132	8.834	.1139	8.777	1.006	.9936	1.4573	30′
40′	.1164	.1161	8.614	.1169	8.556	1.007	.9932	1.4544	20′
50′	.1193	.1190	8.405	.1198	8.345	1.007	.9929	1.4515	10′
7° 00′	.1222	.1219	8.206	.1228	8.144	1.008	.9925	1.4486	83° 00′
10′	.1251	.1248	8.016	.1257	7.953	1.008	.9922	1.4457	50′
20′	.1280	.1276	7.834	.1287	7.770	1.008	.9918	1.4428	40′
30′	.1309	.1305	7.661	.1317	7.596	1.009	.9914	1.4399	30′
40′	.1338	.1334	7.496	.1346	7.429	1.009	.9911	1.4370	20′
50′	.1367	.1363	7.337	.1376	7.269	1.009	.9907	1.4341	10′
8° 00′	.1396	.1392	7.185	.1405	7.115	1.010	.9903	1.4312	82° 00′
10′	.1425	.1421	7.040	.1435	6.968	1.010	.9899	1.4283	50′
20′	.1454	.1449	6.900	.1465	6.827	1.011	.9894	1.4254	40′
30′	.1484	.1478	6.765	.1495	6.691	1.011	.9890	1.4224	30′
40′	.1513	.1507	6.636	.1524	6.561	1.012	.9886	1.4195	20′
50′	.1542	.1536	6.512	.1554	6.435	1.012	.9881	1.4166	10′
9° 00′	.1571	.1564	6.392	.1584	6.314	1.012	.9877	1.4137	81° 00′
		$\cos \alpha$	$\sec \alpha$	$\cot \alpha$	$\tan \alpha$	$\csc \alpha$	$\sin \alpha$	$m(\alpha)$ Radians	$m(\alpha)$ Degrees

Table 6 Values of Trigonometric Functions for Angles in Degrees

$m(\alpha)$ Degrees	Radians	sin α	csc α	tan α	cot α	sec α	cos α		
9° 00′	.1571	.1564	6.392	.1584	6.314	1.012	.9877	1.4137	81 00′
10′	.1600	.1593	6.277	.1614	6.197	1.013	.9872	1.4108	50′
20′	.1629	.1622	6.166	.1644	6.084	1.013	.9868	1.4079	40′
30′	.1658	.1650	6.059	.1673	5.976	1.014	.9863	1.4050	30′
40′	.1687	.1679	5.955	.1703	5.871	1.014	.9858	1.4021	20′
50′	.1716	.1708	5.855	.1733	5.769	1.015	.9853	1.3992	10′
10° 00′	.1745	.1736	5.759	.1763	5.671	1.015	.9848	1.3963	80 00′
10′	.1774	.1765	5.665	.1793	5.576	1.016	.9843	1.3934	50′
20′	.1804	.1794	5.575	.1823	5.485	1.016	.9838	1.3904	40′
30′	.1833	.1822	5.487	.1853	5.396	1.017	.9833	1.3875	30′
40′	.1862	.1851	5.403	.1883	5.309	1.018	.9827	1.3846	20′
50′	.1891	.1880	5.320	.1914	5.226	1.018	.9822	1.3817	10′
11° 00′	.1920	.1908	5.241	.1944	5.145	1.019	.9816	1.3788	79 00
10′	.1949	.1937	5.164	.1974	5.066	1.019	.9811	1.3759	50′
20′	.1978	.1965	5.089	.2004	4.989	1.020	.9805	1.3730	40′
30′	.2007	.1994	5.016	.2035	4.915	1.020	.9799	1.3701	30′
40′	.2036	.2022	4.945	.2065	4.843	1.021	.9793	1.3672	20′
50′	.2065	.2051	4.876	.2095	4.773	1.022	.9787	1.3643	10′
12° 00′	.2094	.2079	4.810	.2126	4.705	1.022	.9781	1.3614	78 00′
10′	.2123	.2108	4.745	.2156	4.638	1.023	.9775	1.3584	50′
20′	.2153	.2136	4.682	.2186	4.574	1.024	.9769	1.3555	40′
30′	.2182	.2164	4.620	.2217	4.511	1.024	.9763	1.3526	30′
40′	.2211	.2193	4.560	.2247	4.449	1.025	.9757	1.3497	20′
50′	.2240	.2221	4.502	.2278	4.390	1.026	.9750	1.3468	10′
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	77 00
10′	.2298	.2278	4.390	.2339	4.275	1.027	.9737	1.3410	50′
20′	.2327	.2306	4.336	.2370	4.219	1.028	.9730	1.3381	40′
30′	.2356	.2334	4.284	.2401	4.165	1.028	.9724	1.3352	30′
40′	.2385	.2363	4.232	.2432	4.113	1.029	.9717	1.3323	20′
50′	.2414	.2391	4.182	.2462	4.061	1.030	.9710	1.3294	10′
14° 00′	.2443	.2419	4.134	.2493	4.011	1.031	.9703	1.3265	76 00′
10′	.2473	.2447	4.086	.2524	3.962	1.031	.9696	1.3235	50
20′	.2502	.2476	4.039	.2555	3.914	1.032	.9689	1.3206	40′
30′	.2531	.2504	3.994	.2586	3.867	1.033	.9681	1.3177	30
40′	.2560	.2532	3.950	.2617	3.821	1.034	.9674	1.3148	20
50′	.2589	.2560	3.906	.2648	3.776	1.034	.9667	1.3119	10′
15° 00′	.2618	.2588	3.864	.2679	3.732	1.035	.9659	1.3090	75 00′
10′	.2647	.2616	3.822	.2711	3.689	1.036	.9652	1.3061	50
20′	.2676	.2644	3.782	.2742	3.647	1.037	.9644	1.3032	40
30′	.2705	.2672	3.742	.2773	3.606	1.038	.9636	1.3003	30′
40′	.2734	.2700	3.703	.2805	3.566	1.039	.9628	1.2974	20
50′	.2763	.2728	3.665	.2836	3.526	1.039	.9621	1.2945	10
16° 00′	.2793	.2756	3.628	.2867	3.487	1.040	.9613	1.2915	74 00
10′	.2822	.2784	3.592	.2899	3.450	1.041	.9605	1.2886	50
20′	.2851	.2812	3.556	.2931	3.412	1.042	.9596	1.2857	40
30′	.2880	.2840	3.521	.2962	3.376	1.043	.9588	1.2828	30
40′	.2909	.2868	3.487	.2994	3.340	1.044	.9580	1.2799	20
50′	.2938	.2896	3.453	.3026	3.305	1.045	.9572	1.2770	10
17° 00′	.2967	.2924	3.420	.3057	3.271	1.046	.9563	1.2741	73 00
10′	.2996	.2952	3.388	.3089	3.237	1.047	.9555	1.2712	50
20′	.3025	.2979	3.357	.3121	3.204	1.048	.9546	1.2683	40
30′	.3054	.3007	3.326	.3153	3.172	1.049	.9537	1.2654	30
40′	.3083	.3035	3.295	.3185	3.140	1.049	.9528	1.2625	20
50′	.3113	.3062	3.265	.3217	3.108	1.050	.9520	1.2595	10
18° 00′	.3142	.3090	3.236	.3249	3.078	1.051	.9511	1.2566	72 00
		cos α	sec α	cot α	tan α	csc α	sin α	Radians	Degrees
								$m(\alpha)$	

Table 6 Values of Trigonometric Functions for Angles in Degrees

m(α) Degrees	m(α) Radians	sin α	csc α	tan α	cot α	sec α	cos α		
18° 00′	.3142	.3090	3.236	.3249	3.078	1.051	.9511	1.2566	72° 00′
10′	.3171	.3118	3.207	.3281	3.047	1.052	.9502	1.2537	50′
20′	.3200	.3145	3.179	.3314	3.018	1.053	.9492	1.2508	40′
30′	.3229	.3173	3.152	.3346	2.989	1.054	.9483	1.2479	30′
40′	.3258	.3201	3.124	.3378	2.960	1.056	.9474	1.2450	20′
50′	.3287	.3228	3.098	.3411	2.932	1.057	.9465	1.2421	10′
19° 00′	.3316	.3256	3.072	.3443	2.904	1.058	.9455	1.2392	71° 00′
10′	.3345	.3283	3.046	.3476	2.877	1.059	.9446	1.2363	50′
20′	.3374	.3311	3.021	.3508	2.850	1.060	.9436	1.2334	40′
30′	.3403	.3338	2.996	.3541	2.824	1.061	.9426	1.2305	30′
40′	.3432	.3365	2.971	.3574	2.798	1.062	.9417	1.2275	20′
50′	.3462	.3393	2.947	.3607	2.773	1.063	.9407	1.2246	10′
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	70° 00′
10′	.3520	.3448	2.901	.3673	2.723	1.065	.9387	1.2188	50′
20′	.3549	.3475	2.878	.3706	2.699	1.066	.9377	1.2159	40′
30′	.3578	.3502	2.855	.3739	2.675	1.068	.9367	1.2130	30′
40′	.3607	.3529	2.833	.3772	2.651	1.069	.9356	1.2101	20′
50′	.3636	.3557	2.812	.3805	2.628	1.070	.9346	1.2072	10′
21° 00′	.3665	.3584	2.790	.3839	2.605	1.071	.9336	1.2043	69° 00′
10′	.3694	.3611	2.769	.3872	2.583	1.072	.9325	1.2014	50′
20′	.3723	.3638	2.749	.3906	2.560	1.074	.9315	1.1985	40′
30′	.3752	.3665	2.729	.3939	2.539	1.075	.9304	1.1956	30′
40′	.3782	.3692	2.709	.3973	2.517	1.076	.9293	1.1926	20′
50′	.3811	.3719	2.689	.4006	2.496	1.077	.9283	1.1897	10′
22° 00′	.3840	.3746	2.669	.4040	2.475	1.079	.9272	1.1868	68° 00′
10′	.3869	.3773	2.650	.4074	2.455	1.080	.9261	1.1839	50′
20′	.3898	.3800	2.632	.4108	2.434	1.081	.9250	1.1810	40′
30′	.3927	.3827	2.613	.4142	2.414	1.082	.9239	1.1781	30′
40′	.3956	.3854	2.595	.4176	2.394	1.084	.9228	1.1752	20′
50′	.3985	.3881	2.577	.4210	2.375	1.085	.9216	1.1723	10′
23° 00′	.4014	.3907	2.559	.4245	2.356	1.086	.9205	1.1694	67° 00′
10′	.4043	.3934	2.542	.4279	2.337	1.088	.9194	1.1665	50′
20′	.4072	.3961	2.525	.4314	2.318	1.089	.9182	1.1636	40′
30′	.4102	.3987	2.508	.4348	2.300	1.090	.9171	1.1606	30′
40′	.4131	.4014	2.491	.4383	2.282	1.092	.9159	1.1577	20′
50′	.4160	.4041	2.475	.4417	2.264	1.093	.9147	1.1548	10′
24° 00′	.4189	.4067	2.459	.4452	2.246	1.095	.9135	1.1519	66° 00′
10′	.4218	.4094	2.443	.4487	2.229	1.096	.9124	1.1490	50′
20′	.4247	.4120	2.427	.4522	2.211	1.097	.9112	1.1461	40′
30′	.4276	.4147	2.411	.4557	2.194	1.099	.9100	1.1432	30′
40′	.4305	.4173	2.396	.4592	2.177	1.100	.9088	1.1403	20′
50′	.4334	.4200	2.381	.4628	2.161	1.102	.9075	1.1374	10′
25° 00′	.4363	.4226	2.366	.4663	2.145	1.103	.9063	1.1345	65° 00′
10′	.4392	.4253	2.352	.4699	2.128	1.105	.9051	1.1316	50′
20′	.4422	.4279	2.337	.4734	2.112	1.106	.9038	1.1286	40′
30′	.4451	.4305	2.323	.4770	2.097	1.108	.9026	1.1257	30′
40′	.4480	.4331	2.309	.4806	2.081	1.109	.9013	1.1228	20′
50′	.4509	.4358	2.295	.4841	2.066	1.111	.9001	1.1199	10′
26° 00′	.4538	.4384	2.281	.4877	2.050	1.113	.8988	1.1170	64° 00′
10′	.4567	.4410	2.268	.4913	2.035	1.114	.8975	1.1141	50′
20′	.4596	.4436	2.254	.4950	2.020	1.116	.8962	1.1112	40′
30′	.4625	.4462	2.241	.4986	2.006	1.117	.8949	1.1083	30′
40′	.4654	.4488	2.228	.5022	1.991	1.119	.8936	1.1054	20′
50′	.4683	.4514	2.215	.5059	1.977	1.121	.8923	1.1025	10′
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	63° 00′
		cos α	sec α	cot α	tan α	csc α	sin α	m(α) Radians	m(α) Degrees

Table 6 Values of Trigonometric Functions for Angles in Degrees

$m(\alpha)$ Degrees	$m(\alpha)$ Radians	$\sin\alpha$	$\csc\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\cos\alpha$		
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	63° 00′
10′	.4741	.4566	2.190	.5132	1.949	1.124	.8897	1.0966	50′
20′	.4771	.4592	2.178	.5169	1.935	1.126	.8884	1.0937	40′
30′	.4800	.4617	2.166	.5206	1.921	1.127	.8870	1.0908	30′
40′	.4829	.4643	2.154	.5243	1.907	1.129	.8857	1.0879	20′
50′	.4858	.4669	2.142	.5280	1.894	1.131	.8843	1.0850	10′
28° 00′	.4887	.4695	2.130	.5317	1.881	1.133	.8829	1.0821	62° 00′
10′	.4916	.4720	2.118	.5354	1.868	1.134	.8816	1.0792	50′
20′	.4945	.4746	2.107	.5392	1.855	1.136	.8802	1.0763	40′
30′	.4974	.4772	2.096	.5430	1.842	1.138	.8788	1.0734	30′
40′	.5003	.4797	2.085	.5467	1.829	1.140	.8774	1.0705	20′
50′	.5032	.4823	2.074	.5505	1.816	1.142	.8760	1.0676	10′
29° 00′	.5061	.4848	2.063	.5543	1.804	1.143	.8746	1.0647	61° 00′
10′	.5091	.4874	2.052	.5581	1.792	1.145	.8732	1.0617	50′
20′	.5120	.4899	2.041	.5619	1.780	1.147	.8718	1.0588	40′
30′	.5149	.4924	2.031	.5658	1.767	1.149	.8704	1.0559	30′
40′	.5178	.4950	2.020	.5696	1.756	1.151	.8689	1.0530	20′
50′	.5207	.4975	2.010	.5735	1.744	1.153	.8675	1.0501	10′
30° 00′	.5236	.5000	2.000	.5774	1.732	1.155	.8660	1.0472	60° 00′
10′	.5265	.5025	1.990	.5812	1.720	1.157	.8646	1.0443	50′
20′	.5294	.5050	1.980	.5851	1.709	1.159	.8631	1.0414	40′
30′	.5323	.5075	1.970	.5890	1.698	1.161	.8616	1.0385	30′
40′	.5352	.5100	1.961	.5930	1.686	1.163	.8601	1.0356	20′
50′	.5381	.5125	1.951	.5969	1.675	1.165	.8587	1.0327	10′
31° 00′	.5411	.5150	1.942	.6009	1.664	1.167	.8572	1.0297	59° 00′
10′	.5440	.5175	1.932	.6048	1.653	1.169	.8557	1.0268	50′
20′	.5469	.5200	1.923	.6088	1.643	1.171	.8542	1.0239	40′
30′	.5498	.5225	1.914	.6128	1.632	1.173	.8526	1.0210	30′
40′	.5527	.5250	1.905	.6168	1.621	1.175	.8511	1.0181	20′
50′	.5556	.5275	1.896	.6208	1.611	1.177	.8496	1.0152	10′
32° 00′	.5585	.5299	1.887	.6249	1.600	1.179	.8480	1.0123	58° 00′
10′	.5614	.5324	1.878	.6289	1.590	1.181	.8465	1.0094	50′
20′	.5643	.5348	1.870	.6330	1.580	1.184	.8450	1.0065	40′
30′	.5672	.5373	1.861	.6371	1.570	1.186	.8434	1.0036	30′
40′	.5701	.5398	1.853	.6412	1.560	1.188	.8418	1.0007	20′
50′	.5730	.5422	1.844	.6453	1.550	1.190	.8403	.9977	10′
33° 00′	.5760	.5446	1.836	.6494	1.540	1.192	.8387	.9948	57° 00′
10′	.5789	.5471	1.828	.6536	1.530	1.195	.8371	.9919	50′
20′	.5818	.5495	1.820	.6577	1.520	1.197	.8355	.9890	40′
30′	.5847	.5519	1.812	.6619	1.511	1.199	.8339	.9861	30′
40′	.5876	.5544	1.804	.6661	1.501	1.202	.8323	.9832	20′
50′	.5905	.5568	1.796	.6703	1.492	1.204	.8307	.9803	10′
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	56° 00′
10′	.5963	.5616	1.781	.6787	1.473	1.209	.8274	.9745	50′
20′	.5992	.5640	1.773	.6830	1.464	1.211	.8258	.9716	40′
30′	.6021	.5664	1.766	.6873	1.455	1.213	.8241	.9687	30′
40′	.6050	.5688	1.758	.6916	1.446	1.216	.8225	.9657	20′
50′	.6080	.5712	1.751	.6959	1.437	1.218	.8208	.9628	10′
35° 00′	.6109	.5736	1.743	.7002	1.428	1.221	.8192	.9599	55° 00′
10′	.6138	.5760	1.736	.7046	1.419	1.223	.8175	.9570	50′
20′	.6167	.5783	1.729	.7089	1.411	1.226	.8158	.9541	40′
30′	.6196	.5807	1.722	.7133	1.402	1.228	.8141	.9512	30′
40′	.6225	.5831	1.715	.7177	1.393	1.231	.8124	.9483	20
50′	.6254	.5854	1.708	.7221	1.385	1.233	.8107	.9454	10′
36° 00′	.6283	.5878	1.701	.7265	1.376	1.236	.8090	.9425	54° 00′
		$\cos\alpha$	$\sec\alpha$	$\cot\alpha$	$\tan\alpha$	$\csc\alpha$	$\sin\alpha$	$m(\alpha)$ Radians	$m(\alpha)$ Degrees

Table 6 Values of Trigonometric Functions for Angles in Degrees

$m(\alpha)$									
Degrees	*Radians*	*sin α*	*csc α*	*tan α*	*cot α*	*sec α*	*cos α*		
36° 00′	.6283	.5878	1.701	.7265	1.376	1.236	.8090	.9425	54° 00′
10′	.6312	.5901	1.695	.7310	1.368	1.239	.8073	.9396	50′
20′	.6341	.5925	1.688	.7355	1.360	1.241	.8056	.9367	40′
30′	.6370	.5948	1.681	.7400	1.351	1.244	.8039	.9338	30′
40′	.6400	.5972	1.675	.7445	1.343	1.247	.8021	.9308	20′
50′	.6429	.5995	1.668	.7490	1.335	1.249	.8004	.9279	10′
37° 00′	.6458	.6018	1.662	.7536	1.327	1.252	.7986	.9250	53° 00′
10′	.6487	.6041	1.655	.7581	1.319	1.255	.7969	.9221	50′
20′	.6516	.6065	1.649	.7627	1.311	1.258	.7951	.9192	40′
30′	.6545	.6088	1.643	.7673	1.303	1.260	.7934	.9163	30′
40′	.6574	.6111	1.636	.7720	1.295	1.263	.7916	.9134	20′
50′	.6603	.6134	1.630	.7766	1.288	1.266	.7898	.9105	10′
38° 00′	.6632	.6157	1.624	.7813	1.280	1.269	.7880	.9076	52° 00′
10′	.6661	.6180	1.618	.7860	1.272	1.272	.7862	.9047	50′
20′	.6690	.6202	1.612	.7907	1.265	1.275	.7844	.9018	40′
30′	.6720	.6225	1.606	.7954	1.257	1.278	.7826	.8988	30′
40′	.6749	.6248	1.601	.8002	1.250	1.281	.7808	.8959	20′
50′	.6778	.6271	1.595	.8050	1.242	1.284	.7790	.8930	10′
39° 00′	.6807	.6293	1.589	.8098	1.235	1.287	.7771	.8901	51° 00′
10′	.6836	.6316	1.583	.8146	1.228	1.290	.7753	.8872	50′
20′	.6865	.6338	1.578	.8195	1.220	1.293	.7735	.8843	40′
30′	.6894	.6361	1.572	.8243	1.213	1.296	.7716	.8814	30′
40′	.6923	.6383	1.567	.8292	1.206	1.299	.7698	.8785	20′
50′	.6952	.6406	1.561	.8342	1.199	1.302	.7679	.8756	10′
40° 00′	.6981	.6428	1.556	.8391	1.192	1.305	.7660	.8727	50° 00′
10′	.7010	.6450	1.550	.8441	1.185	1.309	.7642	.8698	50′
20′	.7039	.6472	1.545	.8491	1.178	1.312	.7623	.8668	40′
30′	.7069	.6494	1.540	.8541	1.171	1.315	.7604	.8639	30′
40′	.7098	.6517	1.535	.8591	1.164	1.318	.7585	.8610	20′
50′	.7127	.6539	1.529	.8642	1.157	1.322	.7566	.8581	10′
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	49° 00′
10′	.7185	.6583	1.519	.8744	1.144	1.328	.7528	.8523	50′
20′	.7214	.6604	1.514	.8796	1.137	1.332	.7509	.8494	40′
30′	.7243	.6626	1.509	.8847	1.130	1.335	.7490	.8465	30′
40′	.7272	.6648	1.504	.8899	1.124	1.339	.7470	.8436	20′
50′	.7301	.6670	1.499	.8952	1.117	1.342	.7451	.8407	10′
42° 00′	.7330	.6691	1.494	.9004	1.111	1.346	.7431	.8378	48° 00′
10′	.7359	.6713	1.490	.9057	1.104	1.349	.7412	.8348	50′
20′	.7389	.6734	1.485	.9110	1.098	1.353	.7392	.8319	40′
30′	.7418	.6756	1.480	.9163	1.091	1.356	.7373	.8290	30′
40′	.7447	.6777	1.476	.9217	1.085	1.360	.7353	.8261	20′
50′	.7476	.6799	1.471	.9271	1.079	1.364	.7333	.8232	10′
43° 00′	.7505	.6820	1.466	.9325	1.072	1.367	.7314	.8203	47° 00′
10′	.7534	.6841	1.462	.9380	1.066	1.371	.7294	.8174	50′
20′	.7563	.6862	1.457	.9435	1.060	1.375	.7274	.8145	40′
30′	.7592	.6884	1.453	.9490	1.054	1.379	.7254	.8116	30′
40′	.7621	.6905	1.448	.9545	1.048	1.382	.7234	.8087	20′
50′	.7650	.6926	1.444	.9601	1.042	1.386	.7214	.8058	10′
44° 00′	.7679	.6947	1.440	.9657	1.036	1.390	.7193	.8029	46° 00′
10′	.7709	.6967	1.435	.9713	1.030	1.394	.7173	.7999	50′
20′	.7738	.6988	1.431	.9770	1.024	1.398	.7153	.7970	40′
30′	.7767	.7009	1.427	.9827	1.018	1.402	.7133	.7941	30′
40′	.7796	.7030	1.423	.9884	1.012	1.406	.7112	.7912	20′
50′	.7825	.7050	1.418	.9942	1.006	1.410	.7092	.7883	10′
45° 00′	.7854	.7071	1.414	1.000	1.000	1.414	.7071	.7854	45° 00′
		cos α	*sec α*	*cot α*	*tan α*	*csc α*	*sin α*	*Radians*	*Degrees*
								$m(\alpha)$	

Table 7 Values of Circular Functions and Trigonometric Functions for Angles in Radians

Real Number x or $m^R(\alpha)$	$m(\alpha)$	sin x or sin α	csc x or csc α	tan x or tan α	cot x or cot α	sec x or sec α	cos x or cos α
0	0°	0	Undefined	0	Undefined	1	1
0.01	0° 34′	0.0100	100.0	0.0100	100.0	1.000	1.000
.02	1° 09′	.0200	50.00	.0200	49.99	1.000	0.9998
.03	1° 43′	.0300	33.34	.0300	33.32	1.000	0.9996
.04	2° 18′	.0400	25.01	.0400	24.99	1.001	0.9992
0.05	2° 52′	0.0500	20.01	0.0500	19.98	1.001	0.9988
.06	3° 26′	.0600	16.68	.0601	16.65	1.002	.9982
.07	4° 01′	.0699	14.30	.0701	14.26	1.002	.9976
.08	4° 35′	.0799	12.51	.0802	12.47	1.003	.9968
.09	5° 09′	.0899	11.13	.0902	11.08	1.004	.9960
0.10	5° 44′	0.0998	10.02	0.1003	9.967	1.005	0.9950
.11	6° 18′	.1098	9.109	.1104	9.054	1.006	.9940
.12	6° 53′	.1197	8.353	.1206	8.293	1.007	.9928
.13	7° 27′	.1296	7.714	.1307	7.649	1.009	.9916
.14	8° 01′	.1395	7.166	.1409	7.096	1.010	.9902
0.15	8° 36′	0.1494	6.692	0.1511	6.617	1.011	0.9888
.16	9° 10′	.1593	6.277	.1614	6.197	1.013	.9872
.17	9° 44′	.1692	5.911	.1717	5.826	1.015	.9856
.18	10° 19′	.1790	5.586	.1820	5.495	1.016	.9838
.19	10° 53′	.1889	5.295	.1923	5.200	1.018	.9820
0.20	11° 28′	0.1987	5.033	0.2027	4.933	1.020	0.9801
.21	12° 02′	.2085	4.797	.2131	4.692	1.022	.9780
.22	12° 36′	.2182	4.582	.2236	4.472	1.025	.9759
.23	13° 11′	.2280	4.386	.2341	4.271	1.027	.9737
.24	13° 45′	.2377	4.207	.2447	4.086	1.030	.9713
0.25	14° 19′	0.2474	4.042	0.2553	3.916	1.032	0.9689
.26	14° 54′	.2571	3.890	.2660	3.759	1.035	.9664
.27	15° 28′	.2667	3.749	.2768	3.613	1.038	.9638
.28	16° 03′	.2764	3.619	.2876	3.478	1.041	.9611
.29	16° 37′	.2860	3.497	.2984	3.351	1.044	.9582
0.30	17° 11′	0.2955	3.384	0.3093	3.233	1.047	0.9553
.31	17° 46′	.3051	3.278	.3203	3.122	1.050	.9523
.32	18° 20′	.3146	3.179	.3314	3.018	1.053	.9492
.33	18° 55′	.3240	3.086	.3425	2.920	1.057	.9460
.34	19° 29′	.3335	2.999	.3537	2.827	1.061	.9428
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
.36	20° 38′	.3523	2.839	.3764	2.657	1.068	.9359
.37	21° 12′	.3616	2.765	.3879	2.578	1.073	.9323
.38	21° 46′	.3709	2.696	.3994	2.504	1.077	.9287
.39	22° 21′	.3802	2.630	.4111	2.433	1.081	.9249
0.40	22° 55′	0.3894	2.568	0.4228	2.365	1.086	0.9211
.41	23° 30′	.3986	2.509	.4346	2.301	1.090	.9171
.42	24° 04′	.4078	2.452	.4466	2.239	1.095	.9131
.43	24° 38′	.4169	2.399	.4586	2.180	1.100	.9090
.44	25° 13′	.4259	2.348	.4708	2.124	1.105	.9048
0.45	25° 47′	0.4350	2.299	0.4831	2.070	1.111	0.9004
.46	26° 21′	.4439	2.253	.4954	2.018	1.116	.8961
.47	26° 56′	.4529	2.208	.5080	1.969	1.122	.8916
.48	27° 30′	.4618	2.166	.5206	1.921	1.127	.8870
.49	28° 05′	.4706	2.125	.5334	1.875	1.133	.8823

Table 7 Values of Circular Functions and Trigonometric Functions for Angles in Radians

Real Number x or $m^R(\alpha)$	$m(\alpha)$	$\sin x$ or $\sin \alpha$	$\csc x$ or $\csc \alpha$	$\tan x$ or $\tan \alpha$	$\cot x$ or $\cot \alpha$	$\sec x$ or $\sec \alpha$	$\cos x$ or $\cos \alpha$
0.50	28° 39′	0.4794	2.086	0.5463	1.830	1.139	0.8776
.51	29° 13′	.4882	2.048	.5594	1.788	1.146	.8727
.52	29° 48′	.4969	2.013	.5726	1.747	1.152	.8678
.53	30° 22′	.5055	1.978	.5859	1.707	1.159	.8628
.54	30° 56′	.5141	1.945	.5994	1.668	1.166	.8577
0.55	31° 31′	0.5227	1.913	0.6131	1.631	1.173	0.8525
.56	32° 05′	.5312	1.883	.6269	1.595	1.180	.8473
.57	32° 40′	.5396	1.853	.6410	1.560	1.188	.8419
.58	33° 14′	.5480	1.825	.6552	1.526	1.196	.8365
.59	33° 48′	.5564	1.797	.6696	1.494	1.203	.8309
0.60	34° 23′	0.5646	1.771	0.6841	1.462	1.212	0.8253
.61	34° 57′	.5729	1.746	.6989	1.431	1.220	.8196
.62	35° 31′	.5810	1.721	.7139	1.401	1.229	.8139
.63	36° 06′	.5891	1.697	.7291	1.372	1.238	.8080
.64	36° 40′	.5972	1.674	.7445	1.343	1.247	.8021
0.65	37° 15′	0.6052	1.652	0.7602	1.315	1.256	0.7961
.66	37° 49′	.6131	1.631	.7761	1.288	1.266	.7900
.67	38° 23′	.6210	1.610	.7923	1.262	1.276	.7838
.68	38° 58′	.6288	1.590	.8087	1.237	1.286	.7776
.69	39° 32′	.6365	1.571	.8253	1.212	1.297	.7712
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
.71	40° 41′	.6518	1.534	.8595	1.163	1.319	.7584
.72	41° 15′	.6594	1.517	.8771	1.140	1.330	.7518
.73	41° 50′	.6669	1.500	.8949	1.117	1.342	.7452
.74	42° 24′	.6743	1.483	.9131	1.095	1.354	.7385
0.75	42° 58′	0.6816	1.467	0.9316	1.073	1.367	0.7317
.76	43° 33′	.6889	1.452	.9505	1.052	1.380	.7248
.77	44° 07′	.6961	1.437	.9697	1.031	1.393	.7179
.78	44° 41′	.7033	1.422	.9893	1.011	1.407	.7109
.79	45° 16′	.7104	1.408	1.009	.9908	1.421	.7038
0.80	45° 50′	0.7174	1.394	1.030	0.9712	1.435	0.6967
.81	46° 25′	.7243	1.381	1.050	.9520	1.450	.6895
.82	46° 59′	.7311	1.368	1.072	.9331	1.466	.6822
.83	47° 33′	.7379	1.355	1.093	.9146	1.482	.6749
.84	48° 08′	.7446	1.343	1.116	.8964	1.498	.6675
0.85	48° 42′	0.7513	1.331	1.138	0.8785	1.515	0.6600
.86	49° 17′	.7578	1.320	1.162	.8609	1.533	.6524
.87	49° 51′	.7643	1.308	1.185	.8437	1.551	.6448
.88	50° 25′	.7707	1.297	1.210	.8267	1.569	.6372
.89	51° 00′	.7771	1.287	1.235	.8100	1.589	.6294
0.90	51° 34′	0.7833	1.277	1.260	0.7936	1.609	0.6216
.91	52° 08′	.7895	1.267	1.286	.7774	1.629	.6137
.92	52° 43′	.7956	1.257	1.313	.7615	1.651	.6058
.93	53° 17′	.8016	1.247	1.341	.7458	1.673	.5978
.94	53° 52′	.8076	1.238	1.369	.7303	1.696	.5898
0.95	54° 26′	0.8134	1.229	1.398	0.7151	1.719	0.5817
.96	55° 00′	.8192	1.221	1.428	.7001	1.744	.5735
.97	55° 35′	.8249	1.212	1.459	.6853	1.769	.5653
.98	56° 09′	.8305	1.204	1.491	.6707	1.795	.5570
.99	56° 43′	.8360	1.196	1.524	.6563	1.823	.5487

Table 7 Values of Circular Functions and Trigonometric Functions for Angles in Radians

Real Number x or $m^R(\alpha)$	$m(\alpha)$	$\sin x$ or $\sin \alpha$	$\csc x$ or $\csc \alpha$	$\tan x$ or $\tan \alpha$	$\cot x$ or $\cot \alpha$	$\sec x$ or $\sec \alpha$	$\cos x$ or $\cos \alpha$
1.00	57° 18′	0.8415	1.188	1.557	0.6421	1.851	0.5403
1.01	57° 52′	.8468	1.181	1.592	.6281	1.880	.5319
1.02	58° 27′	.8521	1.174	1.628	.6142	1.911	.5234
1.03	59° 01′	.8573	1.166	1.665	.6005	1.942	.5148
1.04	59° 35′	.8624	1.160	1.704	.5870	1.975	.5062
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
1.06	60° 44′	.8724	1.146	1.784	.5604	2.046	.4889
1.07	61° 18′	.8772	1.140	1.827	.5473	2.083	.4801
1.08	61° 53′	.8820	1.134	1.871	.5344	2.122	.4713
1.09	62° 27′	.8866	1.128	1.917	.5216	2.162	.4625
1.10	63° 02′	0.8912	1.122	1.965	0.5090	2.205	0.4536
1.11	63° 36′	.8957	1.116	2.014	.4964	2.249	.4447
1.12	64° 10′	.9001	1.111	2.066	.4840	2.295	.4357
1.13	64° 45′	.9044	1.106	2.120	.4718	2.344	.4267
1.14	65° 19′	.9086	1.101	2.176	.4596	2.395	.4176
1.15	65° 53′	0.9128	1.096	2.234	0.4475	2.448	0.4085
1.16	66° 28′	.9168	1.091	2.296	.4356	2.504	.3993
1.17	67° 02′	.9208	1.086	2.360	.4237	2.563	.3902
1.18	67° 37′	.9246	1.082	2.428	.4120	2.625	.3809
1.19	68° 11′	.9284	1.077	2.498	.4003	2.691	.3717
1.20	68° 45′	0.9320	1.073	2.572	0.3888	2.760	0.3624
1.21	69° 20′	.9356	1.069	2.650	.3773	2.833	.3530
1.22	69° 54′	.9391	1.065	2.733	.3659	2.910	.3436
1.23	70° 28′	.9425	1.061	2.820	.3546	2.992	.3342
1.24	71° 03′	.9458	1.057	2.912	.3434	3.079	.3248
1.25	71° 37′	0.9490	1.054	3.010	0.3323	3.171	0.3153
1.26	72° 12′	.9521	1.050	3.113	.3212	3.270	.3058
1.27	72° 46′	.9551	1.047	3.224	.3102	3.375	.2963
1.28	73° 20′	.9580	1.044	3.341	.2993	3.488	.2867
1.29	73° 55′	.9608	1.041	3.467	.2884	3.609	.2771
1.30	74° 29′	0.9636	1.038	3.602	0.2776	3.738	0.2675
1.31	75° 03′	.9662	1.035	3.747	.2669	3.878	.2579
1.32	75° 38′	.9687	1.032	3.903	.2562	4.029	.2482
1.33	76° 12′	.9711	1.030	4.072	.2456	4.193	.2385
1.34	76° 47′	.9735	1.027	4.256	.2350	4.372	.2288
1.35	77° 21′	0.9757	1.025	4.455	0.2245	4.566	0.2190
1.36	77° 55′	.9779	1.023	4.673	.2140	4.779	.2092
1.37	78° 30′	.9799	1.021	4.913	.2035	5.014	.1994
1.38	79° 04′	.9819	1.018	5.177	.1931	5.273	.1896
1.39	79° 39′	.9837	1.017	5.471	.1828	5.561	.1798
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
1.41	80° 47′	.9871	1.013	6.165	.1622	6.246	.1601
1.42	81° 22′	.9887	1.011	6.581	.1519	6.657	.1502
1.43	81° 56′	.9901	1.010	7.055	.1417	7.126	.1403
1.44	82° 30′	.9915	1.009	7.602	.1315	7.667	.1304
1.45	83° 05′	0.9927	1.007	8.238	0.1214	8.299	0.1205
1.46	83° 39′	.9939	1.006	8.989	.1113	9.044	.1106
1.47	84° 14′	.9949	1.005	9.887	.1011	9.938	.1006
1.48	84° 48′	.9959	1.004	10.98	.0911	11.03	.0907
1.49	85° 22′	.9967	1.003	12.35	.0810	12.39	.0807

Table 7 Values of Circular Functions and Trigonometric Functions for Angles in Radians

Real Number x or $m^R(\alpha)$	$m(\alpha)$	sin x or sin α	csc x or csc α	tan x or tan α	cot x or cot α	sec x or sec α	cos x or cos α
1.50	85° 57′	0.9975	1.003	14.10	0.0709	14.14	0.0707
1.51	86° 31′	.9982	1.002	16.43	.0609	16.46	.0608
1.52	87° 05′	.9987	1.001	19.67	.0508	19.70	.0508
1.53	87° 40′	.9992	1.001	24.50	.0408	24.52	.0408
1.54	88° 14′	.9995	1.000	32.46	.0308	32.48	.0308
1.55	88° 49′	0.9998	1.000	48.08	0.0208	48.09	0.0208
1.56	89° 23′	.9999	1.000	92.62	.0108	92.63	.0108
1.57	89° 57′	1.000	1.000	1256	.0008	1256	.0008
$\frac{\pi}{2}$	90°	1	1	Undefined	0	Undefined	0

Glossary

abscissa (p. 71). The x-coordinate of a point.
absolute value (p. 58). The nonnegative (0 or positive) one of the pair a and $-a$, where $a \in \mathcal{R}$.
accuracy of a measurement (p. 403). The relative error of a measurement; that is, the ratio of the maximum possible error in the measurement to the measurement itself.
additive inverse (p. 7). For each $a \in \mathcal{R}$, there exists an additive inverse $-a \in \mathcal{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$.
amplitude of a periodic function (p. 525). When a periodic function attains a maximum value M and a minimum value m, you say the function has an amplitude of $\frac{M - m}{2}$.
angle (p. 502). The union of two noncollinear rays that have the same endpoint. *See also* directed angle.
angle of depression (p. 538). The angle between the line of sight to an object (below the observer) and a horizontal ray through the observer.
angle of elevation (p. 538). The angle between the horizontal ray through the observer and the line of sight to an object (above the observer).
antilogarithm (p. 398). If $\log x = a$, then x is called the antilogarithm of a.
arithmetic mean (or average) (p. 217). A single arithmetic mean inserted between two numbers is the average or *the* arithmetic mean of the two numbers.
arithmetic means (p. 217). The terms between two given terms of an arithmetic sequence.
arithmetic progression (p. 219). *See* arithmetic sequence.
arithmetic sequence (p. 219). Any sequence in which each term after the first is obtained by adding a fixed number, called the common difference, to the preceding term. Also called *arithmetic progression.*
arithmetic series (p. 221). A series whose terms are in arithmetic progression.

bounded sequence (p. 242). A sequence for which there exists a number which equals or exceeds the absolute value of every term of the sequence.

Cartesian coordinate system (p. 70). A rectangular system which establishes a one-to-one correspondence between the set of points in the plane and the set of ordered pairs of real numbers. Also called *plane rectangular coordinate system.*
Cartesian product (p. 422). If a finite set A contains r elements and a finite set B contains s elements, then the set of ordered pairs (a, b) with $a \in A$ and $b \in B$ is called the Cartesian product of A and B (denoted by $A \times B$) and contains rs elements.
characteristic (p. 398). The integral part of a logarithm to base 10 when the fractional part (the part between 0 and 1) is nonnegative.
circle (p. 348). In a plane, a circle is the set of all points at a given distance, called the radius, from a given point, called the center of the circle.
circular functions (p. 513). The trigonometric functions with domain pictured as lengths of arcs on the unit circle, rather than as the set of angles.
circular permutation (p. 426). An arrangement of objects in a circular pattern.
coefficient of a monomial (p. 31). *See under* monomial.
column matrix (or column vector) (p. 462). Matrix with only one column.
combination (p. 431). An r-element subset of a set with n elements is called a combination of n elements taken r at a time.
combined variation (p. 364). A variation defined by an equation where a given variable varies directly with a second variable and inversely with a third variable.
common logarithm (p. 395). Logarithm to the base 10.
complement of A (p. 451). If an event A is in the sample space S, the complement of A is the set of elements of S that are not members of A.
complete factorization (p. 182). The factorization of a polynomial is complete when it has been expressed as the product of a constant and one or more irreducible polynomials each of which has 1 as its greatest monomial factor.
completing the square (p. 286). Transforming a quadratic expression into a square of a binomial.
complex conjugate (p. 300). For any real numbers a and b, the complex conjugate of $a + bi$ is $a - bi$; conversely, the complex conjugate of $a - bi$ is $a + bi$.
complex number (p. 299). Any number of the form $a + bi$ where $a \in \mathcal{R}$ and bi is a pure imaginary number. If $b \neq 0$, $a + bi$ is also called an imaginary number.
conditional probability (p. 450). The probability of an event occurring given that another event has occurred.
conic sections (p. 360). The curves (circle, ellipse, parabola, hyperbola) which are formed by the intersection of a plane with a conical surface of two nappes.
conjunction (p. 50). A compound sentence formed by joining two sentences with the word *and.*
consistent system of equations (p. 108). A system of equations that has at least one solution.
constant function (p. 89). A linear function where $m = 0$ and, therefore, $y = b$ for all $x \in \mathcal{R}$.

constant of proportionality (pp. 89, 256, 363). *See* constant of variation.

constant of variation (pp. 89, 256, 363). In a direct variation $y = mk$ or in an inverse variation $xy = k$ $(k \neq 0)$, k is the constant of variation. Also called *constant of proportionality*.

convergent sequence (p. 241). An infinite sequence which has a limit.

converse (p. 46). Any two "If-then" statements are converses of each other if each can be obtained from the other by interchanging hypothesis and conclusion.

coordinate(s) of a point (pp. 4, 70, 140). The number or ordered pair (ordered triple) of numbers associated with a point.

cosecant function (p. 530). A trigonometric function such that cosecant: $\alpha \rightarrow \frac{1}{\sin \alpha}$, $\sin \alpha \neq 0$.

cosine function (p. 510). If α is an angle in standard position, with $P(u, v)$ any point other than the origin on the terminal side of α, and if $\sqrt{u^2 + v^2} = r$, then cosine: $\alpha \rightarrow \frac{u}{r}$.

cotangent function (p. 530). A trigonometric function such that cotangent: $\alpha \rightarrow \frac{\cos \alpha}{\sin \alpha}$, $\sin \alpha \neq 0$.

coterminal angles (p. 503). Angles that have the same initial side and the same terminal side.

degree (of rotation) (p. 505). Each of the 360 equal parts into which a complete revolution is divided, in one system of measuring angles.

degree of a monomial (p. 32). In the monomial ax^n, $a \neq 0$, the number denoted by n.

degree of a polynomial (p. 32). The degree of the nonzero term of highest degree of a polynomial.

dependent events (p. 451). Events for which the probability of one depends on the occurrence of the other.

depressed equation (p. 329). Whenever r is a root of the polynomial equation $P(x) = 0$, $P(x) \div (x - r) = 0$ is called the depressed equation.

determinant (pp. 116, 155). A square array of numerals, set off with vertical bars, which names a real number. The numerals in the array are called the entries (or elements) of the determinant. The order of the determinant is the number of rows (or columns).

direct variation (p. 89). A linear function of the form $y = mx$, $m \neq 0$.

directed angle (p. 502). An ordered pair of rays with a common endpoint, one ray called the initial side of the angle and the other called the terminal side of the angle, together with a rotation from the initial to the terminal side.

discriminant (p. 305). The number $b^2 - 4ac$, which is named under the radical sign in the quadratic formula, is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

disjunction (p. 50). A compound sentence formed by joining two sentences by the word *or*.

divergent sequence (p. 243). An infinite sequence that does not have a limit.

domain of a function (p. 67). *See* function.

domain of a variable (p. 3). The set whose members may be used as a replacement for the variable. Also called *replacement set*.

ellipse (p. 353). In the plane, the set of points for each of which the sum of the distances from two fixed points, called foci, is a given constant.

equivalent equations (p. 35). Equations that have the same solution set over a given set.

equivalent expressions (p. 33). Two expressions are equivalent if, when they are joined by the = symbol, the resulting equation is a true statement for every numerical replacement of the variable.

equivalent inequalities (p. 48). Inequalities with the same solution set over a given set.

equivalent systems (p. 111). Systems with the same solution set.

equivalent vectors (p. 605). Vectors with the same norm and same direction.

even function (pp. 256, 552). A function such that whenever it contains the ordered pair (a, b), it also contains $(-a, b)$.

event (p. 421). Any subset of a sample space.

exponential form (p. 382). Form of a radical expression when it is written as a power (or product of powers) with rational exponents.

extremes of a proportion (p. 90). In the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, y_1 and x_2 are called the extremes.

factor of a polynomial (p. 181). If a given polynomial is the product of polynomials, each of the latter polynomials is called a factor of the given polynomial.

factor set (p. 181). A designated set to which the factors of a polynomial belong.

finite sequence (p. 213). A sequence which has a last term.

fractional equation (p. 204). An equation involving one or more rational expressions in which a variable appears in the denominator.

function (p. 67). A set of ordered pairs in which each first component is paired with exactly one second component according to a given rule. The set of first components is called the domain, and the set of second components the range, of the function.

fundamental period (p. 517). If there is a least positive period p for a periodic function, p is called the fundamental period of the function.

geometric mean (p. 233). A single geometric mean inserted between two numbers. Also called *geometric proportional.*

geometric means (p. 231). The terms between two given terms of a geometric sequence are called geometric means between the given terms.

geometric progression (p. 227). *See* geometric sequence.

geometric proportional (p. 233). *See* geometric mean.

geometric sequence (p. 227). Any sequence in which each term after the first is the product of the preceding term and a fixed number, called the common ratio. Also called *geometric progression.*

geometric series (p. 235). A series whose terms are in geometric sequence.

hyperbola (p. 358). The two-branched curve formed by the set of points in the plane such that for each point, the absolute value of the difference of its distances, called focal radii, from two fixed points, called foci, is a constant.

identity (p. 512). An equation which is true for all real values of the variable.

imaginary number (p. 299). *See under* complex number.

imaginary unit (p. 296). The number i which is a square root of -1; thus, $i = \sqrt{-1}$.

inconsistent system of equations (p. 108). System of equations whose solution set is the empty set $\emptyset$.

independent events (p. 451). Events such that the probability of one does not depend on the occurrence of the other.

indirect proof (p. 54). A method of proof which begins by assuming that the conclusion of a theorem is false and reasons from this to a contradiction of the hypothesis, an axiom, or a previously proved theorem.

infinite sequence (p. 213). A sequence which has no last term.

inverse function (p. 388). The inverse of a function, which is also a function.

inverse relation (p. 388). The relation (set of ordered pairs) which results when the components of each of the ordered pairs in a relation are interchanged.

inverse variation (p. 363). In general, any function defined by an equation of the form $xy = k$ where k (called the constant of variation) is a nonzero constant.

irrational numbers (p. 262). Real numbers that are not rational.

irreducible polynomial (p. 181). A polynomial that cannot be expressed as a product of polynomials of lower positive degree.

joint variation (p. 364). A variation defined by an equation of the form $y = kxz$, where y varies directly with x and also directly with z. We say y varies jointly as x and z.

limit of a sequence (p. 241). A number such that the absolute value of the difference of a_n (the nth term of a sequence) and this number can be made less than any given positive number, however small, by choosing n large enough.

linear combination (p. 112). When both members of an equation are multiplied by the same nonzero constant and the resulting expressions are added to the corresponding members of another equation, a linear combination of the two equations is obtained.

linear equation in two variables (p. 74). An equation which can be transformed into the form $Ax + By = C$ where $A, B, C \in \mathcal{R}$ and A and B are not both zero. The graph of such an equation is a straight line.

linear function (p. 89). A function f for which the rule for pairing is given by a linear equation of the form $y = mx + b$ $(m, b \in \mathcal{R})$.

linear inequality in two variables (p. 77). An inequality which has the linear equation in two variables $Ax + By = C$ as its associated linear equation.

linear interpolation (p. 335). The process of approximating a value of a polynomial function by using a line segment.

linear programming (p. 128). A means of finding maximum and minimum values of a linear expression over a region (feasibility region) which satisfies a system of inequalities (constraints).

logarithm (p. 390). In the exponential function with base b, $x = b^y$ $(b > 0, b \neq 1)$, the exponent y is called the logarithm of x to the base b.

mantissa (p. 398). The fractional part (the nonnegative part between 0 and 1) of the logarithm of a number.

matrix (plural, **matrices**) (p. 461). In general, a rectangular array of numerals. Each numeral in the array is called an entry of the matrix. The number of rows and the number of columns of entries in the matrix are its dimensions.

maximum possible error (p. 402). The maximum possible error of a measurement is half the precision (unit) of the measurement.

means of a proportion (p. 90). In the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, x_1 and y_2 are called the means.

minor of an element in a determinant (p. 161). The determinant obtained by deleting the row and column containing the element.

monomial (p. 31). A monomial in the variable x is an expression of the form ax^n where $a \in \mathcal{R}$ and n denotes a positive integer. The number denoted by a is called the coefficient (or numerical coefficient) of the monomial.

multiplicative inverse (p. 7). For each nonzero $a \in \mathcal{R}$, there exists a multiplicative inverse $\frac{1}{a} \in \mathcal{R}$ such that $\frac{1}{a} \cdot a = 1$ and $a \cdot \frac{1}{a} = 1$. Also called *reciprocal.*

mutually exclusive events (p. 447). Events that have no outcome in common.

nonsingular (or **invertible**) **matrix** (p. 483). A matrix A such that $\det A \neq 0$.

norm of a vector (p. 605). The length of a vector.

nth root (p. 259). Each solution of the equation $x^n = b$, n a positive integer, is called an nth root of b.
numerical coefficient (p. 31). *See* coefficient of a monomial.

octant (p. 140). Each of the eight regions into which space is separated by the three coordinate planes, the xy-plane, the yz-plane, and the xz-plane.
odd function (pp. 256, 552). A function with the property that whenever it contains (a, b), it also contains $(-a, -b)$.
odds (p. 445). The odds that the event A in sample space S will occur are given by $\frac{P(A)}{P(\bar{A})}$.
one-to-one function (p. 389). A function such that not only is each element in the domain paired with exactly one element in the range, but also each element in the range is paired with exactly one element in the domain.
ordinate (p. 71). The y-coordinate of a point.

parabola (p. 350). A curve consisting of the set of all points P whose distance from a fixed point, called the focus, is equal to the perpendicular distance from P to a line, called the directrix, that does not contain the focus.
partial sum (p. 245). In general, for any infinite series $a_1 + a_2 + \cdots + a_n + \cdots$, $S_n = \sum_{i=1}^{n} a_i$ is called a partial sum.
periodic decimal (p. 267). *See* repeating decimal.
periodic function (p. 517). A function f is periodic if there is some nonzero constant p such that $f(x + p) = f(x)$ for all x in the domain of f; p is called a period of the function.
permutation (p. 424). An arrangement of the elements of a set in a definite order.
perpendicular lines (p. 345). Two lines intersecting at right angles.
plane rectangular coordinate system (p. 70). *See* Cartesian coordinate system.
point-slope form (p. 86). An equation of a line of the form $y - y_1 = m(x - x_1)$, where m is the slope and point $P(x_1, y_1)$ is a point on the line.
polar axis (p. 593). The nonnegative x-axis in the polar coordinate system.
polar coordinates (p. 593). The components of the ordered pair $(r, m(\theta))$, usually written (r, θ), which specifies the location of point P in the plane in terms of r, the distance from the origin to P, and θ, an angle having the nonnegative x-axis as its initial side and the ray $\overrightarrow{OP}$ as its terminal side.
polar form (p. 598). The expression $r(\cos\theta + i\sin\theta)$ is called the polar form for denoting the complex number $x + yi$, where (r, θ) are the polar coordinates of the point (x, y).
pole (p. 593). The origin in the polar coordinate system.
polynomial (p. 32). An expression which consists of a string of monomials connected by plus signs.
polynomial functions (p. 322). Functions whose values are given by polynomials.
power function (p. 255). A function f defined by an equation of the form $f(x) = x^n$.
precision of a measurement (p. 402). The smallest unit on the scale of the measuring device used in making the measurement.
principal-value inverse function (p. 584). An inverse function defined by restricting (according to custom) the range of the inverse of a circular or trigonometric function.
probability (p. 444). Let S be a sample space of an experiment in which there are n possible outcomes, each equally likely. If an event A is a subset of S such that A contains h elements, then the probability of the event A is given by $\frac{h}{n}$.
proportion (p. 90). An equality of ratios.

quadrantal angle (p. 503). An angle in standard position whose terminal side coincides with a coordinate axis.
quadratic function (p. 312). A function f with domain $\mathcal{R}$ and values given by a quadratic polynomial, that is, $f = \{(x, y): y = ax^2 + bx + c,\ a, b, c, \text{ and } x \in \mathcal{R},\ a \neq 0\}$, is called a quadratic function, or a polynomial function of degree two, over $\mathcal{R}$.
quadratic polynomial (p. 32). A polynomial of degree 2 that contains a single variable.

radian (p. 506). Unit which can be used to measure any angle in standard position. If the length of an arc on the unit circle measured from point (1, 0) is a units, then the measure of the angle in standard position intercepting that arc is said to be a radians.
radical (p. 260). The symbol $\sqrt[n]{b}$ is called a radical; b is the radicand and n the index.
radicand (p. 260). *See under* radical.
range of a function (p. 67). *See under* function.
rational algebraic expression (p. 192). The quotient of two polynomials (divisor not 0).
rational number (p. 192). Any number which is the quotient of two integers (divisor not 0).
rationalizing the denominator (p. 277). The process of transforming an expression with a radical (or radicals) in its denominator into an equivalent expression with the denominator free of radicals.
real numbers (p. 4). The set of all the positive numbers, the negative numbers, and zero.
reciprocal (p. 7). *See* multiplicative inverse.
reducible polynomial (p. 181). A polynomial that can be expressed as a product of two or more polynomials of lower positive degree.
reduction formula (p. 555). A formula which can be used to "reduce" a given circular or trigonometric function value in any quadrant to a function value in Quadrant I.
relation (p. 70). Any pairing of the elements of one set, the domain, with those of another, the range, in accordance with some rule.

repeating decimal (p. 267). A decimal numeral for a rational number which consists of an endlessly repeating block of digits (the repetend). Also called *periodic decimal.*

replacement set (p. 3). *See* domain of a variable.

right angle (p. 502). An angle having a rotation of $\frac{1}{4}$ of a revolution.

round-off error (p. 269). The difference between a number and its approximation.

row matrix (or **row vector**) (p. 462). A matrix with only one row.

sample space (p. 441). A set S of elements that correspond one-to-one with the outcomes of an experiment. Also called *universe.*

scalars (p. 468). In dealing with matrices, we often refer to real numbers as scalars.

scientific notation (p. 269). Another name for standard notation when used in the expression of measurements.

secant function (p. 530). A trigonometric function such that secant: $\alpha \rightarrow \frac{1}{\cos \alpha}$, $\cos \alpha \neq 0$.

sequence (p. 213). A set of numbers (some of which can be repeated) in a particular order.

series (p. 220). In general, given any sequence a_1, a_2, . . . with n or more terms, the associated series of n terms, S_n, is $S_n = a_1 + a_2 + \cdots + a_n$.

significant digit (p. 269). In a numeral, each digit reporting the number of units of measure contained in a measurement.

similar, or **like, monomials** (p. 32). Monomials which are exactly the same or which differ only in numerical coefficients.

simple form of a polynomial (p. 32). A form of a polynomial in which no two terms are like (or similar) monomials.

simplifying a polynomial (p. 33). Replacing one polynomial by an equivalent polynomial in simple form.

sine function (p. 510). If α is an angle in standard position, with $P(u, v)$ any point other than the origin on the terminal side of α, and if $\sqrt{u^2 + v^2} = r$, then sine: $\alpha \rightarrow \frac{v}{r}$.

singular matrix (p. 483). A square matrix A such that $\det A = 0$.

slope-intercept form (p. 86). An equation of a line of the form $y = mx + b$, where m is the slope and b is the y-intercept.

slope of a line (p. 80). Ratio of rise to run. Let (x_1, y_1) and (x_2, y_2) be the coordinates of any two different points of a nonvertical line. Then the slope m of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

solution set (p. 3). The set that consists of the values of the variable for which an open sentence is true. Also called *truth set.*

square matrix (p. 462). A matrix with the same number of rows as columns.

standard notation (p. 269). An expression of a number as a product, $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer. *See also* scientific notation.

standard position (p. 503). Position of an angle placed on a rectangular coordinate system with the vertex of the angle at the origin and the initial side as the positive part of the horizontal axis.

straight angle (p. 503). An angle having a rotation of $\frac{1}{2}$ of a revolution.

sum of infinite series (p. 245). If the sequence S_1, S_2, . . . , S_n, . . . of partial sums converges to S, then the sum of the infinite series is defined to be S.

tangent function (p. 530). A trigonometric function such that tangent: $\alpha \rightarrow \frac{\sin \alpha}{\cos \alpha}$, $\cos \alpha \neq 0$.

terminating decimal (p. 266). A decimal numeral containing only a finite number of digits; that is, having 0 as repetend.

terms of a polynomial (p. 32). The monomials in the expression for the polynomial.

trace of a plane (p. 146). The line in which a plane intersects a coordinate plane is called the trace of the given plane in that coordinate plane.

transformations on a system (p. 111). Operations performed on a system of equations to produce an equivalent system.

translation (p. 488). A transformation, or mapping, of the plane in which each point P of the plane is mapped onto a corresponding point P' of the plane. P' is said to be the image of P, and P to be the preimage of P' under the mapping.

trigonometric form (p. 598). *See* polar form.

trigonometric functions (p. 510). The set of functions (sine, cosine, tangent, cotangent, secant, cosecant) with domain of each function a subset of the set of all angles.

truth set (p. 3). *See* solution set.

variable (p. 3). A symbol which may represent any one of the members of a specified set.

variation. *See* combined variation, direct variation, inverse variation, joint variation.

vector (p. 604). A directed line segment in the plane or in space, from one endpoint, called the initial point, to the other endpoint, called the terminal point.

Venn diagram (p. 447). A diagram used to picture set relationships.

x-intercept (p. 86). The abscissa of the point where a graph intersects the x-axis.

y-intercept (p. 86). The ordinate of the point where a graph intersects the y-axis.

zero matrix (p. 463). A matrix each of whose entries is zero.

zero of a function (p. 306). Any value of x in the domain of a function f which satisfies the equation $f(x) = 0$.

Index

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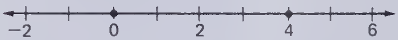
Answers to Self-Tests

Chapter 1, Self-Test 1, page 10

1. $\{0\} \not\subset \{3, 5, 7\}$ 2. $\{4\}$ 3. $\{5\}$

4. 

5. 

6. Axiom of additive inverses
7. Associative axiom for multiplication
8. Commutative axiom for addition
9. Distributive axiom 10. $\{-18\}$

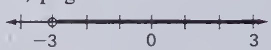
Chapter 1, Self-Test 2, page 26

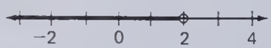
1. 1. $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$ Hypothesis
   2. $a \cdot \frac{1}{c} = b \cdot \frac{1}{c}$ Rel. bet. mult. and div.
   3. $a = b$ Canc. prop. of mult.
2. $-(a + b) = -a + (-b)$ 3. 0 4. 6
5. -36

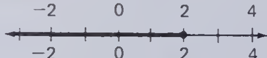
Chapter 2, Self-Test 1, pages 44–45

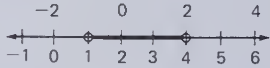
1. $4x^2 - 3x + 4$ 2. $-2y^2 + 6$
3. $3n^2 + 2n - 4$ 4. $2xy + 2x^2$
5. $\{-1\}$ 6. $\{4\}$ 7. $x = \frac{c}{2a} - \frac{b}{2}$
8. 7 by 9 by 9 9. 144 players 10. 2 h

Chapter 2, Self-Test 2, page 61

1. $\{x\colon x > -3\}$ 

2. $\{y\colon y < 2\}$ 

3. $\{n\colon n \leq 2\}$ 

4. $\{k\colon 1 < k < 4\}$ 

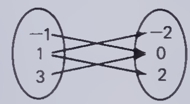
5. 6 cm

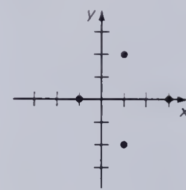
6. Assume $a \not> 1$; consider two cases:
   1. $a = 1$ and 2. $a < 1$.
   Case 1. Assume $a = 1$.
   1. Assume $a = 1$; $c > 0$ Hypothesis
   2. $ac = 1 \cdot c$ Mult. prop. of =
   3. $ac = c$ Ax. of 1
   Case 2. Assume $a < 1$
   1. $a < 1$, $c > 0$ Hypothesis
   2. $ac < 1 \cdot c$ Mult. prop. of order
   3. $ac < c$ Ax. of 1
   Both assumptions lead to contradictions of hyp: $ac > c$. Therefore, assumption $a \not> 1$ is false, and $a > 1$.

7. $\{z\colon -1 \leq z \leq 9\}$
8. $\{x\colon x < -7\} \cup \{x\colon x > -2\}$

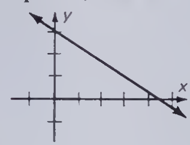
Chapter 3, Self-Test 1, page 73

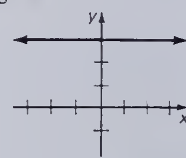
1. 6 2. 5 3. 37 4. -20

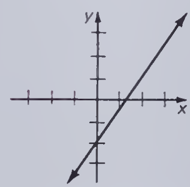
5. 

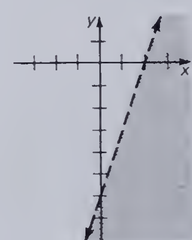
6. 

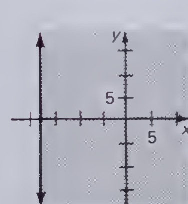
Chapter 3, Self-Test 2, page 80

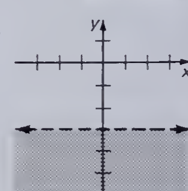
1. 

2. 

3. 

4. 

5. 

6. 

Chapter 3, Self-Test 3, page 88

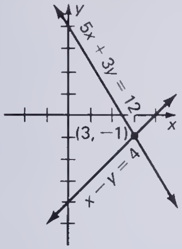
1. $\frac{1}{4}$ 2. $5x - 3y = 15$ 3. $y = -5x + 9$
4. $y = -3x - 13$ 5. $y = 2x - \frac{3}{2}$

Chapter 3, Self-Test 4, page 94

1. $31\frac{1}{2}$ 2. 7 L

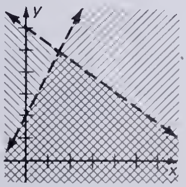
Chapter 4, Self-Test 1, page 125

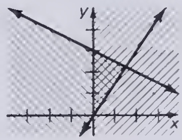
1. $\{(3, -1)\}$

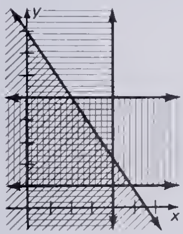


2. a. one sol. b. infinitely many sol's c. no sol.
3. $\{(1, 4)\}$
4. $\{(3, -5)\}$
5. $\{(0, -2)\}$ 6. 110 adults, 40 children

Chapter 4, Self-Test 2, page 133

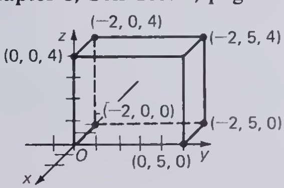
1. 

2. 

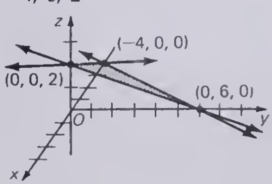
3. 

4. max: 15; min: 7
5. max: 40; min: 19
6. $560

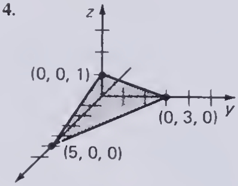
Chapter 5, Self-Test 1, page 154

1. 

2. $-4, 6, 2$



3. $\begin{cases} x = 0 \\ -6y + 2z = 20 \end{cases}$ $\begin{cases} y = 0 \\ 4x + 2z = 20 \end{cases}$ $\begin{cases} z = 0 \\ 4x - 6y = 20 \end{cases}$

4. 

5. $\{(3, 1, -1)\}$

Chapter 5, Self-Test 2, page 167

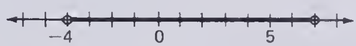
1. $\{(2, -1, 3)\}$ 2. $m°(\angle A) = 20$, $m°(\angle B) = 40$, $m°(\angle C) = 120$ 3. -27

Chapter 6, Self-Test 1, page 185

1. $\dfrac{3y^3}{x^2}$ 2. $\dfrac{12}{xy^4}$ 3. $8x^3 - 36x^2 + 54x - 27$
4. $(2x - 1)(x + 7)$
5. $(3a - b)(9a^2 + 3ab + b^2)$

Chapter 6, Self-Test 2, page 191

1. $\{0, -9\}$ 2. $\left\{\dfrac{4}{5}, -3\right\}$ 3. 7 cm
4. $\{x: -4 < x < 7\}$



Chapter 6, Self-Test 3, page 208

1. $2y - 6$ 2. $3x^2 + x + 4$ 3. $\dfrac{2(a + 2)}{a^2}$
4. $\dfrac{x - 2}{x - 4}$ 5. $\{-2, 5\}$ 6. 17.5 h

Chapter 7, Self-Test 1, page 225

1. 11, 17, 23 2. 15 3. 308 4. 1683
5. 19,250

Chapter 7, Self-Test 2, page 239

1. $25, 10, 4, \dfrac{8}{5}$ 2. 24 3. 9, 3 and 1 or -9, 3 and -1 4. $\dfrac{1042}{125}$ 5. 14.4 km

Chapter 7, Self-Test 3, page 250

1. $3, 2\frac{1}{4}, 2\frac{1}{9}, 2\frac{1}{16}$; $L = 2$; $|L - a_n| = \dfrac{1}{n^2}$
2. $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$; $L = 1$; $|L - a_n| = \dfrac{1}{n}$ 3. $\dfrac{48}{5}$
4. $\dfrac{2}{5}$ 5. $\dfrac{4}{9}$ 6. $\dfrac{6}{11}$

Chapter 8, Self-Test 1, page 264

1. 24   2. a. 14   b. -2 3. $\{2\}$
4. $\{-2, -1, 3\}$

Chapter 8, Self-Test 2, page 275

1. $0.\overline{54}$ 2. $\dfrac{11}{37}$ 3. 4.782×10^3
4. 7.83×10^{-3} 5. 2×10^{-3} 6. 0.445; 0.44505505550 . . .

Chapter 8, Self-Test 3, page 289

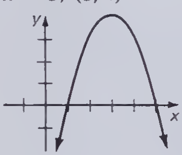
1. $26\sqrt{6}$ 2. $(2x - 3y)\sqrt[3]{2xy}$
3. $-4 - \sqrt{3}$ 4. $\dfrac{21 + 8\sqrt{5}}{11}$ 5. $\{96\}$
6. $\{-2\}$ 7. $\left\{\dfrac{3}{2} + \dfrac{\sqrt{3}}{2}, \dfrac{3}{2} - \dfrac{\sqrt{3}}{2}\right\}$
8. $\left\{\dfrac{4}{3} + \dfrac{\sqrt{37}}{3}, \dfrac{4}{3} - \dfrac{\sqrt{37}}{3}\right\}$

Chapter 9, Self-Test 1, page 311

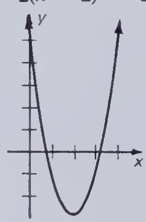
1. $24i$ 2. $\dfrac{4\sqrt{3}}{3}i$ 3. $-2 + i$
4. $-3 + 3i$ 5. $21 - 20i$ 6. $\sqrt{2} + i$
7. -16; imaginary   8. 81; real; rational
9. a. 3   b. $\frac{2}{3}$ 10. $x^2 + 6x + 4 = 0$

Chapter 9, Self-Test 2, page 321

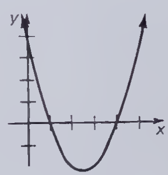
1. $x = 3$; $(3, 4)$ 2. maximum
3.



4. $y = 2(x - 2)^2 - 3$



5. 18 cm^2
6. $\{x: x < 1\} \cup \{x: x > 4\}$



Chapter 9, Self-Test 3, page 336

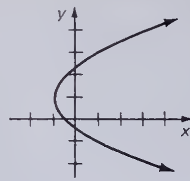
1. a. -30 b. $7 + i$ 2. $x^2 - 4x - 5$; 3
3. $\{-1, 2, \frac{5}{2}\}$ 4. $\{2 - 3i, 2 + 3i, -4\}$
5. 1.43

Chapter 10, Self-Test 1, page 347

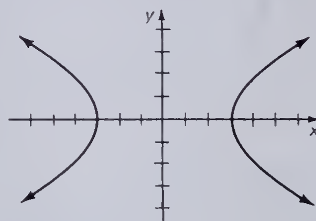
1. $\sqrt{34}$ 2. $(-3, -2)$ 3. $y = \frac{1}{2}x + \frac{7}{2}$

Chapter 10, Self-Test 2, page 368

1.



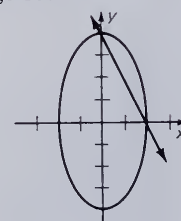
2.



3. $x^2 + y^2 - 6x + 2y - 6 = 0$
4. $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$ 5. 48   6. 8 lx

Chapter 10, Self-Test 3, page 376

1. $\{(2, 0), (0, 4)\}$
See figure at right.



2. $\left\{(5, -2), \left(-\dfrac{35}{9}, \dfrac{22}{9}\right)\right\}$
3. $\{(2, 4), (2, -4), (-2, 4), (-2, -4)\}$

Chapter 11, Self-Test 1, page 387

1. 4   2. $\sqrt{3y}$ 3. 5   4. $\sqrt{2}$ 5. $\{-8\}$
6. $\{3\}$ 7. $\{2\}$ 8. $\left\{-\dfrac{5}{2}\right\}$

Chapter 11, Self-Test 2, page 393

1. $f^{-1}(x) = \sqrt[3]{x - 1}$ 2. No. The inverse is $\{(x, y): y^2 = x\}$, and for every $x > 0$ there are two values of y.   3. $\log_4 64 = 3$
4. $4^{\frac{3}{2}} = 8$ 5. $\{100{,}000\}$ 6. $\{8\}$

Chapter 11, Self-Test 3, page 415

1. 1.4527   2. 0.04832   3. a. 0.1 m   b. 4%
4. 0.1601   5. 10.3   6. $\{1.40\}$ 7. $\{15.1\}$

Chapter 12, Self-Test 1, page 430

1. 75   2. 720   3. 840   4. 420

Chapter 12, Self-Test 2, page 435

1. 210   2. 462   3. 300   4. 420

Chapter 12, Self-Test 3, page 441

1. $a^{10} - 10a^8 + 40a^6 - 80a^4 + 80a^2 - 32$

2. $x^6 - 3x^5y + \frac{15x^4y^2}{4} - \frac{5x^3y^3}{2} + \frac{15x^2y^4}{16} - \frac{3xy^5}{16} + \frac{y^6}{64}$ 3. $3360p^4q^6$

Chapter 12, Self-Test 4, page 455

1. $\{(A, 1), (A, 2), (A, 3), (B, 1), (B, 2), (B, 3), (C, 1), (C, 2), (C, 3)\}$, $\{(A, 1), (A, 3), (B, 1), (B, 3)\}$

2. $\frac{1}{4}$ 3. $\frac{1}{2}$ 4. No. Rolling a 1 on one die makes it impossible to have 8 for the sum of the two dice.

Chapter 13, Self-Test 1, page 480

1. a. $\begin{bmatrix} -1 & -3 \\ -13 & 9 \end{bmatrix}$ b. $\begin{bmatrix} -2 & -3 & 12 \\ -2 & 1 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 7 & -10 \\ 7 & 15 \end{bmatrix}$ 3. a. $\begin{bmatrix} 9 & 0 \\ -6 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 3 & -6 \\ 5 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 0 \\ -4 & 8 \end{bmatrix}$

Chapter 13, Self-Test 2, page 486

1. $\begin{bmatrix} -\frac{4}{3} & -\frac{5}{3} \\ 1 & 1 \end{bmatrix}$ 2. $x = \begin{bmatrix} -18 & 16 \\ 22 & -20 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ 4. $\{(4, 1)\}$

Chapter 13, Self-Test 3, page 496

1. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ 2. $(-4, 12)$

3. $(6, -14)$ 4. $(-17, -25)$ 5. $(-2, 6)$

Chapter 14, Self-Test 1, page 509

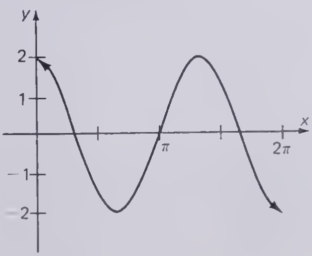
1. 594 cm 2. $\frac{7\pi}{3}^{R}$ 3. 495°

Chapter 14, Self-Test 2, page 529

1. $\sin \alpha = \frac{4\sqrt{2}}{9}$, $\cos \alpha = -\frac{7}{9}$ 2. $\frac{7}{25}$

3. a. $\frac{\sqrt{2}}{2}$ b. $\frac{1}{2}$ 4. a. 0.9365 b. 0.7949 c. -0.8307 d. 0.4176

5.



Chapter 14, Self-Test 3, page 540

1. $\sin \alpha \approx 0.821$, $\cos \alpha \approx -0.571$, $\tan \alpha \approx -1.44$, $\cot \alpha \approx -0.696$, $\sec \alpha \approx -1.75$, $\csc \alpha \approx 1.22$

2. $a \approx 25.9$, $c \approx 30.5$, $m(B) = 31° \, 40'$

Chapter 15, Self-Test 1, page 551

1. $\sec \alpha(\cos \alpha + \sin \alpha \tan \alpha)$

$$= \frac{1}{\cos \alpha}\left(\cos \alpha + \sin \alpha \frac{\sin \alpha}{\cos \alpha}\right)$$
$$= \frac{\cos \alpha}{\cos \alpha} + \frac{\sin^2 \alpha}{\cos^2 \alpha}$$
$$= 1 + \left(\frac{\sin \alpha}{\cos \alpha}\right)^2$$
$$= 1 + \tan^2 \alpha$$
$$= \sec^2 \alpha$$

2. $$\frac{\sin^2 \alpha}{1 - \cos \alpha} = \frac{1 - \cos^2 \alpha}{1 - \cos \alpha}$$
$$= \frac{(1 + \cos \alpha)(1 - \cos \alpha)}{1 - \cos \alpha}$$
$$= 1 + \cos \alpha$$

3. $$\frac{\cos \alpha}{\sec \alpha} + \frac{\sin \alpha}{\csc \alpha} = \frac{\cos \alpha}{\frac{1}{\cos \alpha}} + \frac{\sin \alpha}{\frac{1}{\sin \alpha}}$$
$$= \cos^2 \alpha + \sin^2 \alpha$$
$$= 1$$

4. $\sec \alpha \csc \alpha - \tan \alpha$

$$= \frac{1}{\cos \alpha} \cdot \frac{1}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha}$$
$$= \frac{1}{\sin \alpha \cos \alpha} - \frac{\sin^2 \alpha}{\sin \alpha \cos \alpha}$$
$$= \frac{1 - \sin^2 \alpha}{\sin \alpha \cos \alpha}$$
$$= \frac{\cos^2 \alpha}{\sin \alpha \cos \alpha}$$
$$= \frac{\cos \alpha}{\sin \alpha}$$
$$= \cot \alpha$$

Chapter 15, Self-Test 2, page 571

1. a. $-\sin 45°$ b. $-\cos \frac{\pi}{3}$ 2. $\frac{\sqrt{2} - \sqrt{6}}{4}$

3. $\frac{1}{9}$

4. $$\frac{\csc \alpha}{2 \cos \alpha} = \frac{\frac{1}{\sin \alpha}}{2 \cos \alpha}$$
$$= \frac{1}{2 \sin \alpha \cos \alpha}$$
$$= \frac{1}{\sin 2\alpha}$$
$$= \csc 2\alpha$$

Chapter 15, Self-Test 3, page 579

1. 10.0 2. 52.0 3. 52.5 4. 34.2 cm
5. 170.5 km

Chapter 16, Self-Test 1, page 592

1. $\left\{x: x = \frac{4\pi}{3} + 2k\pi\right\} \cup \left\{x: x = \frac{5\pi}{3} + 2k\pi\right\}$
2. a. $60°$ b. $135°$ 3. $\left\{x: x = \frac{\pi}{4} + \frac{k\pi}{2}\right\}$
4. $\{60°, 90°, 270°, 300°\}$

Chapter 16, Self-Test 2, page 609

1. $(4, 150°)$, $(-4, -30°)$ 2. $(5, -8.66)$
3. a. $-16i$ b. $-2 + 2i\sqrt{3}$ c. $8 - 8i\sqrt{3}$
4. $4(\cos 75° + i \sin 75°)$, $4(\cos 195° + i \sin 195°)$, $4(\cos 315° + i \sin 315°)$
5. $\|\mathbf{u}_x\| = 4.0$, $\|\mathbf{u}_y\| \approx 6.9$, $\|\mathbf{v}_x\| \approx 7.1$, $\|\mathbf{v}_y\| \approx 7.1$, $\|\mathbf{u} + \mathbf{v}\| \approx 14$, $m(\theta) \approx 78°$

Answers to Selected Exercises

Chapter 1 Review of Essentials

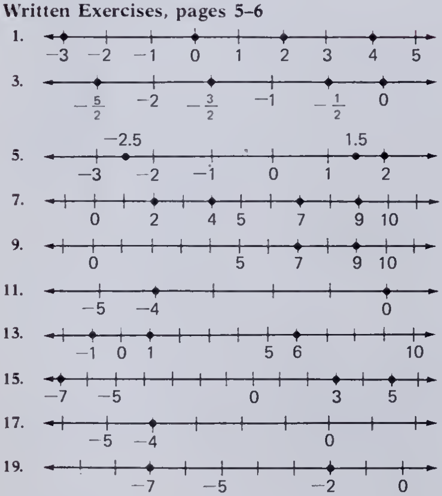
Written Exercises, pages 2–3

1. $\in$, $\not\subset$, or $\neq$ 3. $\in$, $\not\subset$, or $\neq$ 5. $=$, $\notin$, or $\not\subset$
7. $\neq$, $\notin$, or $\not\subset$ 9. $\neq$, $\notin$, or $\not\subset$ 11. $=$, $\subset$, $\notin$
13. $\neq$, $\not\subset$, or $\notin$ 15. $=$, $\subset$, or $\notin$

Written Exercises, page 4

1. $\emptyset$ 3. $\{1\}$ 5. $\{2\}$ 7. $\{3\}$ 9. $\emptyset$ 11. $\{3\}$
13. $\{1, 4\}$ 15. $\emptyset$ 17. $\{1\}$ 19. $\{3, 4, 5\}$

Written Exercises, pages 5–6



Written Exercises, pages 8–9

1. Commutative axiom for multiplication
3. Closure axiom for multiplication
5. Commutative axiom for addition
7. Distributive axiom 9. Transitive property of equality 11. Closure axiom for addition

13. 33 15. -210 17. $\{5\}$ 19. $\left\{\frac{1}{2}\right\}$ 21. $\{2\}$
23. $\{40\}$ 25. $\{-32\}$

Written Exercises, pages 15–16

1. 9 3. t 5. 0 7. -6 9. 25 11. 2
13. -7.3 15. 1 17. 36 19. -66 21. 23
23. -42.4 25. 1. Hypothesis 2. Axiom of additive inverses 3. Associative axiom for add. 4. Axiom of additive inverses 5. Axiom of 0

27.
1.	$c + a = c + b$	Hypothesis
2.	$c + a = a + c$; $c + b = b + c$	Comm. ax. for add.
3.	$a + c = b + c$	Subs. prin.
4.	$a = b$	Theorem, page 12

29.
1.	$b + a = a$	Hypothesis
2.	$a = 0 + a$	Ax. of 0
3.	$b + a = 0 + a$	Trans. prop. of $=$
4.	$b = 0$	Canc. prop. of add.

31.
1.	$a, b \in \mathbb{R}$	Hypothesis
2.	$-a, -b \in \mathbb{R}$	Ax. of add. inv.
3.	$-[(-a) + (-b)] = -(-a) + [-(-b)]$	Prop. of neg. of a sum
4.	$-(-a) = a$, $-(-b) = b$	Canc. prop. of inv.
5.	$-[(-a) + (-b)] = a + b$	Subs. prin.

33.
1.	$a + c = b + d$; $c = d$	Hypothesis
2.	$a + c = b + c$	Substitution principle
3.	$a = b$	Cancellation property of add.

35.
1.	$x + (-5) = -2$	Hypothesis
2.	$= -2 + 0$	Ax. of 0
3.	$= -2 + [5 + (-5)]$	Ax. of add. inv.
4.	$= (-2 + 5) + (-5)$	Assoc. ax. of add.
5.	$= 3 + (-5)$	Subs. prin.
6.	$x + (-5) = 3 + (-5)$	Trans. prop. of $=$
7.	$x = 3$	Canc. prop. of add.

37. 1. $x + b = a$ Hyp.
2. $= a + 0$ Ax. of 0
3. $= a + [(-b) + b]$ Ax. of add. inv.
4. $= [a + (-b)] + b$ Assoc. ax. for add.
5. $x + b = [a + (-b)] + b$ Trans. prop. of =
6. $x = a + (-b)$ Canc. prop. of add.

39. 1. $2x + b = x$ Hyp.
2. $(x + x) + b = x$ Subs. prin.
3. $x + (x + b) = x$ Assoc. ax. for add.
4. $x + (x + b) = x + 0$ Ax. of 0
5. $x + b = 0$ Canc. prop. of add.
6. $x + b = (-b) + b$ Ax. of add. inv.
7. $x = -b$ Canc. prop. of add.

Written Exercises, pages 19–20

1. 168 3. 9 5. 26 7. 6 9. $\frac{1}{2}$

11. -22.5 13. -84 15. $-\frac{3}{2}$ 17. positive

19. 0 21. positive 23. positive.

25. 1. $b, c \in \mathcal{R}, c \neq 0$ Hypothesis
2. $\frac{1}{c}$ is a real number. Ax. of mult. inv.
3. $bc\left(\frac{1}{c}\right) = b\left[c\left(\frac{1}{c}\right)\right]$ Assoc. ax. for mult.
4. $= b \cdot 1$ Ax. of mult. inv.
5. $= b$ Ax. of 1
6. $bc\left(\frac{1}{c}\right) = b$ Trans. prop. of =

27. 1. $ca = cb$ Hypothesis
$c \neq 0$
2. $ca = ac$ Comm. ax. for mult.
$cb = bc$
3. $ac = bc$ Subs. prin.
4. $a = b$ Exercise 26

29. 1. $ab = a$ Hypothesis
$a \neq 0$
2. $a = a \cdot 1$ Ax. of 1
3. $ab = a \cdot 1$ Subs. prin.
4. $b = 1$ Canc. prop. of mult.

31. 1. $a \neq 0$ Hypothesis
2. $\frac{1}{a} \cdot a = 1$ Ax. of mult. inv.
3. $a = \frac{1}{\frac{1}{a}}$ Ex. 30 (uniqueness)
4. $\frac{1}{\frac{1}{a}} = a$ Symm. prop. of =

33. 1. $a, b \in \mathcal{R}$ Hypothesis
2. $a(-b) = a[b \cdot (-1)]$ Mult. prop. of -1
3. $= (ab)(-1)$ Assoc. ax. for mult.
4. $= -ab$ Mult. prop. of -1
5. $a(-b) = -ab$ Trans. prop. of =

35. 1. $a = b$ Hypothesis
$a, b \neq 0$
2. $a\left(\frac{1}{a}\right) = 1$ Ax. of mult. inv.
$b\left(\frac{1}{b}\right) = 1$
3. $a\left(\frac{1}{a}\right) = b\left(\frac{1}{b}\right)$ Trans. prop. of =
4. $b\left(\frac{1}{a}\right) = b\left(\frac{1}{b}\right)$ Subs. prin.
5. $\frac{1}{a} = \frac{1}{b}$ Canc. prop. of mult.

37. 1. $xb = a,\ b \neq 0$ Hypothesis
2. $xb = a \cdot 1$ Ax. of 1
3. $= a\left(\frac{1}{b} \cdot b\right)$ Ax. of mult. inv.
4. $= \left(a \cdot \frac{1}{b}\right) \cdot b$ Assoc. ax. for mult.
5. $xb = \left(a \cdot \frac{1}{b}\right) \cdot b$ Trans. prop. of =
6. $x = a \cdot \frac{1}{b}$ Canc. prop. of mult.

39. 1. $ax + b = c$ Hypothesis
2. $= c + 0$ Ax. of 0
3. $= c + [(-b) + b]$ Ax. of add. inv.
4. $= [c + (-b)] + b$ Assoc. ax. for add.
5. $ax + b = [c + (-b)] + b$ Trans. prop. of =
6. $ax = c + (-b)$ Canc. prop. of add.
7. $xa = c + (-b)$ Comm. ax. for mult.
8. $x = [c + (-b)]\frac{1}{a}$ Exercise 37
9. $x = \frac{1}{a}[c + (-b)]$ Comm. ax. for mult.

41. 1. $\frac{1}{x} = ab$ Hypothesis
2. $\frac{1}{\left(\frac{1}{x}\right)} = \frac{1}{ab}$ Exercise 35
3. $x = \frac{1}{\left(\frac{1}{x}\right)}$ Exercise 31
4. $x = \frac{1}{ab}$ Trans. prop. of =

Written Exercises, page 22

1. 8 3. 288 5. 389 7. -10 9. 21
11. 19 13. -20 15. 0 19. 1. Hypothesis
2. Rel. between add. and sub. 3. Substitution
4. Distributive ax. 5. Prop. of neg. in products
6. Rel. between add. and sub. 7. Transitive prop. of =

21.

1.	$a, b \in \mathcal{R}$	Hypothesis
2.	$(a - b) + b = [a + (-b)] + b$	Rel. between add. and sub.
3.	$= a + [(-b) + b]$	Assoc. ax. for add.
4.	$= a + 0$	Ax. of add. inv.
5.	$= a$	Ax. of 0
6.	$(a - b) + b = a$	Trans. prop. of =

23.

1.	$a, b, c \in \mathcal{R}$	Hyp.
2.	$-a(b - c) = -a[b + (-c)]$	Rel. bet. add. and sub.
3.	$= (-a)b + (-a)(-c)$	Dist. ax.
4.	$= -(ab) + ac$	Prop. of neg. in prod.
5.	$= ac + -(ab)$	Comm. ax. for add.
6.	$= ac - ab$	Rel. bet. add. and sub.
7.	$-a(b - c) = ac - ab$	Transitive prop. of =

25.

1.	$c - a = c - b$	Hypothesis
2.	$c + (-a) = c + (-b)$	Rel. bet. add. and sub.
3.	$-a = -b$	Cancellation prop. of add.
4.	$a(-1) = -a$ $b(-1) = b$	Mult. prop. of (-1)
5.	$a(-1) = b(-1)$	Subs. prin.
6.	$a = b$	Canc. prop. of mult.

27.

1.	$a = b$	Hypothesis
2.	$c - a = c - a$	Refl. prop. of =
3.	$c - a = c - b$	Subs. prin.

29.

1.	$x - b = a$	Hyp.
2.	$x + (-b) = a$	Rel. bet. add. and sub.
3.	$x + (-b) = a + 0$	Ax. of 0
4.	$= a + [b + (-b)]$	Ax. of add. inv.
5.	$= (a + b) + (-b)$	Assoc. ax. for add.
6.	$x + (-b) = (a + b) + (-b)$	Trans. prop. of =
7.	$x = a + b$	Canc. prop. of add.

31.

1.	a, b, c, d real numbers	Hyp.
2.	$(a - b)(c + d)$ $= (a - b)c + (a - b)d$	Dist. ax.
3.	$= [a + (-b)]c + [a + (-b)]d$	Rel. bet. add. and sub.
4.	$= [ac + (-b)c] + [ad + (-b)d]$	Dist. ax.
5.	$[ac + (-bc)] + [ad + (-bd)]$	Prop. of neg. in prod.
6.	$= ac - bc + ad - bd$	Rel. bet. add. and sub.
7.	$(a - b)(c + d)$ $= ac - bc + ad - bd$	Trans. prop. of =

33.

1.	a, b, c, d real numbers	Hypothesis
2.	$a[b - (c + d)] = ab - a(c + d)$	Exercise 19
3.	$a[b - (c + d)] = ab - ac - ad$	Exercise 22

Written Exercises, pages 24–25

1. -3 **3.** 15 **5.** -213 **7.** 15 **9.** -5 **11.** $6\frac{2}{3}$ **13.** -6 **19.** 1. Hypothesis 2. Rel. bet. mult. and div. 3. Dist. ax. 4. Rel. bet. mult. and div. 5. Trans. prop. of =.

21.

1.	$a \in \mathcal{R}$ $a \neq 0$	Hypothesis
2.	$\frac{a}{a} = a\left(\frac{1}{a}\right)$	Rel. bet. mult. and div.
3.	$\frac{a}{a} = 1$	Ax. of mult. inv.

23.

1.	$a, b \in \mathcal{R}$	Hypothesis
2.	$(ab) \div b = (ab)\left(\frac{1}{b}\right)$	Rel. bet. mult. and div.
3.	$= a\left(b \cdot \frac{1}{b}\right)$	Assoc. ax. for mult.
4.	$= a(1)$	Ax. of mult. inv.
5.	$= a$	Ax. of 1
6.	$(ab) \div b = a$	Trans. prop. of =

25.

1.	$x = \frac{b}{a}$	Hypothesis
2.	$1 \cdot x = 1\left(\frac{b}{a}\right)$	Ax. of 1
3.	$\left(\frac{1}{a} \cdot a\right)x = \left(\frac{1}{a} \cdot a\right) \cdot \frac{b}{a}$	Ax. of mult. inv.
4.	$\frac{1}{a}(ax) = \frac{1}{a}\left(a \cdot \frac{b}{a}\right)$	Assoc. ax. for mult.
5.	$ax = \left(\frac{b}{a}\right)a$	Canc. prop. of mult.
6.	$= \left[b\left(\frac{1}{a}\right)\right]a$	Rel. bet. mult. and div.
7.	$= b\left[\frac{1}{a} \cdot a\right]$	Assoc. ax. for mult.
8.	$= b \cdot 1$	Ax. of mult. inv.
9.	$= b$	Ax. of 1
10.	$ax = b$	Transitive prop. of =

27.

1.	$a, b, c, d \in \mathcal{R}$	Hypothesis
2.	$\frac{a}{b} \cdot \frac{c}{d} = \left(a \cdot \frac{1}{b}\right)\left(c \cdot \frac{1}{d}\right)$	Rel. bet. mult. and div.
3.	$= a\left(\frac{1}{b} \cdot c\right)\frac{1}{d}$	Assoc. ax. for mult.
4.	$= a\left(c \cdot \frac{1}{b}\right)\frac{1}{d}$	Comm. ax. for mult.

5. $= (ac)\left(\frac{1}{b} \cdot \frac{1}{d}\right)$ Assoc. ax. for mult.

6. $= ac\left(\frac{1}{bd}\right)$ Prop. of the rec. of a prod.

7. $= \frac{ac}{bd}$ Rel. bet. mult. and div.

8. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ Trans. prop. of =

29. 1. $a, b \in \mathcal{R}$; $a, b \neq 0$ Hypothesis

2. $\frac{b}{ab} = \frac{1}{a}$; $\frac{a}{ab} = \frac{1}{b}$ Exercise 26

3. $\frac{1}{a} + \frac{1}{b} = \frac{b}{ab} + \frac{a}{ba}$ Subs. prin.

4. $= \frac{b}{ab} + \frac{a}{ab}$ Comm. ax. of mult.

5. $= b\left(\frac{1}{ab}\right) + a\left(\frac{1}{ab}\right)$ Rel. bet. mult. and div.

6. $= (b + a)\frac{1}{ab}$ Dist. ax.

7. $= \frac{b + a}{ab}$ Rel. bet. mult. and div.

8. $\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$ Trans. prop. of =

31. 1. $a, b \in \mathcal{R}$; $a, b \neq 0$ Hypothesis

2. $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba}$ Exercise 27

3. $= \frac{ba}{ba}$ Comm. ax. for mult.

4. $= 1$ Exercise 21

5. $\frac{a}{b} \cdot \frac{b}{a} = 1$ Trans. prop. of =

33. 1. $a, b, c, d \in \mathcal{R}$, $b, c, d \neq 0$ Hypothesis

2. $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\frac{1}{\left(\frac{c}{d}\right)}$ Rel. between mult. and div.

3. $= \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$ Exercise 32

4. $= \frac{ad}{bc}$ Exercise 27

5. $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$ Transitive prop. of =

Chapter Review, page 28

1. b **3.** a **5.** d **7.** c **9.** b **11.** b **13.** c **15.** a **17.** b

Chapter 2 Review of Essentials

Written Exercises, page 34

1. $5y^3 - 6y^2 + 2$ **3.** $t^4 + 4t^3 - 7t^2 + t - 1$ **5.** $10w^5 - 3w^3 + w^2 - w - 8$ **7.** $3y^3 - 4y^2 + 2y + 14$ **9.** $t^4 - 6t^3 + t^2 - t + 1$ **11.** $-8w^5 - 5w^3 + w^2 + w - 2$ **13.** $4x + 1$ **15.** $r - s$ **17.** $-z^2 - 10z + 1$ **19.** $-r^2 - 6r + 3$ **21.** $2t^4 + t^3 - 8t^2 - 3t - 4$ **23.** $3x^2 - xy$ **25.** $3q^2 + 2q + 1$ **27.** $19t - 240$ **29.** $-7x^4 + 5x^3 + x + 9$ **31.** $3x^3 - 19x^2 + 52x$ **33.** $2r^3 - 6r^2 - 3r + 20$

Written Exercises, pages 36–37

1. Subtract 5 from both members; divide each member by -8. **3.** Subtract 11 from both members; multiply each member by 3. **5.** Simplify by the distributive axiom; simplify by combining like terms in left member **7.** Simplify by the distributive axiom; subtract $3m$ from each member **9.** $\{8\}$ **11.** $\{11\}$ **13.** $\{5\}$ **15.** $\{-25\}$ **17.** $\{28\}$ **19.** $\{-9\}$ **21.** $\{0\}$ **23.** $\{-14\}$ **25.** $\{4\}$ **27.** $x = \frac{4c}{a}$ **29.** $n = 1$ **31.** $t = \frac{k - v}{g}$ **33.** $d = \frac{bc}{a}$ **35.** $l = \frac{A - \pi r^2}{\pi r}$ **37.** $t = \frac{6m - n}{4m + 2n}$; $t = 1$ **39.** $m = \frac{3c - 6d}{2c + 4d}$; $m = -2$.

41. 1. $a = b$; a, b, c real numbers Hypothesis
2. $a - c = a - c$ Refl. prop. of =
3. $a - b = b - c$ Subs. prin.

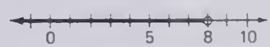
43. 1. $a = b$; a, b, c real numbers Hypothesis
2. $\frac{a}{c} = \frac{a}{c}$ Refl. prop. of =
3. $\frac{a}{c} = \frac{b}{c}$ Subs. prin.

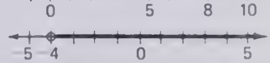
Problems, pages 42–44

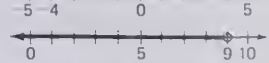
1. 29, 56 **3.** 6 cm **5.** 66, 67, 68 **7.** 13, 15, 17 **9.** 52°, 64°, 64° **11.** 40 cm **13.** 24 L **15.** 45 min **17.** 1 h 36 min **19.** 15 sides **21.** \$1.50 **23.** 80 km/h

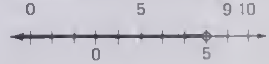
Written Exercises, page 49

1. Add. prop. or trans. 2 **3.** Add prop. or trans. 2 **5.** Not true **7.** Not true **9.** Add. prop. or trans. 2

11. $\{x: x < 8\}$


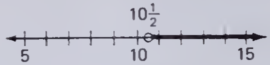
13. $x: \{x > -4\}$


15. $\{y: y < 9\}$


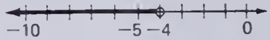
17. $\{z: z < 5\}$


19. $\{t: t > 2\}$

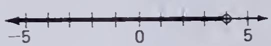
21. $\left\{x: x > 10\frac{1}{2}\right\}$



23. $\{x: x < -4\}$



25. $\{t: t < 4\}$



27.

1. $a > 0, a \in \mathcal{R}$	Hypothesis
2. $a + (-a) > 0 + (-a)$	Add. prop. of order
3. $0 > 0 + (-a)$	Ax. of add. inv.
4. $0 > (-a)$	Ax. of 0
5. $-a < 0$	Rewriting ineq.

29.

1. $a > b, c > 0$; $a, b, c \in \mathcal{R}$	Hypothesis
2. There is a number d, $d > 0$, such that $b + d = a$	Df. of $>$
3. $(b + d)c = ac$	Mult. prop. of $=$
4. $bc + dc = ac$	Dist. ax.
5. dc is a pos. real no.	Closure ax. for $\mathcal{R}_+$
6. $ac > bc$	Df. of $>$

31.

1. $a + c > b + c$	Hyp.
2. $[a + c] + (-c) > [b + c] + (-c)$	Ax. of add. inv.
3. $a + [c + (-c)] > b + [c + (-c)]$	Assoc. ax. for add.
4. $a + 0 > b + 0$	Ax. of add. inv.
5. $a > b$	Ax. of 0

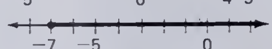
33.

1. $a > b$ $c > d$	Hypothesis
2. $a + c > b + c$ $b + c > b + d$	Add. prop. of order.
3. $a + c > b + d$	Trans. prop. of order.

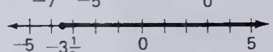
Written Exercises, page 52

1. $\{t: t \leq 4\}$

3. $\{y: y \geq -7\}$

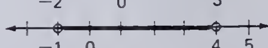


5. $\left\{x: x \geq -3\frac{1}{2}\right\}$



7. $\{a: -2 \leq a \leq 3\}$

9. $\{c: -1 < c < 4\}$

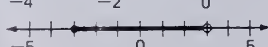


11. $\{s: -4 \leq s \leq 8\}$

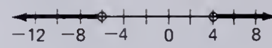
13. $\{n: n \geq -2\}$

15. $\{m: m \leq -2\}$

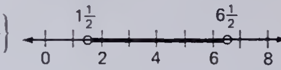
17. $\{x: -3 \leq x < 3\}$



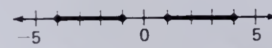
19. $\{y: y < -6\} \cup \{y: y > 4\}$



21. $\left\{t: 1\frac{1}{2} < t < 6\frac{1}{2}\right\}$

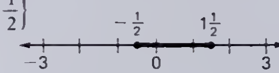


23. $\{x: -4 \leq x \leq -1\} \cup \{x: 1 \leq x \leq 4\}$



25. $\{r: -2 < r < 4\}$

27. $\left\{p: -\frac{1}{2} \leq p \leq 1\frac{1}{2}\right\}$



Problems, pages 52–53

1. 70 m 3. 25 cm 5. 3 min 7. 320 m 9. 60 min 11. 20 sides

Written Exercises, pages 56–57

1. Case 1: 1. Mult. prop. of = 2. Mult. prop. of 0 3. Ax. of mult. inv. 4. $1 > 0$ 3. Case 1: 1. Mult. prop. of eq. 2. Ax. of 1 3. Rel. bet. mult. and div. 4. Assoc. ax. for mult. 5. Ax. of mult. inv. 6. Ax. of 1 7. By hypothesis

5. Assume $\frac{1}{a} \not< 1$.

Case 1: $\frac{1}{a} = 1$

1. $\frac{1}{a} = 1$	Hypothesis
2. $\frac{1}{a} \cdot a = 1 \cdot a$	Mult. prop. of $=$
3. $1 = a$	Ax. of mult. inv.
4. Contradiction	Hyp: $1 < a$

Case 2: $\frac{1}{a} > 1$

1. $\frac{1}{a} > 1, a > 0$	Hypothesis
2. $a \cdot \frac{1}{a} > a \cdot 1$	Mult. prop. of order
3. $1 > a$	Ax. of mult. inv.
4. Contradiction	Hyp: $1 < a$

Therefore, $\frac{1}{a} < 1$

7.

1. $ac > bc$; $c > 0$ $a, b, c \in \mathcal{R}$	Hyp.
2. $ac \cdot \frac{1}{c} > bc \cdot \frac{1}{c}$	Mult. prop. of order
3. $a\left(c \cdot \frac{1}{c}\right) > b\left(c \cdot \frac{1}{c}\right)$	Assoc. ax. for mult.
4. $a \cdot 1 > b \cdot 1$	Ax. of mult. inv.
5. $a > b$	Ax. of 1

9.

1. $0 < a < b$	Hypothesis
2. $\frac{1}{a} > 0$; $\frac{1}{b} > 0$	Theorem, page 54
3. $\frac{1}{a} \cdot \frac{1}{b} > 0$	Closure ax. for $\mathcal{R}_+$
4. $a\left(\frac{1}{a} \cdot \frac{1}{b}\right) < b\left(\frac{1}{a} \cdot \frac{1}{b}\right)$	Mult. prop. of order
5. $a\left(\frac{1}{a} \cdot \frac{1}{b}\right) < b\left(\frac{1}{b} \cdot \frac{1}{a}\right)$	Comm. ax. for mult.
6. $\left(a \cdot \frac{1}{a}\right)\frac{1}{b} < \left(b \cdot \frac{1}{b}\right)\frac{1}{a}$	Assoc. ax. for mult.
7. $1 \cdot \frac{1}{b} < 1 \cdot \frac{1}{a}$	Ax. of mult. inv.
8. $\frac{1}{b} < \frac{1}{a}$	Ax. of 1
9. $\frac{1}{a} > \frac{1}{b}$	Rewriting ineq.

11. Assume $a \not> b$. Then we must show two cases: 1. $a = b$ and 2. $a < b$.

Case 1: Assume $a = b$.

1. $a = b$	Hypothesis
2. $a^2 = ab$, $ab = b^2$	Mult. prop. of $=$
3. $a^2 = b^2$	Trans. prop. of $=$

Case 2: Assume $a < b$.

1. $a < b$; $a > 0$; $b > 0$	Hypothesis
2. $a^2 < ab$; $ab < b^2$	Mult. prop. of order
3. $a^2 < b^2$	Trans. prop. of order

Both assumptions lead to contradictions of hyp: $b^2 > a^2$. Therefore, $a \not> b$ is false, and $a > b$.

13. Assume that $a \not< b$. Then we must consider two cases: (1) $a = b$ and (2) $a > b$.

Case 1: Assume that $a = b$.

1. $a = b$	Hypothesis
2. $a^2 = ab$, $ab = b^2$	Mult. prop. of $=$
3. $a^2 = b^2$	Trans. prop. of $=$

Case 2: Assume that $a > b$.

1. $a > b$; $a, b < 0$	Hypothesis
2. $a^2 < ba$, $ba < b^2$	Mult. prop. of order with $a, b < 0$
3. $a^2 < b^2$	Trans. prop. of order

Both assumptions lead to contradictions of hyp: $a^2 > b^2$. Therefore, the assumption that $a \not< b$ is false, and $a < b$.

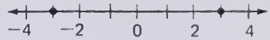
15.

$a, b, c, d > 0$, $ab < cd$, $b > d$	Hypothesis
2. $ab > ad$	Mult. prop. of order
3. $cd > ab$	Rewriting inequality
4. $cd > ad$	Trans. prop. of order
5. $c > a$	Exercise 7
6. $a < c$	Rewriting inequality

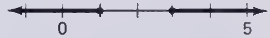
Written Exercises, page 60

1. -2 3. 33 5. 0

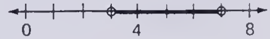
7. $\{-3, 3\}$



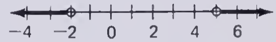
9. $\{b: b \le 1\} \cup \{b: b \ge 3\}$



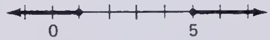
11. $\{y: 3 < y < 7\}$



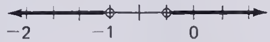
13. $\{n: n < -2\} \cup \{n: n > 5\}$



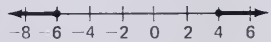
15. $\{x: x \le 1\} \cup \{x: x \ge 5\}$



17. $\{x: x < -1\} \cup \left\{x: x > -\frac{1}{3}\right\}$

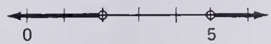


19. $\{a: a \le -6\} \cup \{a: a \ge 4\}$



21. $\left\{y: -25\frac{1}{2} < y < 19\frac{1}{2}\right\}$

23. $\{p: p < 2\} \cup \{p: p > 5\}$

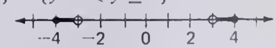


25. False. Let $a = -1$, $b = -1$ 27. True

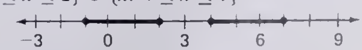
29. False. Let $a = 1$, $b = -1$ 31. False. Let $a = -1$, $b = c = 1$

33. True.

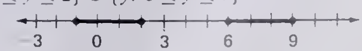
35. $\{y: -4 \le y < -3\} \cup \{y: 3 < y \le 4\}$



37. $\{x: -1 \le x \le 2\} \cup \{x: 4 \le x \le 7\}$



39. $\{y: -1 \le y \le 2\} \cup \{y: 6 \le y \le 9\}$



Chapter Review, page 64

1. b 3. c 5. b 7. b 9. a 11. b 13. d

Chapter 3 Linear Functions and Relations

Written Exercises, page 69

1. 7 3. 3 5. -25 7. $-3c^2 - 4c + 7$

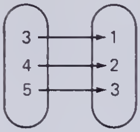
9. $y = 3x$ 11. $y = \frac{x}{3}$ 13. $y = 5x - 1$

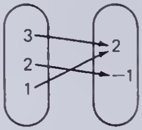
15. $y = x^2$ 17. $y = x^2 + 1$ 19. 9 21. 8

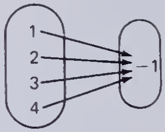
23. 0

Written Exercises, pages 72–73

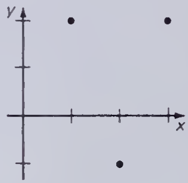
1. {(1, 1), (1, 2), (2, 2), (3, 1)} Not a function
3. {(−5, 4), (0, 6), (5, 2), (10, 6)} A function
5. A function 7. A function 9. A function.

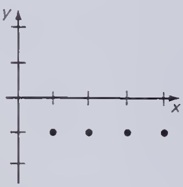


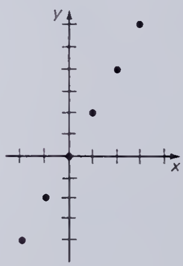


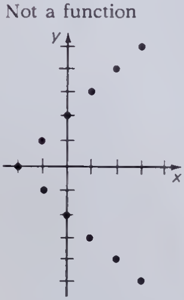


11.


13.


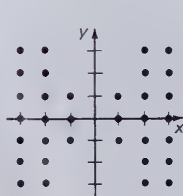
15.


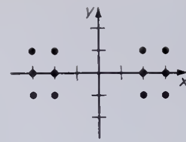
17. A function


19. Not a function


21. A function


23. Not a function


25.


27.


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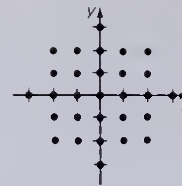
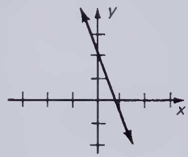
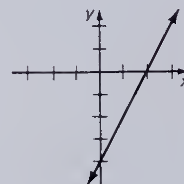


Written Exercises, pages 76–77

1.


3.


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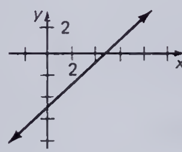

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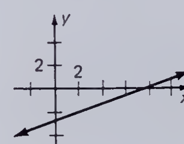

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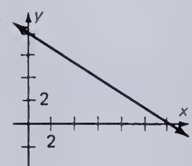

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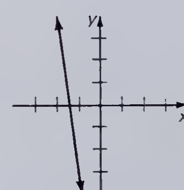

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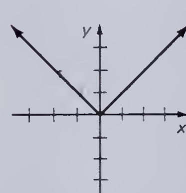

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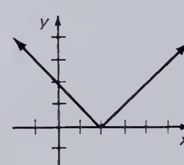
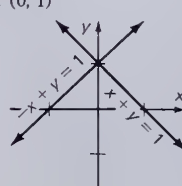
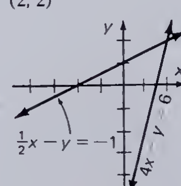
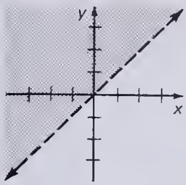


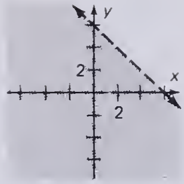
17. -4 19. $-\frac{3}{2}$

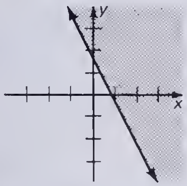
21. (0, 1)


23. (2, 2)


Written Exercises, page 79

1.


3.


5.


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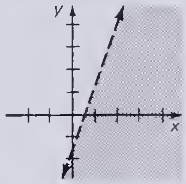

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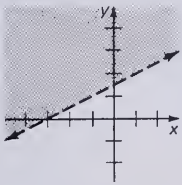

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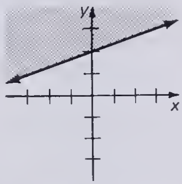

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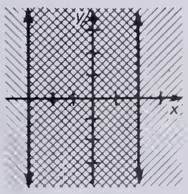

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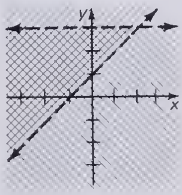

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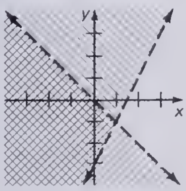
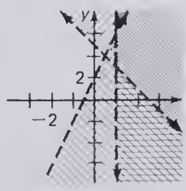
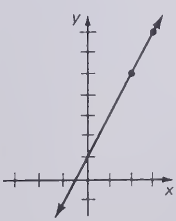
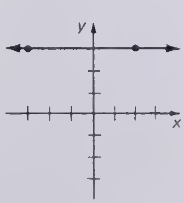
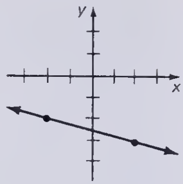


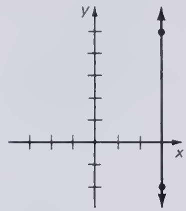
19.


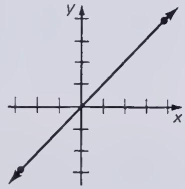
Written Exercises, pages 83–84

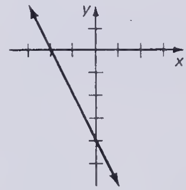
1. $m = 2$


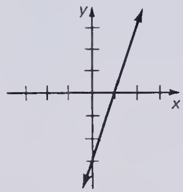
3. $m = 0$


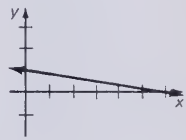
5. $m = -\frac{1}{4}$


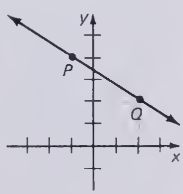
7. Line has no slope.


9. $m = 1$


11. $m = -2$


13. $m = 3$


15. $m = -\frac{1}{6}$


17.


19. $b = \frac{3}{2}$

21. $b = -\frac{2}{3}$

23. $b = 3$

25. $b = 2$

Written Exercises, pages 87–88

1. $y = 5x - 7$ 3. $y = -\frac{1}{2}x + 10$ 5. $y = -\frac{3}{2}$

7. $y = 3x + 5$ 9. $y = -2$ 11. $y = -\frac{5}{2}x + \frac{15}{2}$

13. $y = 9$ 15. $y = 3x + 7$ 17. $y = -\frac{2}{5}x$

19. $y = -1$ 21. $y = \frac{3}{4}x - \frac{11}{4}$ 23. $y = 4x - 1$

25. $y = -7x + 9$ 27. $y = -4x + 5$

29. $y = -\frac{3}{2}x + \frac{9}{2}$ 31. $y = \frac{1}{5}x - 4$

33. $y = -\frac{2}{3}x + 3$ 35. $y = \frac{1}{2}x - \frac{3}{2}$

37. $y = \frac{3}{4}x - 3$ 39. $a = 6$ 41. $k = -2$

Written Exercises, pages 91–92

1. $f: x \to 3x - 1$ 3. $f: x \to -x + 7$ 5. $f: x \to -\frac{3}{2}x$ 7. $f: x \to \frac{2}{3}x + 1$ 9. In direct variation; -2 11. In direct variation; 0 13. In direct variation; $\frac{5}{2}$ 15. Not in direct variation 17. $\frac{5}{2}$ 19. $4\frac{2}{3}$ 21. $\frac{5}{6}$ 23. $6\frac{2}{3}$ 25. $12\frac{1}{2}$ 27. \$2.25 29. $-4\frac{1}{2}$

Problems, page 93

1. 200 hits 3. 18 g of H, 288 g of O 5. 13 times 7. 450 9. 1428 m

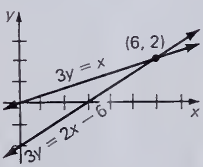
Chapter Review, page 95

1. c 3. a 5. a 7. c 9. c 11. b 13. b 15. c

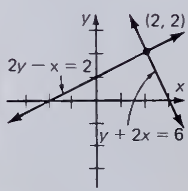
Chapter 4 Systems of Linear Equations or Inequalities

Written Exercises, pages 110–111

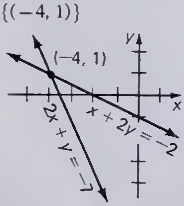
1. Slopes: $\frac{1}{3}, \frac{2}{3}$; $\{(6, 2)\}$



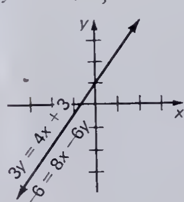
3. Slopes: $-2, \frac{1}{2}$; $\{(2, 2)\}$



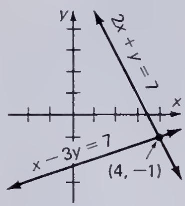
5. Slopes: $-2, -\frac{1}{2}$; $\{(-4, 1)\}$



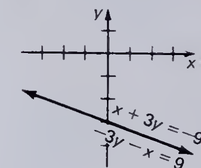
7. Slopes: $\frac{4}{3}, \frac{4}{3}$; y-int: 1; infinite sol. set: $\{(x, y)$: $3y = 4x + 3\}$



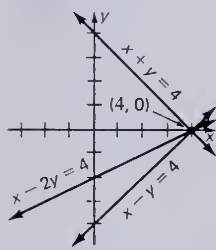
9. Slopes: $\frac{1}{3}, -2$; $\{(4, -1)\}$



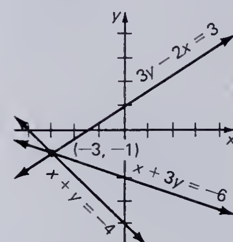
11. Slopes: $-\frac{1}{3}$; y-int: -3; infinite sol. set: $\{(x, y)$: $x + 3y = -9\}$



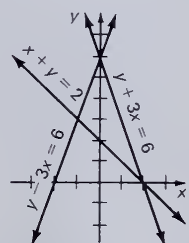
13. $\{(4, 0)\}$



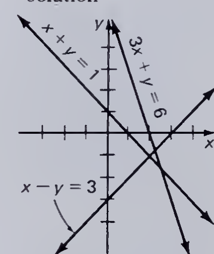
15. $\{(-3, -1)\}$



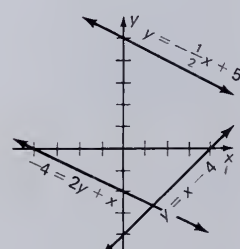
17. No common solution



19. No common solution



21. No common solution



Written Exercises, pages 114–115

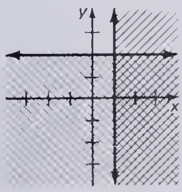
1. $\{(1, -3)\}$ 3. $\left\{\left(\frac{1}{2}, -\frac{1}{4}\right)\right\}$ 5. $\{(2, 0)\}$

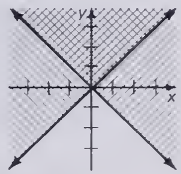
7. $\left\{\left(-\frac{2}{5}, \frac{3}{5}\right)\right\}$ 9. $\left\{\left(-\frac{1}{2}, -5\right)\right\}$ 11. $\{(44, 33)\}$
13. $\{(-1, -1)\}$ 15. $\{(7, 1)\}$ 17. $\{(3, -2)\}$
19. $\{(29, 36)\}$ 21. $\left\{\left(\frac{3}{4}, -\frac{1}{2}\right)\right\}$ 23. $\{(8, 6)\}$
25. $\{(0, 3)\}$ 27. $\{(5, 6)\}$ 29. $x = 1, y = \frac{1}{5}$
31. $x = -\frac{1}{2}, y = \frac{1}{9}$ 33. $A = 2, B = -1$

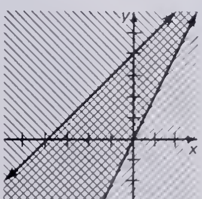
Written Exercises, pages 118–119
1. 8 3. 0 5. $\{3\}$ 7. $\{6, -6\}$
9. $\{(-2, -5)\}$ 11. $\{(37, -97)\}$
13. $\{(1, -2)\}$ 15. $\{(-4, 3)\}$ 17. inconsistent
19. infinite sol. set: $\{(x, y): 8x + 10y = 4\}$
21. $\{(2, -3)\}$

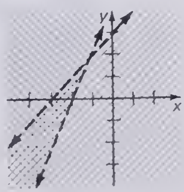
Problems, pages 122–124
1. 13 dimes, 18 quarters 3. rate of boat in still water: 7 km/h; rate of current: 2.5 km/h
5. 5, -1 7. 40°, 50° 9. 54° 11. 560 km/h
13. 160 15. 81 17. $A = 3, B = -5$
19. $A = -1, B = 4$ 21. $V_0 = 25$ m/s; $h_0 = 15$ m 23. 2 km/h (east); 10 km/h
25. 14 quarters, 5 dimes, 8 nickels

Written Exercises, pages 127–128

1.


3.


5.


7.


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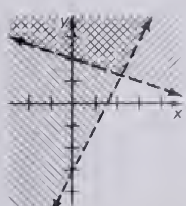

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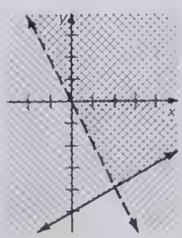

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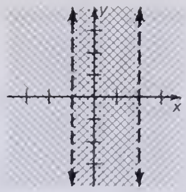

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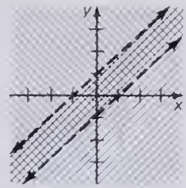

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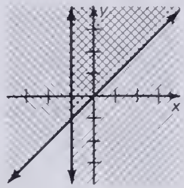

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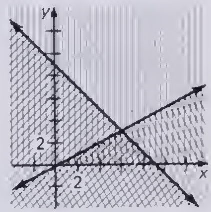

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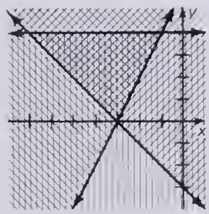

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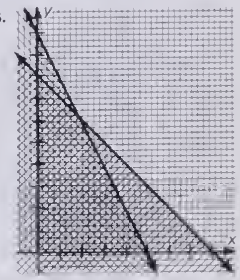

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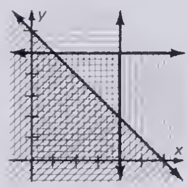

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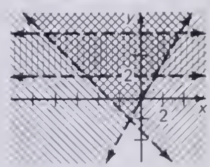

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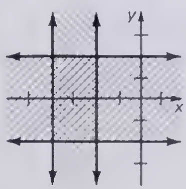

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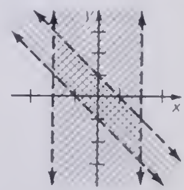

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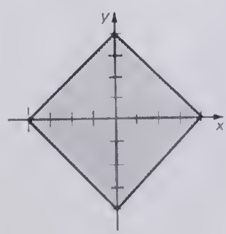
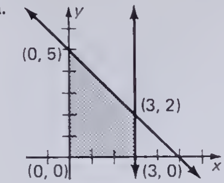


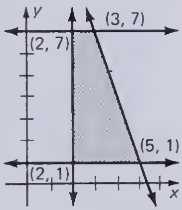
Written Exercises, pages 130–131

1. a.



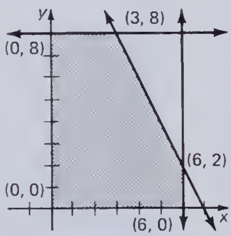
b. (0, 0), (3, 0), (3, 2), (0, 5)
c. 0, 3, 7, 10
d. max: 10; min: 0

3. a.



b. (2, 1), (5, 1), (2, 7), (3, 7)
c. 12, 27, 24, 29
d. max: 29; min: 12

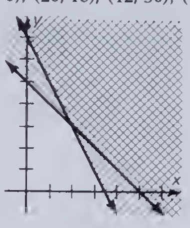
5. a.



b. (0, 0), (0, 8), (3, 8), (6, 2), (6, 0)
c. 0, -16, -7, 14, 18
d. max: 18; min: -16

7. $0 \le x \le 20$, $0 \le y \le 30$, $3x + 2y \le 96$ 9. (0, 0), (20, 0), (20, 18), (12, 30), (0, 30)

11.



13. 5 g of A; 0 g of B
15. $a + bc$

17.

Chapter Review, pages 134–135

1. a 3. b 5. a 7. b 9. c 11. d.

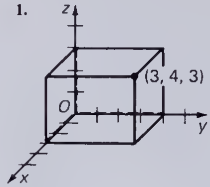
Cumulative Review, pages 136–137

1. c 3. a 5. c 7. a 9. b 11. c 13. c
15. b 17. d 19. c

Chapter 5 Graphs in Space; Determinants

Written Exercises, page 142

1.



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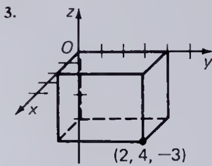
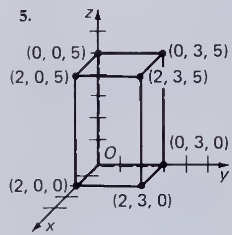


5.



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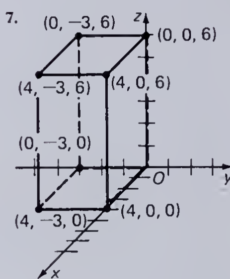

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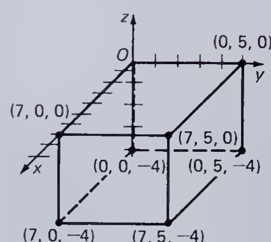
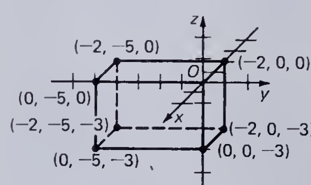


11.



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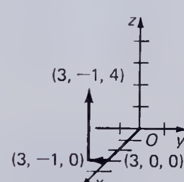

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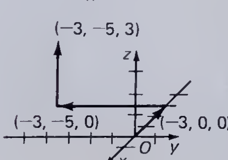

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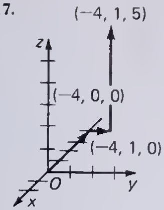

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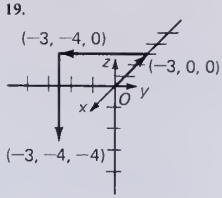
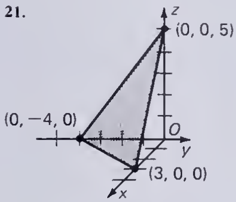


21.



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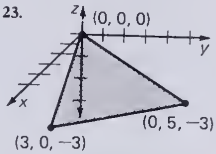
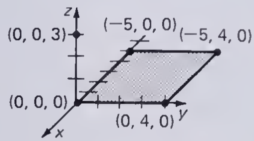
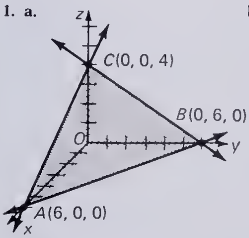


25.



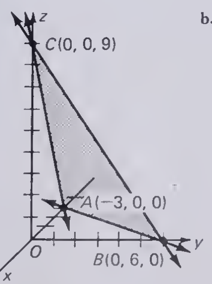
Written Exercises, pages 149–150

1. a.



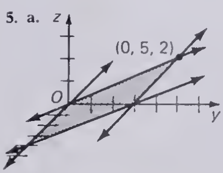
b. $\begin{cases} z = 0 \\ 2x + 2y = 12 \end{cases}$ $\begin{cases} y = 0 \\ 2x + 3z = 12 \end{cases}$ $\begin{cases} x = 0 \\ 2y + 3z = 12 \end{cases}$

3. a.



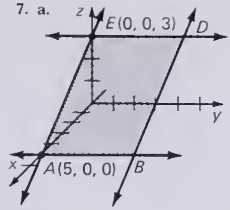
b. $\begin{cases} z = 0 \\ -6x + 3y = 18 \end{cases}$ $\begin{cases} y = 0 \\ -6x + 2z = 18 \end{cases}$ $\begin{cases} x = 0 \\ 3y + 2z = 18 \end{cases}$

5. a.



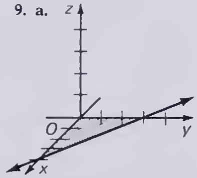
b. $\begin{cases} x = 0 \\ 2y - 5z = 0 \end{cases}$ $\begin{cases} y = 0 \\ z = 0 \end{cases}$
The graph contains the x-axis

7. a.



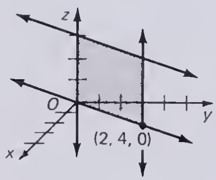
b. $\begin{cases} y = 0 \\ 3x + 5z = 15 \end{cases}$ $\begin{cases} x = 0 \\ z = 3 \end{cases}$ $\begin{cases} z = 0 \\ x = 5 \end{cases}$

9. a.



b. $\begin{cases} z = 0 \\ y = 0 \end{cases}$ $\begin{cases} z = 0 \\ x = 0 \end{cases}$
The graph of $z = 0$ is the xy-plane, so there is no trace in this plane.

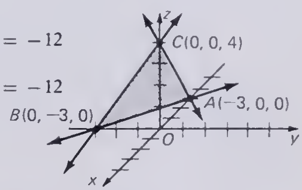
11. a.



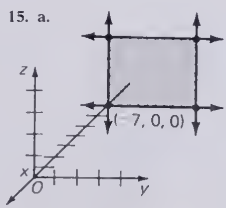
b. $\begin{cases} z = 0 \\ 4x - 2y = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$
The graph contains the z-axis.

13. a. See figure.

b. $\begin{cases} x = 0 \\ 4y - 3z = -12 \end{cases}$ $\begin{cases} y = 0 \\ 4x - 3z = -12 \end{cases}$ $\begin{cases} z = 0 \\ 4x + 4y = -12 \end{cases}$

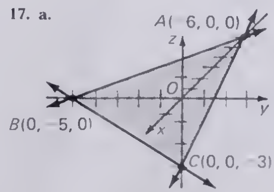


15. a.



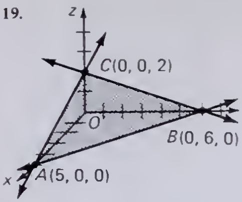
b. There is no trace in the yz-plane.
$\begin{cases} z = 0 \\ 2x = -14 \end{cases}$ $\begin{cases} y = 0 \\ 2x = -14 \end{cases}$

17. a.



b. $\begin{cases} x = 0 \\ -6y - 10z = 30 \end{cases}$ $\begin{cases} y = 0 \\ -5x - 10z = 30 \end{cases}$ $\begin{cases} z = 0 \\ -5x - 6y = 30 \end{cases}$

19.



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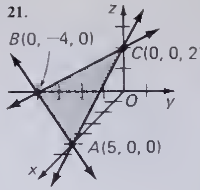
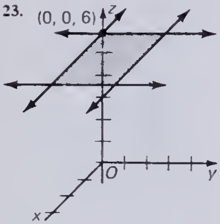
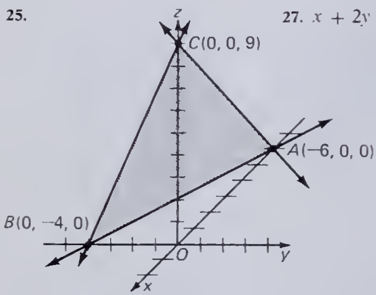


23.

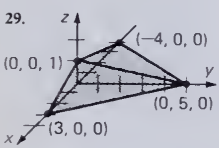


25.



27. $x + 2y = 4$, $z = 0$

29.



$\begin{cases} x = 0 \\ y + 5z = 5 \end{cases}$

Written Exercises, pages 153–154

1. $\{(2, -3, 1)\}$ 3. $\{(-2, 2, 5)\}$ 5. $\left\{\left(\frac{23}{18}, \frac{17}{18}, \frac{1}{9}\right)\right\}$ 7. $\{(5, 1, -1)\}$ 9. $\{(-8, -25, 14)\}$ 11. $\{(2, -2, -2)\}$ 13. $\{(-1, 0, 3)\}$ 15. $\left\{\left(-\frac{42}{17}, \frac{5}{17}, \frac{47}{17}\right)\right\}$ 17. $\{(8, 9, -1)\}$ 19. $\{(0, -3, 9)\}$ 21. $\emptyset$ (no solution) 23. $z = c$

Written Exercises, page 158

1. -27 3. -18 5. $\{(1, 1, 0)\}$ 7. $\{(5, -1, -3)\}$ 9. $\{(-2, 6, 5)\}$ 11. $D = 0$; infinite solution set 13. $D = 0$; inconsistent

Problems, pages 159–161

1. 10 nickels, 15 dimes, 80 quarters 3. 11, 3, 28 5. 80 ones, 25 fives, 15 tens 7. Bob took 10 min, Jim 3 min, and Hal 16 min. 9. *AB*: 15; *BC*: 18; BE: 12 11. 47 pullovers, 46 hooded, and 44 zipper-fronted 13. $a = 20\text{ m/s}^2$, $v_0 = 5\text{ m/s}$, $c_0 = 5\text{ m}$

Written Exercises, page 166

1. 40 3. 0 5. -110 7. $x = 1$, $y = -1$, $z = -2$, $w = 3$

Chapter Review, pages 168–169

1. c 3. b 5. b 7. c 9. c 11. c

Chapter 6 Polynomials and Rational Expressions

Written Exercises, pages 174–175

1. $-15r^6$ 3. $\frac{28x^4}{y^2}$ 5. $\frac{4}{c^2}$ 7. $\frac{-32u^3}{vw^4}$ 9. $-\frac{x^3}{y}$ 11. $\frac{a^2}{125b^6}$ 13. 1 15. $\frac{p^3}{q^5}$ 17. $8a^4b^7$ 19. $\frac{x^2y^2}{x + y}$ 21. $\frac{c^2d^2}{c^2 + d^2}$ 23. pq

Written Exercises, pages 177–178

1. $3x^4 - 12x^2$ 3. $4c^2 + 28cd + 49d^2$ 5. $0.36z^2 - 81$ 7. $n^{10} + 8n^5 + 16$ 9. $9m^6 - 12m^3p^2 + 4p^4$ 11. $x^{2n} - 2yx^n + y^2$ 13. $12x^3 - 7x^2 - x + 6$ 15. $k^3 - 27$ 17. $27b^3 - 27b^2c + 9bc^2 - c^3$ 19. $16x^4 - 8x^2y^2 + y^4$ 21. $c^4 - 8c^3d + 24c^2d^2 - 32cd^3 + 16d^4$ 25. $9b^{-10} - 49$ 27. $r^{10n} - 16r^{-6n}$ 29. $c^{3n} + d^{3n}$ 31. $y^{3n} + 3y^n + 3y^{-n} + y^{-3n}$

Written Exercises, page 184

1. $(2x - 3)^2$ 3. $(2k + 5)(k + 3)$ 5. $(3y - 1)(9y^2 + 3y + 1)$ 7. $(3x - 2y)(x + 4y)$ 9. $(3a + 4b)(2a - 5b)$ 11. $(5t - 2)(2t - 3)$ 13. irreducible 15. $(3x - 4y)(2x + 3y)$ 17. $(3x^2 + 7)(x^2 - 5)$ 19. $(y^3 - 5z^2)^2$ 21. $(b + 3)(b - 3)(b + 2)(b - 2)$ 23. $3x(2xy + 1)(2xy + 1)$ 25. $(x + y)(x - y - 2)$ 27. $(a + 2)(a - 3)$ 29. $(x^n + y^{3n})(x^{2n} - x^ny^{3n} + y^{6n})$ 31. $(a^nb^{3n} - 5)(a^nb^{3n} + 5)$ 33. $(3x^{2y} - 1)(x^{2y} - 3)$ 35. $(3x^2 - xy - y^2)(3x^2 + xy - y^2)$

Written Exercises, page 188

1. $\{-5, 3\}$ 3. $\{-13, 13\}$ 5. $\{6\}$ 7. $\{-4, 4\}$ 9. $\{-2, 0\}$ 11. $\{-2, 6\}$ 13. $\left\{-\frac{3}{5}, \frac{5}{2}\right\}$ 15. $\{7, 8\}$ 17. $\left\{1, 2\frac{1}{4}\right\}$ 19. $\{-7, 4\}$ 21. $\{-4, 0, 4\}$ 23. $\{-3, 3\}$ 25. $\{-5, -2, 2, 5\}$ 27. $x^2 - 15x = 0$ 29. $x^2 + 12x + 32 = 0$ 31. $x^2 + 4x + 4 = 0$ 33. $35x^2 + x - 12 = 0$

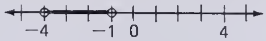
Problems, pages 188–189

1. 9 3. -15 or 24 5. 8 m by 15 m 7. 5 m 9. Either 5 m or $5\frac{1}{3}$ m 11. 10 cm

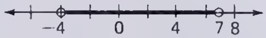
Written Exercises, page 191

1. $\{x: x \leq -3\} \cup \{x: x \geq 2\}$

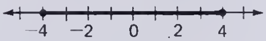
3. $\{a: -4 < a < -1\}$



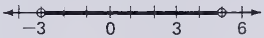
5. $\{x: -4 < x < 7\}$



7. $\{c: -4 \leq c \leq 4\}$

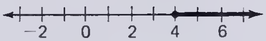


9. $\{x: -3 < x < 5\}$



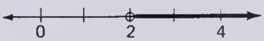
11. $\{k: k \neq 2\}$

13. $\{y: y \geq 4\}$



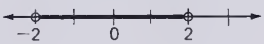
15. $\left\{x: x < \frac{1}{2} \text{ and } x \neq 0\right\}$

17. $\{z: z > 2\}$



19. $\{x: x \leq 0\}$

21. $\{x: -2 < x < 2\}$



Written Exercises, pages 193–194

1. $\dfrac{2}{x-5}$ 3. $\dfrac{3+5b}{2b}$ 5. $\dfrac{1}{2}c^2$ 7. $-\dfrac{3}{2}n^2$

9. $\dfrac{1}{4-r}$ 11. $\dfrac{5}{3cd}$ 13. $\dfrac{p^2+p+1}{2(p+1)}$

15. $\dfrac{a-3}{a(a+3)}$ 17. $\dfrac{9y^2+3y+1}{y+2}$ 19. $\dfrac{1}{t(t+2)}$

21. $(x-y)^2$ 23. 1 25. $(r+3)(r+1)$

27. b^2+2b+4

Written Exercises, pages 196–197

1. $3y^2-5y+2$ 3. $d-cd-1$

5. $x+10-\dfrac{16}{x-5}$ 7. $-4x+3-\dfrac{4}{2x+5}$

9. $3z^2-z+4$ 11. $r^2+2r-3+\dfrac{1}{3r+1}$

13. c^2-4c+6 15. $3x^2-5+\dfrac{2}{4x-1}$

17. $4d^2-10d+25$ 19. $2y^2+3y-1$

21. $5t^2-3t+2-\dfrac{6t+3}{t^2-2}$ 23. $2x^2-yx+3y^2$

25. The quotient will be $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+ab^{n-2}+b^{n-1}$.

Written Exercises, pages 198–199

1. 3 3. $\dfrac{x}{6x+2}$ 5. $\dfrac{7}{2r+10}$ 7. $\dfrac{3}{y-4}$

9. $\dfrac{2x+1}{x}$ 11. $\dfrac{1}{z-2}$ 13. x 15. $\dfrac{a+b}{a-b}$

17. $-(r+s)$ 19. $n+1$

21. $\dfrac{(k^2x^2-1)(k^2x^4+kx^2+1)}{k^3}$

Written Exercises, pages 200–201

1. 3 3. $\dfrac{2(b-a+2)}{a^2b^2}$ 5. 1

7. $\dfrac{c^2+12c-27}{c^2-9}$ 9. $\dfrac{k^2+5k-15}{k-5}$

11. $-\dfrac{3x+4}{x^3+x^2-4x-4}$ 13. $-\dfrac{16}{(x+3)^2(x-1)}$

15. $c+d$ 17. $\dfrac{4a}{a-b}$

19. $-\dfrac{x^2+xy+y^2}{(x+y)(x-y)(x-2y)}$ 21. 1 23. -1

Problems, pages 203–204

1. 2 h 24 min 3. 30 min 5. 12 km 7. 160 g

9. 21 min 11. 32.5 km

Written Exercises, page 206

1. $\left\{\frac{1}{12}\right\}$ 3. $\left\{1\frac{2}{5}\right\}$ 5. $\left\{-\frac{1}{3}, 2\right\}$ 7. $\left\{\frac{3}{2}\right\}$

9. $\left\{-2, 1\frac{2}{5}\right\}$ 11. $\left\{-\frac{1}{4}, 4\right\}$ 13. $\{2, -6\}$

15. $\{-4, 0\}$ 17. $\{-1, 5\}$ 19. $\left\{2\frac{1}{2}\right\}$ 21. $\emptyset$

23. $\{(2, -7)\}$ 25. $\left\{-2\frac{1}{2}\right\}$

Problems, pages 206–207

1. 60 3. 20 Ω 5. 360 m/min 7. 11 h

9. 5 km/h 11. 10 km/h 13. 18 h

Chapter Review, page 210

1. c 3. b 5. b 7. b 9. c 11. d

Chapter 7 Sequences and Series

Written Exercises, pages 215–216

1. $-1, 9, 19$ 3. $-12, -16, -20$ 5. $-\frac{1}{2}, 0, \frac{1}{2}$

7. 3, 9, 27, 81; $a_n = 3^n$ 9. 2, 4, 16, 256; $a_n = 2^{2^{n-1}}$ 11. 4, $4+2k$, $4+4k$, $4+6k$; $a_n = 4+2(n-1)k$; arith. seq. with $d = 2k$ 13. $a_1 = 15$, $a_{n+1} = a_n + 3$, arith. seq. 15. $a_1 = 5$, $a_{n+1} = -2a_n$ 17. $a_1 = -\frac{1}{2}$, $a_{n+1} = \frac{7}{2} + a_n$, arith. seq. 19. $a_1 = a$, $a_{n+1} = a_n b$ 23. Not in arith. prog. 25. Not in arith. prog. 27. $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$; 34, 55, 89

Written Exercises, pages 218–219

1. -21 3. 93 5. 67 7. $-\frac{43}{6}$ 9. 703

11. -4 13. -5 15. -25 17. 16 19. 29

21. 21, 34, 47 23. $12\frac{1}{2}$, 16, $19\frac{1}{2}$ 25. $-\frac{7}{5}$, $-\frac{9}{5}$, $-\frac{11}{5}$, $-\frac{13}{5}$, -3, $-\frac{17}{5}$, $-\frac{19}{5}$, $-\frac{21}{5}$, $-\frac{23}{5}$
27. $d = 2$, $a_1 = -11$ 29. $d = 10$, $a_1 = -112$
31. $d = -\frac{1}{6}$, $a_1 = \frac{1}{2}$ 33. 2 35. 7

Problems, pages 219–220

1. \$735 3. $44\frac{1}{2}$ bricks; 45 rows. 5. \$276.80
7. 14 yr 9. 81 m; 147 m 11. 7 units

Written Exercises, pages 223–224

1. 165 3. 408 5. $15\frac{3}{4}$ 7. 336 9. -474
11. 595 13. 57 15. -4860 17. 1377
19. -544 21. $\sum_{k=1}^{4}(4 + 2k)$ 23. $\sum_{k=1}^{5}(-8 + 5k)$
25. $\frac{4}{15}$ 27. 28 29. -37 31. 99

Problems, pages 224–225

1. \$1000 3. \$19,700 5. \$696 7. 10 d
9. 37 yr

Written Exercises, pages 229–230

1. 3, 9, 27, 81 3. 6, 3, $\frac{3}{2}$, $\frac{3}{4}$ 5. -2, $-\frac{2}{3}$, $-\frac{2}{9}$, $-\frac{2}{27}$ 7. $\frac{25}{4}$, $-\frac{5}{2}$, 1, $-\frac{2}{5}$ 9. $\frac{4}{9}$, $-\frac{4}{3}$, 4, -12 11. -160 13. 24 15. $\frac{1}{1250}$
17. 0.000002 19. arithmetic; -44
21. geometric; $\frac{1}{300}$ 23. arithmetic; 1.36 27. 10

Problems, pages 230–231

1. 256 3. 14,641 5. 80 7. 0.125 mg
9. Common ratio: $\left(1 + \frac{r}{100d}\right)$ 11. \$1221.02

Written Exercises, pages 234–235

1. 2 3. $-\frac{1}{3}$ 5. $\frac{4}{3}$ or $-\frac{4}{3}$ 7. 3, 6 9. $-\frac{3}{4}$, $-\frac{3}{2}$ 11. $\frac{5}{64}$, $\frac{5}{32}$ 13. 15, 75 15. -36, 12
17. 88, 44, 22 or -88, 44, -22 19. $\frac{5}{9}$, $\frac{5}{3}$, 5 or $-\frac{5}{9}$, $\frac{5}{3}$, -5 21. 6 23. -8 25. p^2q^3
27. $\frac{3}{2}$ 29. $-\frac{1}{2}$ or 4

Written Exercises, pages 237–238

1. $\frac{364}{81}$ 3. 1705 5. 1984.375 7. 363
9. 39,062 11. 93 13. $23\frac{5}{8}$ 15. $\frac{43}{40}$
17. $-\frac{105}{32}$ 19. 10 21. 1215 23. 7
25. 0.0625 27. $\frac{2}{3}$ or $-\frac{5}{3}$ 29. $\frac{1}{2}$

Problems, pages 238–239

1. 81.9 m 3. 12,354 5. 480 g 7. 1.24 cm

Written Exercises, pages 243–244

1. 1, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$; $L = 2$ 3. $\frac{1}{100}$, $\frac{1}{25}$, $\frac{9}{100}$, $\frac{4}{25}$; not convergent 5. $-\frac{1}{2}$, $\frac{4}{5}$, $-\frac{9}{10}$, $\frac{16}{17}$; not convergent 7. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; $\frac{1}{n}$ 9. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$; $\frac{1}{2n}$
11. $\frac{3}{20}$, $\frac{1}{12}$, $\frac{3}{52}$, $\frac{3}{68}$; $\frac{3}{4}\left(\frac{1}{4n+1}\right)$ 13. 11, $\frac{1}{11}$
15. 6, $\frac{1}{12}$ 17. 2, $\frac{1}{12}$

Written Exercises, page 248

1. 81 3. $12\frac{1}{4}$ 5. $\frac{9}{10}$ 7. not convergent
9. $68\frac{3}{5}$ 11. 4 13. not convergent 15. 6
17. -20 19. $-\frac{2}{3}$ 21. $\frac{5}{9}$ 23. $\frac{37}{99}$ 25. $\frac{2}{11}$
27. $\frac{20}{333}$ 29. $\frac{2}{3}$ 31. $\frac{3}{4}$ 33. $-\frac{2}{3}$, $\frac{1}{2}$

Problems, page 249

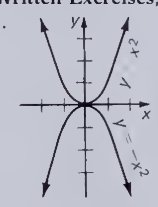
1. 100 cm 3. 4 m 5. 1000 7. 12 cm 9. $\frac{2\sqrt{3}}{5}$

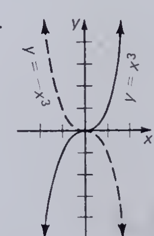
Chapter Review, pages 252–253

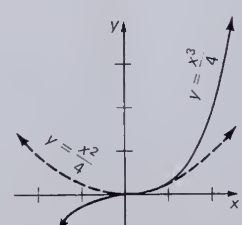
1. c 3. b 5. a 7. d 9. c 11. b 13. c

Chapter 8 Radicals and Irrational Numbers

Written Exercises, page 257

1.


3.


5.


7. $-\frac{8}{9}$ 9. $\frac{3}{4}$
11. $-\frac{1}{3}$ 13. 12
15. -3

17. Value of y is quadrupled. 19. Value of y is multiplied by 32. 21. even

Problems, page 258

1. 67.5 kW·h 3. 0.3125 cm 5. 0.057 g 7. 5 (J/s)/m² 9. 64 h

Written Exercises, pages 260–261

1. 2.6 3. 2.9 5. −3.3 7. −3.6 9. not defined 11. $\{3, -3\}$ 13. $\{2, -2\}$ 15. $\emptyset$ 17. $\left\{\frac{2}{5}\right\}$ 19. $\{7\}$ 21. $\mathcal{R}$ 23. $\{5\}$ 25. $\{5\}$ 27. $\{-3\}$

Written Exercises, pages 263–264

13. $\{-1\}$ 15. $\{1, -2\}$ 17. $\{3, -3\}$ 19. $\left\{1, -1, -\frac{1}{2}\right\}$ 21. $\left\{\frac{1}{2}, -1, -3\right\}$ 23. $\left\{-\frac{1}{2}, 3\right\}$

Written Exercises, page 268

1. 0.3125 3. $0.8\overline{3}$ 5. $0.65\overline{3}$ 7. $0.\overline{243}$ 9. $\frac{13}{40}$ 11. $3\frac{7}{250}$ 13. $\frac{2}{11}$ 15. $\frac{8}{27}$ 17. $1\frac{5}{18}$ 19. $\frac{7}{12}$ 21. $2\frac{37}{999}$ 23. $-1\frac{4}{5}$

Written Exercises, page 270

1. 4.5×10^4 3. 1.5×10^{-4} 5. 7.258×10^3 7. 1.032×10^1 9. 43,000 11. 0.0569 13. 0.808 15. 4,625,000 17. $\frac{(6 \times 10^4)(4 \times 10^{-2})}{8 \times 10^{-3}} = 3 \times 10^5$ 19. $\frac{(4.5 \times 10^5)(2.8 \times 10^3)}{(2 \times 10^{-2})(2.1 \times 10^{-4})} = 3.0 \times 10^{14}$ 21. $\frac{(3.3 \times 10^{-1})(2.8 \times 10^{-5})(2 \times 10^3)}{8.8 \times 10^{-2}} = 2.1 \times 10^{-1}$ 23. 3×10^{-4} 25. 1×10^6 27. 6, 0, 9, 1; 6.122, 0.7080, 9.244, 1.262; subtraction

Written Exercises, pages 274–275

1. $\sqrt{3}$ 3. 1.225 5. 1.291 (In Exs. 7, 8, 9, and 15 other ans. are possible.) 7. a. 1.45 b. 1.414114111 . . . 9. a. 0.67 b. 0.676776777 . . . 11. a. 1.414 b. 1.414114111 . . . 13. $\frac{bc + 1}{bd}$ 15. $\sqrt{\frac{41}{48}}\left(= \frac{1}{4}\sqrt{\frac{41}{3}}\right)$

Written Exercises, pages 278–279

1. $6\sqrt{3}$ 3. $\frac{7}{3}\sqrt{2}$ 5. −9 7. $-\frac{1}{32}$ 9. $\frac{1}{3}\sqrt{6}$ 11. $\frac{1}{2}\sqrt{10}$ 13. 52.92 15. 9.45 17. 27.95 19. $\frac{1}{2}\sqrt{7} = 1.32$ 21. $\frac{a^2}{b^2}\sqrt{b}$ 23. $2c\sqrt[5]{c^2}$ 25. $\frac{y\sqrt{y-3}}{y-3}$ 27. $\frac{\sqrt{9b^2 - a^2}}{3ab}$ 29. $\frac{\sqrt[3]{xy}}{x-y}$ 31. $\frac{a}{a+b}$

Problems, pages 279–280

1. $2\sqrt{30} \approx 10.95$ A 3. $\frac{10^4}{90}\sqrt{7} \approx 294$ Hz 5. $\sqrt[3]{63} \approx 3.98$ cm 7. $\sqrt[3]{\frac{88}{21}} \approx 1.61$ 9. $\frac{\sqrt{6}}{36} \approx 0.07$

Written Exercises, pages 281–282

1. $2\sqrt{3}$ 3. $7\sqrt{5}$ 5. $2\sqrt[3]{2}$ 7. $8a\sqrt{a}$ 9. $\sqrt{x}(9x^2\sqrt{2} + 11)$ 11. $-60\sqrt{2}$ 13. $\frac{8}{5}\sqrt{2}$ 15. $8 - 2\sqrt{15}$ 17. $31 + 4\sqrt{21}$ 19. $2 - 4\sqrt[3]{2}$ 21. $3\sqrt{5} + 6$ 23. $\frac{23 - 10\sqrt{2}}{-47}$ 25. 1 27. $a + b$ 29. $a - \sqrt[3]{ab^2}$ 31. $\frac{3\sqrt{x-1} - 6}{x-5}$ 33. $(x\sqrt{2} - y)(x\sqrt{2} + y)$ 35. $(x + \sqrt{3})^2$ 37. $(b - \sqrt[3]{7})(b^2 + b\sqrt[3]{7} + \sqrt[3]{7^2})$ 41. $(x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$

Written Exercises, page 284

1. $\{79\}$ 3. $\left\{\frac{4}{3}\right\}$ 5. $\{33\}$ 7. $\left\{\frac{27}{2}\right\}$ 9. $\{3\}$ 11. $\{48\}$ 13. $\{5, -5\}$ 15. $\left\{\frac{1}{2}, \frac{3}{2}\right\}$ 17. $\{9\}$ 19. $\emptyset$ 21. $\{3\}$ 23. $\{3, -1\}$ 25. $\{2\}$ 27. $\{4\}$ 29. $\left\{4, \frac{1}{4}\right\}$ 31. $\{8\}$

Written Exercises, pages 287–288

1. $\{3 + 2\sqrt{2}, 3 - 2\sqrt{2}\}$ 3. $\left\{-\frac{3}{2} + \frac{\sqrt{21}}{2}, -\frac{3}{2} - \frac{\sqrt{21}}{2}\right\}$ 5. $\emptyset$ 7. $\left\{\frac{3}{2} + \frac{\sqrt{11}}{2}, \frac{3}{2} - \frac{\sqrt{11}}{2}\right\}$ 9. $\left\{\frac{5}{4} + \frac{\sqrt{13}}{4}, \frac{5}{4} - \frac{\sqrt{13}}{4}\right\}$ 11. $\left\{\frac{3}{2} + \frac{\sqrt{15}}{2}, \frac{3}{2} - \frac{\sqrt{15}}{2}\right\}$ 13. $\left\{-\frac{5}{8} + \frac{\sqrt{29}}{8}, -\frac{5}{8} - \frac{\sqrt{29}}{8}\right\}$ 15. $\left\{\frac{3}{4} + \frac{\sqrt{10}}{4}, \frac{3}{4} - \frac{\sqrt{10}}{4}\right\}$ 17. $\emptyset$ 19. $\left\{-1 + \frac{\sqrt{15}}{3}, -1 - \frac{\sqrt{15}}{3}\right\}$ 21. $\left\{-\frac{3}{10} + \frac{\sqrt{13}}{10}, -\frac{3}{10} - \frac{\sqrt{13}}{10}\right\}$ 23. $\left\{\frac{5}{2}, -2\right\}$

25. $\left\{\frac{4}{3}, -1\right\}$ 27. $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$
29. $\{\sqrt{1 + \sqrt{3}}, -\sqrt{1 + \sqrt{3}}\}$
31. $(x^2 + 2x + 2)(x^2 - 2x + 2)$

Problems, page 288

1. 10 cm 3. $\frac{3\sqrt{26}}{5} - \frac{3}{5} \approx 2.46$ cm
5. $84 + 48\sqrt{3} \approx 167.14$ cm^2, $112 + 64\sqrt{3} \approx 222.85$ cm^2 7. width: $60 + 10\sqrt{66} \approx 141$ m; length: $140 + 10\sqrt{66} \approx 221$ m 9. $3 + \frac{\sqrt{6}}{2} \approx 4.22$ m and $3 - \frac{\sqrt{6}}{2} \approx 1.78$ m

Chapter Review, page 291

1. c 3. a 5. c 7. b 9. d 11. c 13. d

Cumulative Review, pages 292–293

1. c 3. b 5. b 7. d 9. a 11. b 13. c

Written Exercises, page 307

1. -4, two imaginary roots
3. 9, two real rational roots
5. 76, two real irrational roots
7. -144, two imaginary roots
9. 0, one real rational double root
11. $\left\{-\frac{1}{5} + \frac{2}{5}i, -\frac{1}{5} - \frac{2}{5}i\right\}$
13. $\left\{\frac{3}{2} + \frac{\sqrt{21}}{6}, \frac{3}{2} - \frac{\sqrt{21}}{6}\right\}$ 15. $\left\{\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right\}$
17. $\left\{\frac{\sqrt{2}}{3}\right\}$ 19. $\frac{4}{3}$ 21. 8, -4

Written Exercises, page 310

1. $x^2 + 2x - 15 = 0$ 3. $x^2 - 5x + \frac{25}{4} = 0$
5. $x^2 - 48 = 0$ 7. $x^2 - 8x + 14 = 0$
9. $x^2 - 4x + 40 = 0$ 11. $x^2 + (i - 3)x + 4 = 0$
13. -2 15. 17 17. $\frac{1}{9}$ 19. -1 21. -5

Chapter 9 Complex Numbers and Polynomial Functions

Written Exercises, pages 298–299

1. $-i$ 3. i 5. $6i\sqrt{2}$ 7. $20i\sqrt{3}$ 9. $\frac{\sqrt{2}}{3}i$
11. $\frac{-\sqrt{6}}{4}i$ 13. $-4\sqrt{15}$ 15. 2 17. $-\frac{4\sqrt{5}}{5}i$
19. $-i$ 21. -5 23. 4 25. $9i\sqrt{2}$ 27. $12i\sqrt{3}$
29. 0 31. $\frac{2\sqrt{6} - \sqrt{30}}{6}i$ 33. $\frac{6\sqrt{3} - 2\sqrt{6}}{9}i$

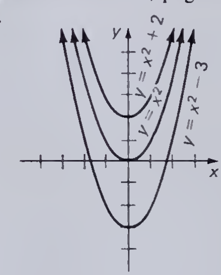
Written Exercises, page 301

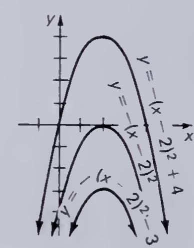
1. $-2 - 5i$ 3. $3 - 16i$ 5. $9 - 11i$ 7. $\frac{1}{2} + 2i$
9. $\frac{10}{9} - \frac{1}{4}i$ 11. $2 + \frac{3}{4}i$ 13. 10 15. i
17. $\frac{1}{18} + \frac{3}{2}i$ 19. $-7\frac{1}{6} - 4\frac{1}{2}i$

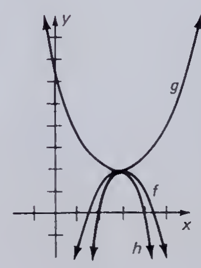
Written Exercises, page 304

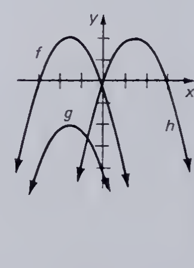
1. $19 - 17i$ 3. $-7 - 24i$ 5. 1 7. $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
9. $\frac{12}{5} + \frac{4}{5}i$ 11. $\frac{24}{29} + \frac{27}{29}i$ 13. $\frac{41}{25} + \frac{52}{25}i$
15. -1 17. $(3y + 5i)(3y - 5i)$
19. $4(4a + bi\sqrt{3})(4a - bi\sqrt{3})$
21. $(z\sqrt{3} + 2i\sqrt{2})(z\sqrt{3} - 2i\sqrt{2})$
25. $\left(x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(x + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \cdot \left(x - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\left(x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

Written Exercises, pages 314–315

1.


3.


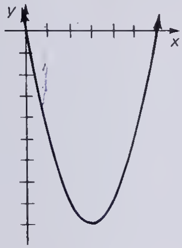
5.


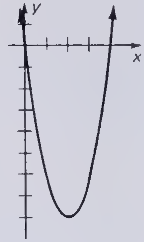
7.


9. $y = 2(x - 2)^2 + 1$ 11. $y = -\frac{1}{4}(x - 6)^2 + 1$
13. $y = (x + 3)^2 + 7$ 15. $y = \frac{2}{3}x^2 - 2$
17. $y = (x - 1)^2 + 5$ or $y = (x - 5)^2 + 5$
19. $y = 3(x - 2)^2 + 5$ 21. $y = 2x^2 - 7$

Written Exercises, pages 317–318

1. axis: $x = 3$, vertex: $(3, -9)$
3. axis: $x = 2$, vertex: $(2, -8)$





5. axis: $x = 0$, vertex: $(0, -4)$
7. axis: $x = 4$, vertex: $(4, -1)$
9. axis: $x = 1$, vertex: $(1, -3)$
11. axis: $x = 2$, vertex: $(2, 3)$
13. $y = -x^2 + 1$ 15. $y = x^2 + 2x - 5$
17. $y = r(x - 0)^2 - rs$, axis: $x = 0$, vertex: $(0, -rs)$ 19. $y = r(x - s)^2 - s^2(r + 2)$, axis: $x = s$, vertex: $(s, -s^2(r + 2))$

Problems, pages 318–319

1. 8 and 8 3. 12 cm 5. 39.6 m
7. 22 stories 9. at the 20 cm point
11. a. $y = -\frac{3}{5}x + 6$
b. $A = x\left(-\frac{3}{5}x + 6\right) = 6x - \frac{3}{5}x^2$ c. $x = 5$

Written Exercises, page 321

1. a. $x^2 - 9 > 0$ b. $\{3, -3\}$
d. $\{x: x < -3\} \cup \{x: x > 3\}$
3. a. $3x - x^2 < 0$ b. $\{0, 3\}$
d. $\{x: x < 0\} \cup \{x: x > 3\}$
5. a. $x^2 - x - 6 \geq 0$ b. $\{3, -2\}$
d. $\{x: x \leq -2\} \cup \{x: x \geq 3\}$
7. a. $x^2 - 4x + 2 \leq 0$ b. $\{2 + \sqrt{2}, 2 - \sqrt{2}\}$
d. $\{x: 2 - \sqrt{2} \leq x \leq 2 + \sqrt{2}\}$
9. a. $x^2 + 2x - 2 > 0$ b. $\{-1 + \sqrt{3}, -1 - \sqrt{3}\}$
d. $\{x: x < -1 - \sqrt{3}\} \cup \{x: x > \sqrt{3} - 1\}$
11. $x^2 - 4x < 0$ and $x^2 - 4x + 3 \geq 0$
$\{x: 0 \leq x \leq 1\} \cup \{x: 3 \leq x \leq 4\}$

Written Exercises, page 324

1. 0, a zero 3. 0, a zero 5. -30 7. -3
9. 53 11. 30 13. 0, a zero 15. $12 + 66i$
17. -16 19. 32 21. -3 23. -4
25. $a = 2$, $b = 1$
27. $P(x) = [(ax + b)x + c]x + d$,
$P(0) = [(a \cdot 0 + b)0 + c]0 + d = d$

Written Exercises, pages 329–330

1. a. $P(x) = (x - 2)(2x^2 - x - 5) - 8$
b. $P(x) = (x - 3)(2x^2 + x) + 2$
3. a. $P(x) = (x + 2)(4x^2 - 12x + 29) - 66$
b. $P(x) = \left(x - \frac{1}{2}\right)(4x^2 - 2x + 4) - 6$
5. a. $\frac{P(x)}{x + 2} = 3x^2 - 13x + 6 - \frac{9}{x + 2}$
b. $\frac{P(x)}{x - 4} = 3x^2 + 5x + \frac{3}{x - 4}$
7. a. $\frac{P(x)}{x + 3} = x^3 - 4x^2 + 12x - 40 + \frac{104}{x + 3}$
b. $\frac{P(x)}{x - 2i} = x^3 + (-1 + 2i)x^2 + (-4 - 2i)x - 8i$
9. $x + 3$ is a factor of $P(x)$ 11. No; $R = -6$
13. No; $R = -2$ 15. $r_2 = -2, r_3 = 4$
17. $r_2 = \frac{3}{2}, r_3 = 1$ 19. $r_2 = 1, r_3 = 3$
21. $r_2 = 3, r_3 = -1, r_4 = -2$ 23. $\{3, 1, -3\}$
25. $\{3, 2 + \sqrt{6}, 2 - \sqrt{6}\}$ 27. $\{-\frac{1}{2}, -2, 4\}$
29. 7 31. -5

Written Exercises, page 333

1. $x^3 - 3x^2 - 4x + 12$ 3. $x^3 - 5x^2 + 4x - 20$
5. $x^3 - 4x^2 - 2x + 20$ 7. $\{1 + 2i\}$;
$(x + 3)(x - (1 - 2i))(x - (1 + 2i))$ 9. $\{-3i, -2\}$;
$(x - 3i)(x + 3i)(x + 2)$ 11. $\{2 - i, -3\}$;
$(x - (2 + i))(x - (2 - i))(x + 3)$
13. $\{1 - i, i\sqrt{5}, -i\sqrt{5}\}$;
$(x - (1 + i))(x - (1 - i))(x - i\sqrt{5})(x + i\sqrt{5})$

Written Exercises, page 336

1. $-2, 0, 2$ 3. 0 5. 2 7. $-3, -1, 1, 3$
9. $-2, \frac{3}{2}, \frac{5}{2}$ 11. 1.77 13. 2.65 15. For example, $P(x) = 40x^3 - 74x^2 + 45x - 9$

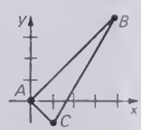
Chapter Review, pages 338–339

1. b 3. d 5. c 7. d 9. d 11. a
13. b 15. b

Chapter 10 Quadratic Relations and Systems

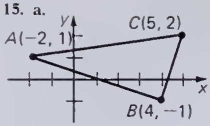
Written Exercises, page 344

1. 5 3. 5 5. $\sqrt{5}$ 7. $\frac{5}{4}$ 9. $|a - b|\sqrt{2}$
13. a.



b. isosceles: $m(\overline{AB}) = m(\overline{BC}) = 5$
c. not right

15. a.



b. not isosceles
c. right: $m(\overline{BC}) = \sqrt{10}$, $m(\overline{AB}) = 2\sqrt{10}$, $m(\overline{AC}) = 5\sqrt{2}$, $m(BC)^2 + m(AB)^2 = m(AC)^2$

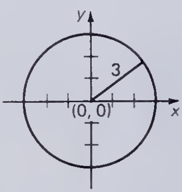
17. $(1, 3)$ 19. $(-3, 2)$ 21. $5\sqrt{5}$
27. $3x - 2y = 1$

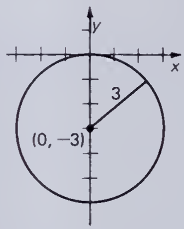
Written Exercises, pages 346–347

1. $y + 2x = 7$ 3. $2x + 5y = 35$
5. $4x + y = -6$ 7. $3x + 8y = 24$
9. $x - 8y = -6$ 11. $5x + 3y = 18$

Written Exercises, pages 349–350

1. $x^2 + y^2 - 49 = 0$
3. $x^2 + y^2 + 4x - 10y + 13 = 0$
5. $x^2 + y^2 - x + 3y - \frac{3}{2} = 0$
7. $x^2 + y^2 - 2ax - 2by + b^2 = 0$
9. $(x - 0)^2 + (y - 0)^2 = 3^2$ 11. $(x - 0)^2 + (y - (-3))^2 = 3^2$

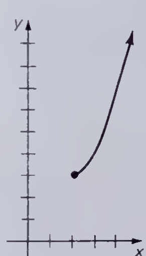




13. $(x - (-1))^2 + (y - 2)^2 = 3^2$
15. $\left(x - \left(-\frac{1}{2}\right)\right)^2 + \left(y - \frac{3}{2}\right)^2 = 2^2$
17. $x^2 + y^2 - 4x + 2y - 20 = 0$
19. $x^2 + y^2 - 2kx = 0$
21. $(x - 3)^2 + (y - 3)^2 = 4$
23. $(x - 2)^2 + (y - 4)^2 = 25$
25. $(x - 3)^2 + (y + 2)^2 = 9$ 29. $C(0, 0)$, $r = 4$

Written Exercises, pages 352–353

1. $y = \frac{1}{4}(x - 0)^2 - 1$
3. $x = (y - 3)^2 - 9$
5. $y = \frac{1}{2}(x - 3)^2 - \frac{9}{2}$
7. $y = 2(x - 3)^2 - 4$
9. $y = -\frac{1}{2}(x - 4)^2 - 1$
11. $y = (x - 2)^2 + 3$, $x \geq 2$

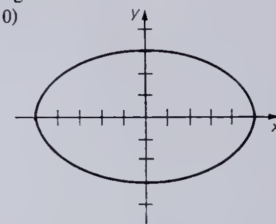


Ex. 11

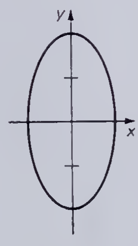
13. $y = \frac{x^2}{8} + 1$ 15. $x = \frac{1}{8}y^2 - \frac{3}{4}y + \frac{9}{8}$
17. $y = \frac{1}{2}x^2 - 2x + 3$ 19. $y = x^2 - x$

Written Exercises, page 356

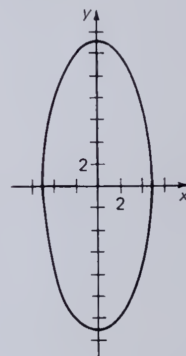
1. Foci: $(4, 0)$, $(-4, 0)$



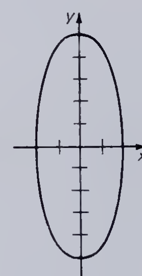
3. Foci: $(0, -\sqrt{3})$, $(0, \sqrt{3})$
5. Foci: $(0, -12)$, $(0, 12)$



Ex. 3



Ex. 5

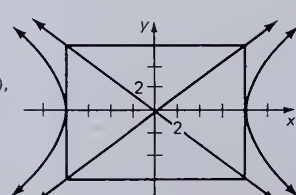


Ex. 7

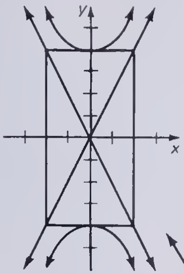
7. Foci: $(0, -\sqrt{21})$, $(0, \sqrt{21})$
9. Foci: $(-10\sqrt{2}, 0)$, $(10\sqrt{2}, 0)$
11. Foci: $\left(0, -\frac{3}{2}\right)$, $\left(0, \frac{3}{2}\right)$
13. $\frac{x^2}{100} + \frac{y^2}{36} = 1$ 15. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
17. $\frac{x^2}{25} + \frac{y^2}{36} = 1$ 19. $\frac{x^2}{36} + \frac{y^2}{20} = 1$

Written Exercises, pages 361–362

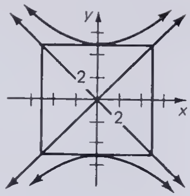
1. Foci: $(-10, 0)$, $(10, 0)$



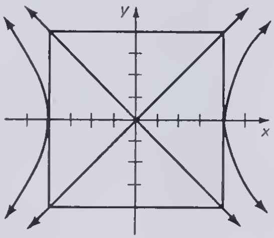
3. Foci: $(0, -2\sqrt{5})$, $(0, 2\sqrt{5})$



5. Foci: $(0, 5\sqrt{2})$, $(0, -5\sqrt{2})$



7. Foci: $(4\sqrt{2}, 0)$, $(-4\sqrt{2}, 0)$



9. Foci: $\left(\frac{5}{4}, 0\right), \left(-\frac{5}{4}, 0\right)$

11. Foci: $(5, 0)$, $(-5, 0)$

13. $\frac{y^2}{16} - \frac{x^2}{9} = 1$ 15. $\frac{y^2}{64} - \frac{x^2}{25} = 1$

17. $\frac{y^2}{32} - \frac{x^2}{32} = 1$ 19. $\frac{x^2}{64} - \frac{y^2}{36} = 1$

21. $\frac{y^2}{4} - \frac{x^2}{12} = 1$

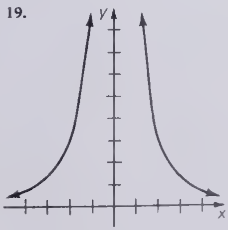
Written Exercises, pages 365–366

1. $y = \frac{18}{x}$ 3. $y = \frac{9}{5x^2}$ 5. $y = -\frac{15}{4\sqrt{x}}$

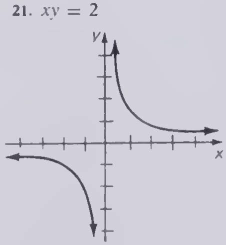
7. $x_2 = \frac{3}{2}$ 9. $y_2 = \frac{1}{6}$ 11. $y_2 = \frac{2}{9}$ 13. $z = \frac{25}{2}$

15. $z = \frac{32}{5}$ 17. $z = \frac{2}{5}$

19.



21. $xy = 2$



Problems, pages 366–367

1. 15,500 V 3. 0.01 cm 5. 120 V

7. 5.292×10^{24} kg

Written Exercises, pages 369–370

1. $\{(-1, -3), (4, 12)\}$ 3. $\{(4, 3), (9, -2)\}$

5. $\{(2, 2), (2, -2), (-2, -2), (-2, 2)\}$

7. $\{(4, 0), (-4, 0)\}$

9. $\{(0, -13), (5, 12), (-5, 12)\}$

11. $\{(4, 0), (-4, 0), (5, 3), (-5, 3)\}$

13. $\{(1.5, 1.5), (1.5, -1.5), (-1.5, 1.5), (-1.5, -1.5)\}$

15. $\{(-2, 0), (3, 4.5), (3, -4.5)\}$

Written Exercises, page 372

1. $\{(1, 5), (4, 20)\}$ 3. $\{(-\sqrt{3}, -\sqrt{3})\}$

5. $\{(\sqrt{2}i, -3\sqrt{2}i), (-\sqrt{2}i, 3\sqrt{2}i)\}$

7. $\{(0, 2), (-1, 0)\}$ 9. $\left\{(4, 1), \left(-\frac{40}{11}, -\frac{17}{11}\right)\right\}$

11. $\{(-2, 1), (1, 7)\}$ 13. $\left\{\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right\}$

Problems, page 373

1. 16 cm $\times$ 3 cm 3. (8, 0) and (0, 15) or (0, 8) and (15, 0) 5. Outside: 6 m $\times$ 6 m; inside: 4 m $\times$ 4 m 7. 9 and 6 9. $y = 2(x - 2)^2 + 5$ or $y = 2(x + 1)^2 - 1$

Written Exercises, page 375

1. $\{(3, 1), (-3, 1), (3, -1), (-3, -1)\}$

3. $\{(0, 4), (0, -4)\}$

5. $\{(\sqrt{17}, i\sqrt{15}), (-\sqrt{17}, i\sqrt{15}), (\sqrt{17}, -i\sqrt{15}), (-\sqrt{17}, -i\sqrt{15})\}$ 7. $\{(3, 0), (-3, 0)\}$

9. $\left\{\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{1}{2}, -\frac{3}{2}\right), \left(-\frac{1}{2}, \frac{3}{2}\right), \left(-\frac{1}{2}, -\frac{3}{2}\right)\right\}$

11. $\{(-1, -2), (1, -2), (2, 1), (-2, 1)\}$

13. $\left\{(4, i), (4, -i), \left(-\frac{9}{2}, \frac{\sqrt{21}}{2}i\right), \left(-\frac{9}{2}, -\frac{\sqrt{21}}{2}i\right)\right\}$

15. $\{(2, 5), (-2, -5), (-5, -2), (5, 2)\}$

Problems, page 376

1. 3 m $\times$ 4 m 3. $\{(2, 2\sqrt{3}), (2, -2\sqrt{3}), (-2, 2\sqrt{3}), (-2, -2\sqrt{3})\}$ 5. 5 cm $\times$ 12 cm

Chapter Review, pages 378–379

1. c 3. c 5. b 7. a 9. a 11. d 13. c 15. c

Chapter 11 Exponents and Logarithms

Written Exercises, pages 383–384

1. 2 3. $\frac{1}{5}$ 5. 343 7. $\frac{1}{16}$ 9. 27 11. 5

13. $x\sqrt{x}$ 15. $\sqrt{2n}$ 17. $\sqrt{2}$ 19. $2\sqrt{2}$

21. $\sqrt[4]{5}$ 23. $\frac{4}{9}$ 25. 4 27. 3 29. $\sqrt[3]{2}$ 31. $\frac{1}{3}$

33. $\sqrt[3]{5}$ 35. $\sqrt[3]{49}$ 37. $\sqrt[3]{5}$ 39. $\{-8, 8\}$

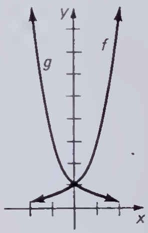
41. $\{16\}$ 43. $\{124, -126\}$ 45. $\{1, 4\}$

47. $\{-1, 1, -27, 27\}$ 49. $\left\{-\frac{1}{32}, \frac{1}{32}, -1, 1\right\}$

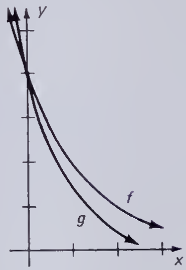
Written Exercises, pages 386–387

1. $2^{\sqrt{2}+\sqrt{3}}$ 3. $2^{3\sqrt{2}}$ 5. $2^{\frac{\sqrt{2}}{2}}$ 7. $2^{\sqrt{2}+2\sqrt{3}}$
9. $2^{2-\frac{\sqrt{3}}{3}}$ 11. $\{4\}$ 13. $\{-2\}$ 15. $\{2\}$
17. $\left\{\frac{1}{3}\right\}$

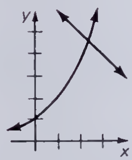
19.



21.

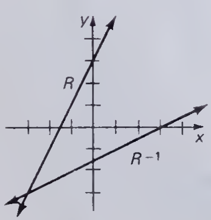


23. $\{(2, 5)\}$

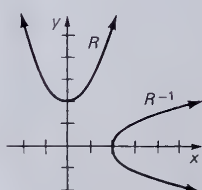


Written Exercises, page 390

1. $f^{-1}(x) = \frac{1}{3}x + 1$; $f^{-1}(f(x)) = \frac{1}{3}(3x - 3) + 1 = x$;
$f(f^{-1}(x)) = 3\left(\frac{1}{3}x + 1\right) - 3 = x$
3. $f^{-1}(x) = -2x + 10$;
$f^{-1}(f(x)) = -2\left(-\frac{1}{2}x + 5\right) + 10 = x$;
$f(f^{-1}(x)) = -\frac{1}{2}(-2x + 10) + 5 = x$
5. $f^{-1}(x) = \sqrt[3]{x}$; $f^{-1}(f(x)) = \sqrt[3]{x^3} = x$;
$f(f^{-1}(x)) = (\sqrt[3]{x})^3 = x$
7. $R^{-1} = \left\{(x, y)\text{: } y = \frac{x}{2} - \frac{3}{2}\right\}$, a function
9. $R^{-1} = \{(x, y)\text{: } y = \pm\sqrt{x - 2},\ x \geq 2\}$, not a function

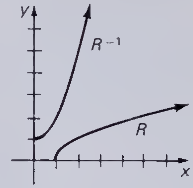


Ex. 7

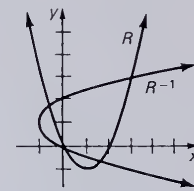


Ex. 9

11. $R^{-1} = \{(x, y)\text{: } y = x^2 + 1,\ x \geq 0\}$, a function
13. $R^{-1} = \{(x, y)\text{: } y = 1 \pm \sqrt{1 + x},\ x \geq -1\}$, not a function

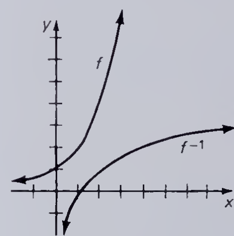


Ex. 11



Ex. 13

15. $f^{-1}(1) = 0$;
$f^{-1}(2) = 1$;
$f^{-1}(4) = 2$;
$f^{-1}(8) = 3$;
$f^{-1}\left(\frac{1}{2}\right) = -1$



Written Exercises, pages 392–393

1. 4 3. -3 5. $\frac{1}{2}$ 7. 3 9. 6 11. $\frac{2}{3}$
13. $\{4\}$ 15. $\{49\}$ 17. $\{81\}$ 19. $\left\{\frac{\sqrt{3}}{3}\right\}$
21. $\{25\}$ 23. $\{125\}$ 25. 10 27. 4 29. $\left\{\frac{3}{4}\right\}$
31. $\{3\}$ 33. $\left\{\frac{5}{3}\right\}$ 35. $\{36\}$ 37. $\{0\}$

Written Exercises, pages 396–397

1. 1.079181 3. 2.602060 5. 0.669007
7. 4.176091 9. 1.58496 11. 2.58496 13. 3
15. $\left\{\frac{1}{2}\right\}$ 17. $\{200\}$ 19. $\{6\}$ 21. $\{5\}$
23. $\{9\}$ 25. $\{10\}$

Written Exercises, page 399

1. 1.4314 3. 9.8028 − 10 5. 7.5798 − 10
7. 5.6911 − 10 9. 1.9154 11. 7.8451 − 10
13. 2.2788 15. 143 17. 1.26 19. 115,000
21. 70.2 23. 0.000637 25. 53,500
27. 4.1931 29. 3.1139 31. $\{18\}$
33. $\{0.06, -0.06\}$ 35. $\{x\text{: } x > 2\} \cup \{x\text{: } 0 < x < 1\}$

Written Exercises, page 402

1. 0.1623 3. 2.7296 5. 8.7730 − 10
7. 4.8147 9. 3.7783 11. 5.0817 13. 14.52
15. 0.09228 17. 321.5 19. 0.007668
21. 0.5634 23. 0.01572 25. 0.0588; 0.0586; interpolating between log 1.14 and log 1.15 produces the greater answer.

Written Exercises, page 405

1. 1300 3. -223 5. 18 7. -0.0270
9. 0.0291 11. 49.4 13. -5.73 15. 14.6
17. 112.1 19. a. $\log 4 = 2 \log 2 \approx 0.6020$
b. $\log 8 = 3 \log 2 \approx 0.9030$
c. $\log 12 = 2 \log 2 + \log 3 \approx 1.0791$
d. $\log \frac{1}{2} = -\log 2 \approx -0.3010$
e. $\log 5 = \log 10 - \log 2 \approx 0.6990$
f. $\log 15 = \log 10 - \log 2 + \log 3 \approx 1.1761$
21. $\log 7 \approx \frac{1}{2}(4 \log 2 + \log 3) \approx 0.8406$;
$\log 11 \approx \frac{1}{2}(\log 3 + 2 \log 2 + \log 10) \approx 1.0396$

Written Exercises, pages 408–409

1. 200 3. 3.65 5. 25.7 7. 40.2 9. 0.676
11. 1.53 13. 3540 15. 0.00400 17. 0.746
19. 1.08 21. 423 23. 0.870 25. $\{100\}$
27. $\{10{,}000\}$ 29. $\left\{\frac{1}{10}\right\}$ 31. $\{75\}$ 33. $\left\{\frac{2\sqrt{3}}{5}\right\}$

Problems, pages 409–410

1. \$35,240 3. 517,000 bacteria 5. 12,490

Written Exercises, pages 413–414

1. $\{5.76\}$ 3. $\{0.367\}$ 5. $\{8.71\}$ 7. $\{5.00\}$
9. $\{1.05\}$ 11. $\{1.17\}$ 13. $\{4.88\}$ 15. $\{0.314\}$
17. 1.95 19. 2.31 21. $\{4.55\}$ 23. $\{1.15\}$
25. $\{-0.735\}$

Problems, page 414

1. 14 yr 3. 9 min 5. 4%

Chapter Review, pages 418–419

1. b 3. c 5. d 7. b 9. a 11. b 13. c
15. a

Chapter 12 Permutations, Combinations, and Probability

Written Exercises, page 423–424

1. 25 3. 360 5. 18 7. 56 9. 649,350
11. 107 13. 45

Written Exercises, pages 427–428

1. 40,320 3. 362,880 5. 3024 7. 11,880
9. 362,880 11. 360 13. 240 15. 720

Written Exercises, page 429

1. 180 3. 1260 5. 2,494,800 7. 39,916,800
9. 360 11. 302,400 13. 12 15. a. 120 b. 240
c. 360

Written Exercises, pages 432–433

1. 330 3. 45 5. 210 7. 120 9. 27
11. 140 13. 31

Written Exercises, pages 434–435

1. 17,640 3. 11,760 5. 10 7. 150 9. 24
11. 1320 13. 1024 15. 1584 17. 123,552
19. 3720 21. 1320; 60

Written Exercises, page 438

1. $a^4 + 12a^3 + 54a^2 + 108a + 81$
3. $243 - 405r + 270r^2 - 90r^3 + 15r^4 - r^5$
5. $32c^5 + 40c^4 + 20c^3 + 5c^2 + \frac{5}{8}c + \frac{1}{32}$
7. $64 - 96b + 60b^2 - 20b^3 + \frac{15b^4}{4} - \frac{3}{8}b^5 + \frac{b^6}{64}$
9. $x^{16} - 8x^{14} + 28x^{12} - 56x^{10} + 70x^8 - 56x^6 + 28x^4 - 8x^2 + 1$ 11. $a^{12} + 12a^{10}b + 60a^8b^2 + 160a^6b^3 + 240a^4b^4 + 192a^2b^5 + 64b^6$ 13. $120a^7b^3$
15. $324p^7q^2$ 17. $924c^6d^6$
19. $x + \frac{1}{2}x^{-1}y - \frac{1}{8}x^{-3}y^2$ 21. $27 - \frac{9}{2}r + \frac{1}{8}r^2$

Written Exercises, page 440

1. $c^5 + 5c^4 + 10c^3 + 10c^2 + 5c + 1$
3. $128x^7 + 448x^6y + 672x^5y^2 + 560x^4y^3 + 280x^3y^4 + 84x^2y^5 + 14xy^6 + y^7$
5. $1 - 6y^2 + 15y^4 - 20y^6 + 15y^8 - 6y^{10} + y^{12}$
7. $r^{16} - 8r^{14}t^2 + 28r^{12}t^4 - 56r^{10}t^6 + 70r^8t^8 - 56r^6t^{10} + 28r^4t^{12} - 8r^2t^{14} + t^{16}$ 9. $84m^3$
11. $7p^3q^5$ 13. $-8064x^5$ 17. $f(n) = 2^{n-1}$. The sum of the entries in the eighth row is $2^7 = 128$.

Written Exercises, page 443

1. a. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ b. $\{3, 6, 9\}$
3. a. {HH, HT, TH, TT} b. {HH, HT, TH}
5. a. $\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$
b. $\{(1, 3), (3, 1)\}$ 7. a. {{A, B}, {A, C}, {A, E}, {B, C}, {B, E}, {C, E}} b. {{A, B}, {A, C}, {A, E}, {B, E}, {C, E}} 9. 52; 13
11. 1326; 325 13. 1326; 78 15. 2,598,960; 22,308

Written Exercises, pages 445–447

1. a. $\frac{3}{10}$ b. $\frac{7}{10}$ c. $\frac{2}{5}$ d. $\frac{2}{5}$ 3. a. $\frac{1}{6}$ b. $\frac{1}{6}$
c. $\frac{5}{6}$ d. $\frac{2}{9}$ 5. a. $\frac{1}{22}$ b. $\frac{5}{33}$ c. $\frac{1}{11}$ d. $\frac{6}{11}$
e. $\frac{7}{22}$ f. $\frac{14}{33}$ 7. a. $\frac{1}{16}$ b. $\frac{1}{4}$ c. $\frac{3}{8}$ d. $\frac{1}{4}$
e. $\frac{15}{16}$ 9. a. $\frac{4}{7}$ b. $\frac{1}{7}$ c. $\frac{2}{7}$ d. $\frac{11}{21}$ 11. a. $\frac{1}{3}$
b. $\frac{2}{15}$ c. $\frac{4}{15}$ d. $\frac{13}{15}$ 13. a. $\frac{5}{18}$ b. $\frac{1}{6}$
15. $\frac{1}{35}$

Written Exercises, pages 448–449

1. a. $\frac{7}{64}$ b. $\frac{11}{32}$ c. $\frac{21}{32}$ d. $\frac{1}{64}$ e. 1
3. $\frac{7}{128}$ 5. a. $\frac{3}{5}$ b. $\frac{4}{5}$ 7. $\frac{3}{50}$ 9. $\frac{24}{35}$

Written Exercises, pages 452–453

1. 0.045 3. a. $\frac{1}{24}$ b. $\frac{1}{6}$ c. $\frac{5}{8}$ d. $\frac{1}{2}$

5. a. $\frac{1}{3757}$ b. $\frac{11}{48{,}841}$ c. $\frac{1}{289}$ d. $\frac{241}{10{,}404}$

7. $P(A) = \frac{5}{36}$; $P(B) = \frac{11}{36}$; $P(A/B) = \frac{2}{11}$;
$P(B/A) = \frac{2}{5}$ 9. a. $\frac{14}{33}$ b. $\frac{35}{99}$ c. $\frac{1}{55}$
d. $\frac{4}{165}$ 11. a. $P(A) \cdot P(B/A) = P(A \cap B) = 0$
b. They are not independent since
$P(A \cap B) \neq P(A) \cdot P(B)$ 13. No. 15. No.

Chapter Review, pages 456–457

1. d 3. a 5. a 7. b 9. d 11. a 13. b
15. c

Cumulative Review, pages 458–459

1. c 3. a 5. b 7. a 9. c 11. b 13. c
15. b 17. d 19. b 21. b 23. c 25. a

Chapter 13 Matrices

Written Exercises, pages 464–465

1. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ 3. $\begin{bmatrix} 8 & -2 \\ 4 & 6 \end{bmatrix}$
5. $[5 \quad 2 \quad 3]$
7. $w = -3, x = -3, y = 6, z = 2$ 9. $x = -5$, $y = -7$ 11. $w = 2, x = -3, y = 1$
13. $x = -3, y = 1, w = -2$ 15. $|A| = -13$, $|B| = 18$, $|A + B| = -19$

Written Exercises, pages 467–468

1. $\left\{\begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix}\right\}$ 3. $\left\{\begin{bmatrix} 5 & 0 \\ -1 & -7 \end{bmatrix}\right\}$
5. $\left\{\begin{bmatrix} 2 & -2 \\ -8 & 9 \end{bmatrix}\right\}$ 7. $\left\{\begin{bmatrix} -5 & 3 & -2 \\ -6 & -7 & 5 \end{bmatrix}\right\}$
9. $X = [a - c \quad b - d]$

Written Exercises, pages 470–471

1. $\begin{bmatrix} 9 & -27 \\ 63 & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 13 & -29 \\ 41 & 10 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & -3 \\ 7 & 0 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 7 \\ -43 & 10 \end{bmatrix}$ 9. $\begin{bmatrix} -12 & 6 \\ 66 & -30 \end{bmatrix}$ 11. $\begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}$
13. $\begin{bmatrix} -2 & 16 \\ -20 & 4 \end{bmatrix}$ 15. $\begin{bmatrix} -8 & 34 \\ -50 & 16 \end{bmatrix}$ 17. $\begin{bmatrix} 4 & 7 \\ 1 & -8 \end{bmatrix}$

Written Exercises, pages 475–476

1. $\begin{bmatrix} -18 \\ 10 \end{bmatrix}$ 3. $\begin{bmatrix} -11 \\ -9 \\ -10 \end{bmatrix}$ 5. $\begin{bmatrix} -9 & -16 \\ -15 & -22 \end{bmatrix}$
7. $\begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$ 9. $\begin{bmatrix} 15 & -13 \\ -1 & -5 \end{bmatrix}$ 11. $\begin{bmatrix} 1 & -16 \\ 8 & -7 \end{bmatrix}$
13. $\begin{bmatrix} 23 & -42 \\ -6 & 11 \end{bmatrix}$ 15. $x = 3, y = -1$
17. $x = 0, y = 4$ 19. $x = 3, y = 1, z = -1$

21. $AB = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$
23. $\begin{bmatrix} 890 & 965 & 1040 \\ 120 & 125 & 130 \end{bmatrix}$

The entries in the top row represent the number of jewels used in producing both models on Monday, Tuesday and Wednesday, respectively. The entries in the second row represent the number of straps used in producing both models.

Written Exercises, page 479

1. $AB \neq BA$ 3. $AD \neq DA$ 5. $CD = DC$
7. $A(B + C) = AB + AC$ 9. $A(BC) \neq (BC)A$
11. $C(A + B) \neq (A + B)C$
13. $(A + B)^2 = A^2 + AB + BA + B^2$
15. $(A + B)(C + D) = AC + BC + AD + BD$

Written Exercises, pages 485–486

1. $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ 3. A is singular. 5. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
7. $\begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{bmatrix}$ 9. $\begin{bmatrix} -\frac{3}{2} & -2 \\ \frac{5}{2} & 3 \end{bmatrix}$ 11. $\begin{bmatrix} -7 & -17 \\ 5 & 12 \end{bmatrix}$
13. $\begin{bmatrix} 12 & 17 \\ -5 & -7 \end{bmatrix}$ 15. $\begin{bmatrix} -2 & -\frac{3}{2} \\ 1 & 1 \end{bmatrix}$ 17. $\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$
19. $\{(3, -1)\}$ 21. $\{(-2, 1)\}$ 23. $\left\{\left(\frac{1}{2}, 0\right)\right\}$

Written Exercises, pages 490–491

1. $(-1, 3)$ 3. $(-6, 3)$ 5. $(0, 2b)$
7. $(2, -2)$ 9. $(-6, -1)$ 11. $(-a, -b)$
13. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ 15. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$
17. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c - a \\ d - b \end{bmatrix}$ 19. $(13, 1)$
21. $(6, 16)$ 23. $(6, 1)$ 25. $(2, -7)$
27. $(a + h_1 + h_2, b + k_1 + k_2)$ 29. Yes

Written Exercises, pages 495–496

1. $(3, 6)$ 3. $(6, -3)$ 5. $(1, 2)$ 7. $(-1, -4)$
9. $(-19, 12)$ 11. $(-3, 4)$ 13. $x' = x + 3y$, $y' = 2x + 6y$, $m = 2$ 15. $x' = -6x - 10y$, $y' = 3x + 5y$, $m = -\frac{1}{2}$ 17. Square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$ 19. Square with vertices $(0, 0)$, $(0, -1)$, $(-1, -1)$, $(-1, 0)$
21. $(x', y') = (2x, 2y)$, an expansion by a factor of 2
23. $(x', y') = (-y, -x)$ a reflection in the line $y = -x$
25. $(x', y') = (x, 0)$, a projection onto the x-axis
27. $(x', y') = (0, 2x)$, a projection onto the y-axis such that the ordinate of each image point is twice that of the abscissa of the preimage.

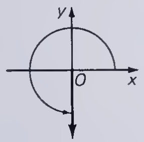
Chapter Review, pages 498–499

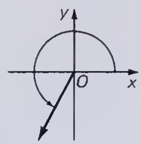
1. b 3. b 5. a 7. b 9. b 11. a

Chapter 14 Trigonometric and Circular Functions

Written Exercises, page 504

1. 94.2 cm 3. 193.424 m 5. 20 cm 7. 1.88 m

9.


11.


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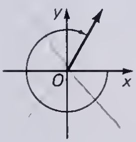


15. $\frac{1}{4}$ revolution clockwise

17. $\frac{1}{3}$ revolution clockwise

19. $\frac{1}{6}$ revolution counterclockwise

21. 10.99 cm 23. 7326.67 km/h 25. 3

Written Exercises, page 508

1. $\frac{2\pi^R}{3}$ 3. $\frac{7\pi^R}{6}$ 5. $\frac{5\pi^R}{3}$ 7. $\frac{5\pi^R}{6}$ 9. $-\frac{4\pi^R}{5}$

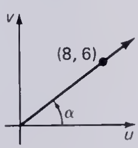
11. 270° 13. −45° 15. 330° 17. 315°

19. 75° 21. 132 cm 23. 27.5 cm

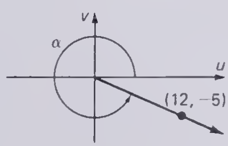
25. 38.5 cm 27. 9 29. 3 31. $\frac{5}{22}$

Written Exercises, pages 513–514

1. $\sin \alpha = \frac{3}{5}$, $\cos \alpha = \frac{4}{5}$



3. $\sin \alpha = -\frac{5}{13}$, $\cos \alpha = \frac{12}{13}$



5. $\sin \alpha = 1$, $\cos \alpha = 0$

7. $\sin \alpha = \frac{2}{\sqrt{5}}$, $\cos \alpha = \frac{1}{\sqrt{5}}$

9. $\sin \alpha = -\frac{4}{5}$, $\cos \alpha = -\frac{3}{5}$

11. $\cos \alpha = \frac{12}{13}$ 13. $\sin \alpha = \frac{15}{17}$

15. $\sin \alpha = \frac{-2\sqrt{2}}{3}$ 17. $\cos \alpha = \frac{-1}{2}$

19. $\sin \alpha = -\frac{1}{\sqrt{5}}$, $\cos \alpha = -\frac{2}{\sqrt{5}}$

21. $\sin \alpha = \frac{1}{\sqrt{10}}$, $\cos \alpha = -\frac{3}{\sqrt{10}}$

23. $\sin \alpha = \frac{1}{2}$, $\cos \alpha = \frac{\sqrt{3}}{2}$

Written Exercises, pages 517–518

1. $-\frac{1}{\sqrt{2}}$ 3. $-\frac{1}{\sqrt{2}}$ 5. $\frac{1}{\sqrt{2}}$ 7. $\frac{1}{2}$ 9. $-\frac{\sqrt{3}}{2}$

11. −1 13. a. $30° + k \cdot 360°$ b. $\frac{\pi}{6} + 2k\pi$

15. a. $240° + k \cdot 360°$ b. $\alpha = \frac{4\pi}{3} + 2k\pi$

17. a. $210° + k \cdot 360°$ b. $\frac{7\pi}{6} + 2k\pi$

19. $225° + k \cdot 360°$ b. $\frac{5\pi}{4} + 2k\pi$

Written Exercises, page 520

1. 0.8809 3. 0.4914 5. 0.6435 7. 0.5255

9. 0.8709 11. 0.1413 13. 30°16′ 15. 14°46′

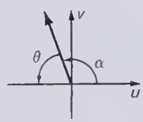
17. 61°38′ 19. 0.622 21. 0.308 23. 0.167

25. 53°10′ 27. 62° 29. 70°30′ 31. 16°40′

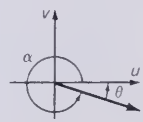
33. 71°34′

Written Exercises, page 523

1. $\sin 110° = 0.9397$



3. $\cos 345° = 0.9659$



5. $\cos 207° = -0.8910$ 7. $\sin 291°10' = -0.9325$

9. $\cos 4.26^R = -0.4357$ 11. $\cos 1.93^R = -0.3530$

13. 0.6604 15. −0.7173 17. −0.7431

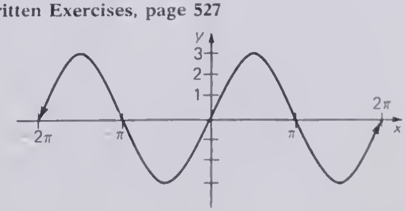
19. 0.9336 21. −0.1045 23. −0.8192

25. −0.9359 27. 0.7956 29. 197° 31. 251°20′

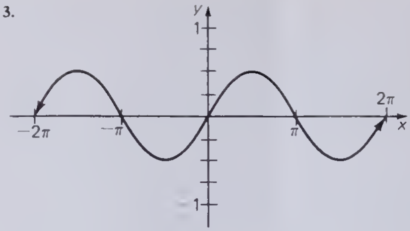
33. 160°30′ 35. 121°20′ 37. 233°10′ 39. 118°

41. 131°50′ 43. 327°40′ 45. 5.64^R 47. 4.51^R

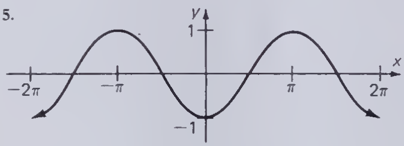
Written Exercises, page 527

1.


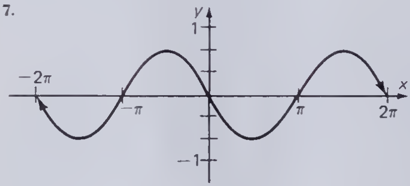
3.



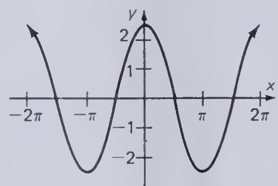
5.



7.



9.

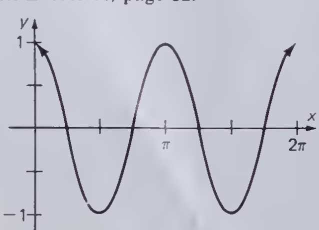


11. maximum: 1, minimum: -5, $|A| = 3$

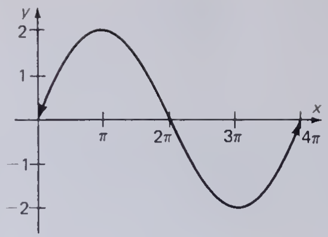
13. maximum: 5, minimum: -3, $|A| = 4$

Written Exercises, page 529

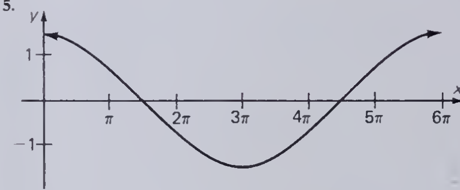
1.



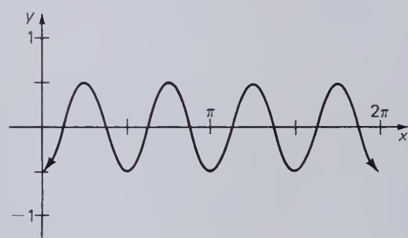
3.



5.



7.



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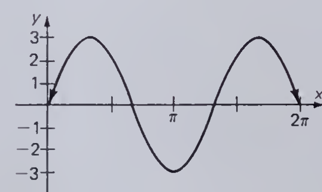

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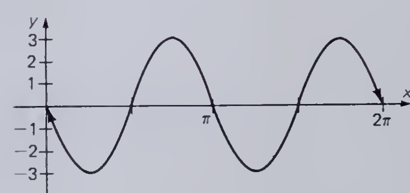
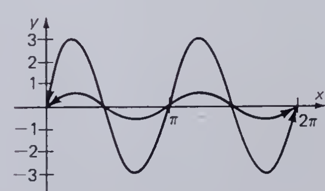
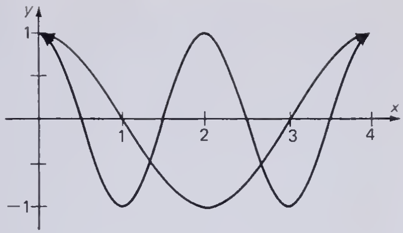


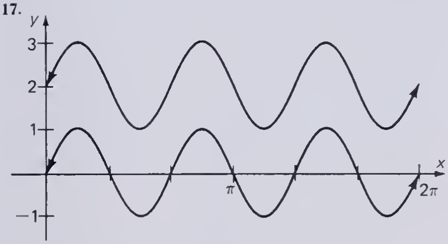
13.



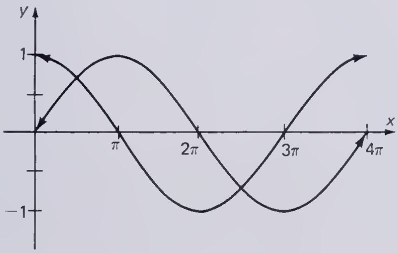
15.



17.



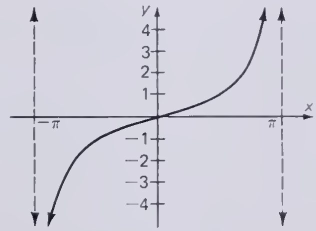
19.



Written Exercises, pages 533–534

1. $\tan 127° = -1.3270$ 3. $\csc 209°10' = -2.052$
5. $\cot 256°48' = 0.2345$ 7. $\sec 2.52^R = -1.229$
9. -0.4831 11. 0.8609
13. $18°40'$ 15. $50°50'$ 17. $84°17'$ 19. 0.70^R
21. 0.573^R 23. 0.516^R 25. $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\tan\alpha = \frac{3}{4}$, $\csc\alpha = \frac{5}{3}$, $\sec\alpha = \frac{5}{4}$, $\cot\alpha = \frac{4}{3}$
27. $\sin\alpha = -1$, $\cos\alpha = 0$, $\tan\alpha$ undefined, $\csc\alpha = -1$, $\sec\alpha$ undefined, $\cot\alpha = 0$
29. $\sin\alpha = 0$, $\cos\alpha = -1$, $\tan\alpha = 0$, $\csc\alpha$ undefined, $\sec\alpha = -1$, $\cot\alpha$ undefined
31. $\sin\alpha = \frac{2\sqrt{6}}{5}$, $\cos\alpha = \frac{1}{5}$, $\tan\alpha = 2\sqrt{6}$, $\csc\alpha = \frac{5}{2\sqrt{6}}$, $\sec\alpha = 5$, $\cot\alpha = \frac{1}{2\sqrt{6}}$
33. $\cos\alpha = \frac{\sqrt{3}}{2}$, $\tan\alpha = -\frac{1}{\sqrt{3}}$, $\csc\alpha = -2$, $\sec\alpha = \frac{2}{\sqrt{3}}$, $\cot\alpha = -\sqrt{3}$
35. $\cos\alpha = -\frac{\sqrt{5}}{3}$, $\tan\alpha = -\frac{2}{\sqrt{5}}$, $\csc\alpha = \frac{3}{2}$, $\sec\alpha = -\frac{3}{\sqrt{5}}$, $\cot\alpha = -\frac{\sqrt{5}}{2}$
37. $\cos\alpha = -\frac{3}{4}$, $\tan\alpha = -\frac{\sqrt{7}}{3}$, $\sin\alpha = \frac{\sqrt{7}}{4}$, $\sec\alpha = -\frac{4}{3}$, $\cot\alpha = -\frac{3}{\sqrt{7}}$
39. $\sin\alpha = -\frac{2}{\sqrt{29}}$, $\cos\alpha = -\frac{5}{\sqrt{29}}$, $\cot\alpha = \frac{5}{2}$, $\csc\alpha = -\frac{\sqrt{29}}{2}$, $\sec\alpha = -\frac{\sqrt{29}}{5}$

43.



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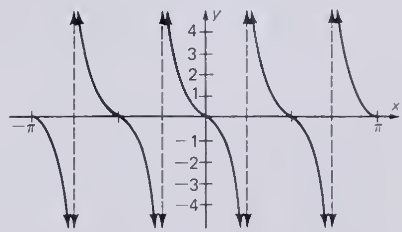
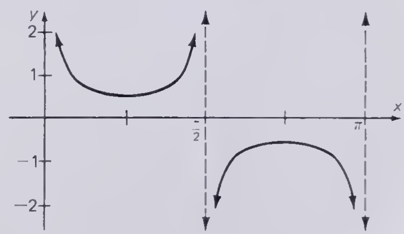


47.



Written Exercises, page 539

1. $m(A) = 65°10'$, $a \approx 36.3$, $b \approx 16.8$
3. $m(B) = 59°10'$, $a \approx 35.8$, $c \approx 69.9$
5. $m(B) = 31°20'$, $b \approx 39.0$, $c \approx 74.9$
7. $m(A) = 19°20'$, $b \approx 71.3$, $c \approx 75.5$
9. $m(B) = 26°$, $a \approx 67.4$, $b \approx 32.9$
11. $m(A) \approx 58°$, $m(B) \approx 32°$, $c \approx 28.3$
13. $m(A) \approx 46°20'$, $m(B) \approx 43°40'$, $a \approx 21$
15. $m(A) \approx 30°30'$, $m(B) \approx 59°30'$, $c \approx 65$
17. $m(A) \approx 12°40'$, $m(B) \approx 77°20'$, $c \approx 410$
19. $m(A) \approx 45°10'$, $m(B) \approx 44°50'$, $a \approx 19.7$

Problems, pages 539–540

1. $\alpha = 55°30'$, $x = 9.62$ m 3. $x = 20.5$ m
5. $3°40'$ 7. 62.1 m 9. 4.2 km 11. 10.2 m

Chapter Review, pages 542–543

1. a 3. b 5. b 7. c 9. d 11. a 13. b

Chapter 15 Trigonometric Identities and Formulas

Written Exercises, pages 547–548

1. $1 - \sin^2 \alpha$, $\sin \alpha \neq 0$ 3. $\frac{1}{\sin^2 x}$, $\sin x \neq 0$, $\cos x \neq 0$ 5. $\pm 2\sqrt{1 - \sin x^2}$, $\sin x \neq 0$ 7. $\sin^2 \alpha$, $\cos \alpha \neq 0$ 9. $\frac{1}{\sin^2 x - 1}$, $\cos x \neq 0$, $\cos x \neq 1$, $\cos x \neq -1$ 11. $\sqrt{1 - \cos^2 \alpha}$ if α lies in Quad. I or II; $-\sqrt{1 - \cos^2 \alpha}$ if α lies in Quad. III or IV, $\cos \alpha \neq 0$, $\sin \alpha \neq 0$ 13. $\frac{1}{1 - \cos^2 x}$, $\sin x \neq 0$
15. $\frac{1}{\sqrt{1 - \cos^2 x}}$ if x lies in Quad. I or II; $-\frac{1}{\sqrt{1 - \cos^2 x}}$ if x lies in Quad. III or IV, $\sin x \neq 0$
17. $\frac{1}{1 - \cos^2 \alpha}$, $\cos \alpha \neq 0$, $\sin \alpha \neq 0$ 19. $\tan^2 \alpha$, $\cos \alpha \neq 0$ 21. $\cot x$, $\sin x \neq 0$, $\cos x \neq 0$
23. $\tan^2 x$, $\cos x \neq 0$ 25. $\tan x$, $\cos x \neq 0$
27. $\frac{1}{\sin x \cos x}$, $\sin x \neq 0$, $\cos x \neq 0$
29. $\frac{1}{\sin^2 x \cos^2 x}$, $\sin x \neq 0$, $\cos x \neq 0$
31. $\frac{1}{\sin^2 x \cos^2 x}$, $\sin x \neq 0$, $\cos x \neq 0$
33. $-\frac{1}{\sqrt{1 + \tan^2 \alpha}}$

Written Exercises, pages 556–558

1. $\frac{1}{4}(\sqrt{6} - \sqrt{2})$ 3. $-\frac{1}{4}(\sqrt{6} - \sqrt{2})$
5. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ 7. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ 9. $-\cos \frac{\pi}{8}$
11. $-\cos \frac{2\pi}{7}$ 13. $-\cos \frac{3\pi}{8}$ 15. $\sin 0.1\pi$ or $\cos 0.4\pi$ 17. $\frac{1}{2}$ 19. $-\frac{1}{2}$ 21. 0 23. $-\frac{1}{2}$
39. $\frac{63}{65}$ 41. $\frac{44}{125}$ 43. $\frac{1}{6}(\sqrt{5} + 2\sqrt{3})$
45. $\frac{1}{12}(5\sqrt{3} - 2)$

Written Exercises, pages 562–564

1. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ 3. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ 5. $\frac{1}{4}(\sqrt{2} - \sqrt{6})$
7. $-\frac{1}{4}(\sqrt{2} + \sqrt{6})$ 9. $\sin \frac{\pi}{8}$ 11. $-\sin 15°$
13. $-\sin \frac{2\pi}{9}$ 15. $-\sin 50°$ 17. $-\tan \frac{7\pi}{20}$
19. $\tan 65°$ 21. $-\tan \frac{2\pi}{7}$ 23. $\tan 50°$
25. $2 + \sqrt{3}$ 27. $-2 - \sqrt{3}$ 29. $-2 + \sqrt{3}$
31. $-2 - \sqrt{3}$ 33. $\frac{\sqrt{3}}{2}$ 35. $-\frac{1}{2}$ 37. $-\frac{1}{2}$
39. $-\sqrt{3}$ 41. $-\frac{1}{\sqrt{3}}$ 55. $\sqrt{2} - \sqrt{6}$
57. $2 + \sqrt{3}$ 59. $-2 + \sqrt{3}$ 61. $\sqrt{6} - \sqrt{2}$
63. $\frac{56}{65}$ 65. $\frac{84}{85}$ 67. $-\frac{3}{5}$ 69. $\frac{12 + 3\sqrt{7}}{20}$

Written Exercises, pages 567–568

1. $-\frac{\sqrt{2 - \sqrt{3}}}{2}$ 3. $\frac{\sqrt{2 - \sqrt{3}}}{2}$ 5. $1 - \sqrt{2}$
7. $\frac{\sqrt{2 - \sqrt{2}}}{2}$ 9. $2 - \sqrt{3}$ 11. $\frac{\sqrt{2 - \sqrt{3}}}{2}$
13. $\frac{24}{25}$ 15. $-\frac{120}{169}$ 17. $-\frac{12}{13}$ 19. $\frac{119}{169}$
21. $-\frac{7}{8}$ 23. $-\frac{1}{9}$ 25. $\sin \frac{\alpha}{2} = \frac{3}{5}$, $\cos \frac{\alpha}{2} = \frac{4}{5}$
27. $\sin \frac{\alpha}{2} = \frac{1}{\sqrt{5}}$, $\cos \frac{\alpha}{2} = \frac{2}{\sqrt{5}}$ 29. $\sin \frac{\alpha}{2} = \frac{4}{\sqrt{17}}$, $\cos \frac{\alpha}{2} = -\frac{1}{\sqrt{17}}$

Written Exercises, page 573

1. 14 3. 7 5. 7.5 7. $29°0'$ 9. $17°40'$
11. $c \approx 9$, $m(A) \approx 28°$, $m(B) \approx 93°$ 13. $a \approx 9$, $m(B) \approx 56°$, $m(C) \approx 64°$ 15. $m(A) \approx 30°$, $m(B) \approx 61°$, $m(C) \approx 89°$

Problems, pages 573–574

1. 14.0 cm 3. $50°30'$ 5. 52.0 cm 7. 9.5
9. 8.2 km/h, $14°0'$

Written Exercises, pages 577–578

1. $m(C) = 114°$, $b \approx 23.5$, $c \approx 36.5$
3. $m(B) \approx 33°40'$, $m(C) \approx 26°20'$, $c \approx 25.6$
5. $m(C) = 37°$, $b \approx 306.4$, $c \approx 240.7$
7. $m(A) = 22°20'$, $b \approx 127.5$, $c \approx 199.7$
9. $m(A) \approx 24°20'$, $m(C) \approx 124°40'$, $c \approx 79.9$
11. 11 13. 6 15. 570 or 123 17. $b \sin A$

Problems, pages 578–579

1. 170.5 cm^2 3. 54.5 km 5. 4.3 m
7. $22°20'$ 9. 218.1 m

Chapter Review, pages 580–581

1. c 3. d 5. a 7. a 9. b 11. b 13. c

Chapter 16 Inverses; Polar Coordinates

Written Exercises, pages 588–589

1. $\{\alpha: \alpha = 180k\}$ 3. $\{\alpha: \alpha = 90 + 360k\}$
5. $\{\alpha: \alpha = 45 + 360k\} \cup \{\alpha: \alpha = 135 + 360k\}$
7. $\{x: x = \frac{\pi}{4} + 2k\pi\} \cup \{x: x = \frac{3\pi}{4} + 2k\pi\}$
9. $\{x: x = \frac{7\pi}{6} + 2k\pi\} \cup \{x: x = \frac{11\pi}{6} + 2k\pi\}$
11. $\{x: x = \frac{3\pi}{4} + 2k\pi\} \cup \{x: x = \frac{7\pi}{4} + 2k\pi\}$
13. $30°, \frac{\pi}{6}$ 15. $-90°, -\frac{\pi}{2}$ 17. $-60°, -\frac{\pi}{3}$
19. $21°, \frac{7\pi}{60}$ 21. $-70°, -\frac{7\pi}{18}$ 23. $12°, \frac{\pi}{15}$
25. $\frac{\pi}{6}$ 27. $-90°$ 29. $\frac{\pi}{4}$ 31. $\frac{\pi}{3}$ 33. $-\frac{\pi}{3}$
35. $\frac{12}{13}$ 37. $\frac{8}{17}$ 39. $\frac{1}{\sqrt{15}}$ 41. $\frac{2\sqrt{6}}{5}$
43. $\frac{1}{10}(3 + 4\sqrt{3})$ 45. $\frac{63}{65}$ 47. $\frac{4\sqrt{2}}{9}$
49. a. Yes

Written Exercises, page 591

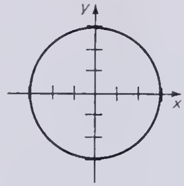
1. a. $\{\alpha: \alpha = 210 + 360k\} \cup \{\alpha: \alpha = 330 + 360k\}$
 b. $\{210°, 330°\}$
3. a. $\{\alpha: \alpha = 225 + 360k\} \cup \{\alpha: \alpha = 315 + 360k\}$
 b. $\{225°, 315°\}$
5. a. $\{\alpha: \alpha = 210 + 360k\} \cup \{\alpha: \alpha = 330 + 360k\}$
 b. $\{210°, 330°\}$
7. a. $\{\alpha: \alpha = 60 + 180k\} \cup \{\alpha: \alpha = 120 + 180k\}$
 b. $\{60°, 120°, 240°, 300°\}$
9. a. $\left\{x: x = \frac{3\pi}{4} + k\pi\right\}$ b. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$
11. a. $\left\{x: x = \frac{\pi}{2} + k\pi\right\} \cup \left\{x: x = \frac{\pi}{6} + 2k\pi\right\}$
 $\cup \left\{x: x = \frac{5\pi}{6} + 2k\pi\right\}$ b. $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$
13. a. $\left\{x: x = \frac{\pi}{2} + k\pi\right\}$ b. $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$
15. $\{90°, 210°, 330°\}$ 17. $\{90°, 120°, 240°, 270°\}$
19. $\left\{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$ 21. $\left\{\frac{\pi}{2}, 3.48, 5.94\right\}$
23. $\left\{0.85, \frac{\pi}{2}, 2.29\right\}$ 25. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
27. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ 29. $\{0°, 60°, 120°, 180°, 240°, 300°\}$
31. $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$
33. $\{30°, 45°, 135°, 150°, 210°, 225°, 315°, 330°\}$
35. $\{90°\}$ 37. $\{30°, 120°, 210°, 300°\}$

Written Exercises, pages 596–597

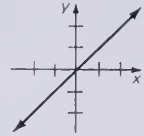
1. $(4\sqrt{2}, 45°)$ 3. $(-6, 45°)$ 5. $(3, 135°)$
7. $(-5, 53°)$ 9. $(-3, 3)$ 11. $(-1, -\sqrt{3})$
13. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ 15. $\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ 17. $r = \frac{2}{\cos\theta}$

19. $r = \frac{2}{\sin\theta + \cos\theta}$ 21. $r^2 = \frac{16}{\cos 2\theta}$

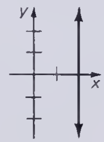
23. $x^2 + y^2 = 9$



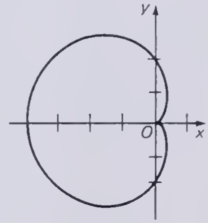
25. $y = x$



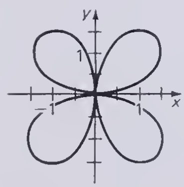
27. $x = 2$



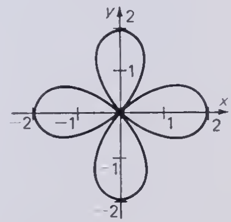
29. $(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$



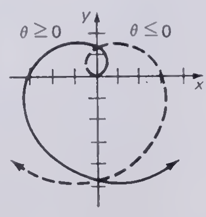
31. $(x^2 + y^2)^3 = 16x^2y^2$



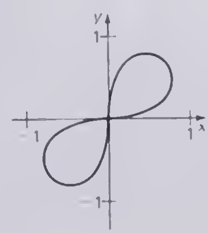
33. $(x^2 + y^2)^3 = 4(y^2 - x^2)^2$



35. $\sqrt{x^2 + y^2} = \cos^{-1}\frac{x}{\sqrt{x^2 + y^2}} + k2\pi$



37. $(x^2 + y^2)^2 = 2xy$



Written Exercises, pages 603–604

1. $2\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$
3. $5(\cos 180^\circ + i \sin 180^\circ)$
5. $2(\cos 315^\circ + i \sin 315^\circ)$
7. $4\sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$

9. $1 + i$ 11. $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$ 13. $-4\sqrt{3} - 4i$

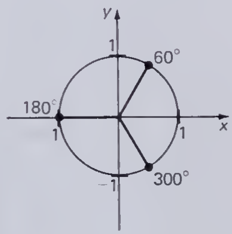
15. $\frac{1}{2} - \frac{1}{2}i$ 17. $z_1z_2 = 12, \frac{z_1}{z_2} = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$

19. $z_1z_2 = -16 - 16\sqrt{3}i, \frac{z_1}{z_2} = \sqrt{3} + i$ 21. -8

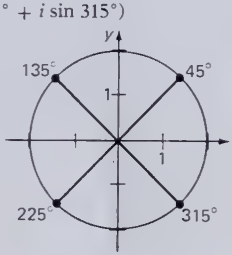
23. $-16 - 16\sqrt{3}i$ 25. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ 27. $-\frac{1}{4}$

29. $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

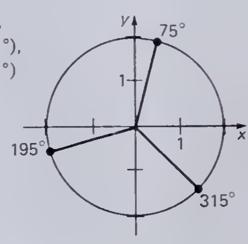
31. $\cos 60^\circ + i \sin 60^\circ$,
$\cos 180^\circ + i \sin 180^\circ$,
$\cos 300^\circ + i \sin 300^\circ$



33. $2(\cos 45^\circ + i \sin 45^\circ)$,
$2(\cos 135^\circ + i \sin 135^\circ)$,
$2(\cos 225^\circ + i \sin 225^\circ)$,
$2(\cos 315^\circ + i \sin 315^\circ)$



35. $2(\cos 75^\circ + i \sin 75^\circ)$,
$2(\cos 195^\circ + i \sin 195^\circ)$,
$2(\cos 315^\circ + i \sin 315^\circ)$



Written Exercises, page 608

1. 7 3. 19 5. $5\sqrt{2}$ 7. $4\sqrt{5}$ 9. $\|\mathbf{u}_x\| = 16$, $\|\mathbf{u}_y\| \approx 27.7$, $\|\mathbf{v}_x\| = 10$, $\|\mathbf{v}_y\| = 0$, $\|(\mathbf{u} + \mathbf{v})_x\| = 26$, $\|(\mathbf{u} + \mathbf{v})_y\| \approx 27.7$ 11. $\|\mathbf{u}_x\| \approx 14.1$, $\|\mathbf{u}_y\| \approx 14.1$, $\|\mathbf{v}_x\| = 5$, $\|\mathbf{v}_y\| \approx 8.7$, $\|(\mathbf{u} + \mathbf{v})_x\| \approx 9.1$, $\|(\mathbf{u} + \mathbf{v})_y\| \approx 22.8$ 13. $\|\mathbf{u}_x\| \approx 10.4$, $\|\mathbf{u}_y\| = 6$, $\|\mathbf{v}_x\| \approx 5.7$, $\|\mathbf{v}_y\| \approx 5.7$, $\|(\mathbf{u} + \mathbf{v})_x\| \approx 16.1$, $\|(\mathbf{u} + \mathbf{v})_y\| \approx 0.3$ 15. $\|\mathbf{u}_x\| \approx 3.4$, $\|\mathbf{u}_y\| \approx 9.4$, $\|\mathbf{v}_x\| \approx 18.4$, $\|\mathbf{v}_y\| \approx 15.4$, $\|(\mathbf{u} + \mathbf{v})_x\| \approx 21.8$, $\|(\mathbf{u} + \mathbf{v})_y\| \approx 24.8$ 17. 47°, 38 19. 112°, 25
21. 1°, 16 23. 49°, 33

Chapter Review, pages 610–611

1. d 3. b 5. b 7. c

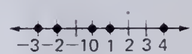
Cumulative Review, pages 612–613

1. b 3. a 5. a 7. b 9. d 11. a
13. b 15. c 17. d 19. a 21. c

Extra Practice

Chapter 1, page 618

1. $\{6\}$ 3. $\emptyset$ 5. $\{x: x \in \mathbb{R}\}$

7. 

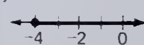
9. 

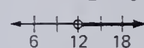
11. -9.6 13. -11 15. $\frac{3}{2}$ 17. 24 19. 49
21. -21 23. 21 25. 11

Chapter 2, pages 618–619

1. $-3x^3 - x^2 - 3x + 7$ 3. $-4y^2 + 2$ 5. $\left\{\frac{3}{2}\right\}$

7. $\{-2\}$ 9. $\left\{\frac{9}{2}\right\}$ 11. 78°, 78°, 24°

13. $\{x: x \geq -4\}$ 

15. $\{z: z > 12\}$ 

17. $\emptyset$

Chapter 3, page 619

1. 5 3. 0 5. $-2a^2 + 3a + 5$
7. Not a function
21. 4 23. -1 25. $\frac{1}{2}$ 27. $-\frac{3}{2}$ 29. 7
31. $y = 4x + 5$ 33. $2y + x = -16$
35. $3y - 2x = -27$ 37. $y + 3x = -1$
39. $y - 2x = 5$ 41. $2y + 5x = 56$
43. $y + x = -2\frac{1}{2}$ 45. 15 47. $10\frac{1}{2}$ 49. $5\frac{1}{3}$
51. \$10.28

Chapter 4, page 620

1. $\{(1, 4)\}$ 3. Inconsistent
5. $\{(x, y): x - 3y = 2\}$ 7. $\{(3, -7)\}$ 9. $\{(3, 1)\}$
11. $\left\{\left(\frac{5}{3}, \frac{2}{3}\right)\right\}$ 13. $\{(2, -3)\}$ 15. $\emptyset$
17. 14 of the \$5 shovels and 10 of the \$8 shovels

Chapter 5, pages 620–621

1. $2x + 3y = 12; z = 0$
$x + 2z = 6; y = 0$
$3y + 4z = 12; x = 0$

3. $2x + 3y = -12; z = 0$
$3x - 2z = -18; y = 0$
$9y - 4z = -36; x = 0$

5. Inconsistent 7. $\{(-1, 2, -2)\}$
9. $\{(3, -4, 2)\}$ 11. 12 13. -24 15. -75

17. $\left\{\left(-1, \frac{1}{2}, 2\right)\right\}$

19. 60 \$1-bills, 12 \$20-bills, 18 \$50-bills 21. 26

Chapter 6, pages 621–623

1. $\frac{30x}{y}$ 3. $2rs^4$ 5. $\frac{2}{y^4}$ 7. $\frac{-2m^5}{n^5}$ 9. $\frac{d^3}{4a}$
11. $x^4 - 6x^2 + 9$ 13. $n^6 - 2n^3m^2 + m^4$
15. $1 - b^{10}$ 17. $t^8 - 6t^4 + 9$ 19. $(6n - 5)^2$
21. $(3a + 7)(a - 5)$ 23. $(4x - 9y)(x + 2y)$
25. $(4r - s^2)(16r^2 + 4rs^2 + s^4)$
27. $(4 - 7bc^2)(4 + 7bc^2)$
29. $(2ab + c)(4a^2b^2 - 2abc + c^2)$ 31. $\{3, 13\}$
33. $\left\{-3, 2\frac{1}{2}\right\}$ 35. $\left\{1\frac{2}{3}\right\}$ 37. $\left\{-\frac{4}{3}, \frac{1}{2}\right\}$
39. $\left\{-\frac{5}{4}, \frac{9}{2}\right\}$ 41. 9 cm, 12 cm
43. $\{x: x \le -3\} \cup \{x: x \ge 6\}$
45. $\{y: y < -5\} \cup \{y: y > 5\}$
47. $\{x: x \in \Re \text{ and } x \ne 7\}$
49. $\frac{x - y}{x + y}$ 51. $\frac{a - 1}{2a + 1}$ 53. $x^2 - x + 5 + \frac{2}{2x - 3}$
55. $x^2 - 6x - 2 + \frac{1}{3x - 1}$
57. $x^3 - 4x^2 - 3x - 1 - \frac{2}{4x - 3}$ 59. $\frac{a^2 + ab + b^2}{a^2 - ab + b^2}$
61. $\frac{z(z - 16)}{2(z - 1)(z - 4)}$ 63. $\frac{4x - 5}{(x - 2)(x - 1)^2}$
65. $\frac{t^2}{(r + t)^3}$ 67. $\{3\}$ 69. $\{0, 4\}$ 71. 4 km/h

Chapter 7, pages 623–624

1. -34 3. -4.1 5. $\frac{77}{6}$ 7. $-3, 1, 5$
9. $-38.8, -9.6, 19.6, 48.8$ 11. $d = -3$, $a_1 = 15$ 13. $d = \frac{1}{6}$, $a_1 = 0$ 15. 90 17. 91
19. -57 21. 7350 23. 80 25. $\frac{64}{81}$
27. -162 29. $\frac{2}{3}, 2$ 31. $\frac{1}{3}, \frac{1}{2}$ 33. $63\frac{3}{4}$
35. $-19{,}531.5$ 37. 42 39. $127\frac{31}{32}$ 41. $1\frac{1}{2}, 1\frac{3}{4}, 1\frac{5}{6}, 1\frac{7}{8}$; $L = 2$ 43. $3, 6, -\frac{27}{7}, \frac{24}{7}$; not convergent 45. $416\frac{2}{3}$ 47. $\frac{10}{33}$ 49. $\frac{37}{999}$

Chapter 8, pages 624–625

1. 24 3. $\frac{8}{9}$ 5. $-\frac{8}{9}$ 7. $\left\{-\frac{7}{5}, \frac{7}{5}\right\}$ 9. $\emptyset$
11. $\left\{\frac{1}{2}\right\}$ 13. $\{-3\}$ 15. $\{-2\}$ 17. $\frac{26}{33}$
19. $2\frac{1}{37}$ 21. 16 23. $\frac{9\sqrt{30}}{25}$ 25. $x\sqrt{3x}$
27. 23 29. $-\frac{8\sqrt{3} + 20}{13}$ 31. $\{5\}$ 33. $\left\{\frac{9}{4}\right\}$
35. $\emptyset$ 37. $\{4, 12\}$ 39. $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$
41. $\left\{\frac{3 - \sqrt{7}}{2}, \frac{3 + \sqrt{7}}{2}\right\}$
43. $\left\{\frac{-5 - 2\sqrt{5}}{5}, \frac{-5 + 2\sqrt{5}}{5}\right\}$
45. $\left\{\frac{7 - \sqrt{69}}{2}, \frac{7 + \sqrt{69}}{2}\right\}$ 47. $\left\{\frac{2 - \sqrt{19}}{5}, \frac{2 + \sqrt{19}}{5}\right\}$
49. $\left\{\frac{1 - \sqrt{19}}{3}, \frac{1 + \sqrt{19}}{3}\right\}$
51. width: $-1 + \sqrt{26}$ cm, length: $1 + \sqrt{26}$ cm

Chapter 9, pages 626–627

1. $4\sqrt{5}i$ 3. -6 5. $-2i$ 7. $-\frac{1}{2}i$
9. $-4 - 3i$ 11. $5 - \frac{8}{3}i$ 13. $-34 - 13i$
15. 34 17. $\frac{12}{17} + \frac{3}{17}i$ 19. $-\frac{16}{13} + \frac{15}{13}i$
21. $\{2 + i, 2 - i\}$ 23. $\{-3 - i\sqrt{3}, -3 + i\sqrt{3}\}$
25. $\left\{\frac{2 - 2i\sqrt{3}}{3}, \frac{2 + 2i\sqrt{3}}{3}\right\}$
27. $\left\{\frac{\sqrt{15} - 9i}{6}, \frac{\sqrt{15} + 9i}{6}\right\}$ 29. $x^2 + 5x - 14 = 0$
31. $x^2 - 10x + 13 = 0$ 33. $x^2 + 9 = 0$
35. $x = 4$; $(4, 2)$ 37. $x = 3$; $(3, 9)$
39. $x = 0$; $(0, -8)$ 41. $\{x: x \le -4\} \cup \{x: x \ge 4\}$
43. $\left\{x: -\frac{3}{2} < x < 0\right\}$
45. $\left\{x: x < -\frac{1}{2}\right\} \cup \{x: x > 2\}$ 47. -32
49. 32 51. 0 53. $-i, 3, -2$

Chapter 10, pages 627–628

1. 10; $(1, 1)$ 3. 13; $\left(-2, -\frac{1}{2}\right)$ 5. $2\sqrt{6}$; $(3\sqrt{5}, -6)$ 7. $y - 2x = 11$
9. $3y + 4x = -39$ 11. $5y + 3x = -2$
21. $y = \frac{1}{4}x^2 + 2$ 23. Foci: $(0, \sqrt{51})$, $(0, -\sqrt{51})$
25. Foci: $\left(0, \frac{3}{2}\sqrt{3}\right)$, $\left(0, -\frac{3}{2}\sqrt{3}\right)$
27. $\frac{x^2}{16} + \frac{y^2}{7} = 1$
29. Foci: $\left(\frac{3}{2}\sqrt{29}, 0\right)$, $\left(-\frac{3}{2}\sqrt{29}, 0\right)$
31. $\frac{x^2}{7} - \frac{y^2}{9} = 1$ 33. $-\frac{1}{5}$ 35. $\frac{2}{9}$ 37. $\{(9, 8), (9, -8), (-9, 8), (-9, -8)\}$ 39. $\{(5, 3), (-5, 3), (5, -3), (-5, -3)\}$ 41. $\{(3, 5), (-3, 5), (0, -4)\}$

Chapter 11, pages 628–629

1. $\sqrt{2}$ 3. $\frac{8}{27}$ 5. $2\sqrt{2}$ 7. $\sqrt[3]{2}$ 9. $\{3\}$

11. $\{5\}$ 13. $\{3\}$ 15. $\left\{-\frac{11}{3}\right\}$ 17. $\left\{\frac{1}{14}\right\}$

19. $\left\{\frac{1}{125}\right\}$ 21. $\{\sqrt[3]{4}\}$ 23. $\left\{\frac{1}{9}\right\}$ 25. $\left\{\frac{3}{4}\right\}$

27. $\{5\}$ 29. $\{-30, 30\}$ 31. $\{3\}$ 33. $\left\{\frac{1}{400}\right\}$

35. 0.3458 37. 8.6957 − 10 39. 1.474

41. 0.8504 43. 0.00259 45. 19,600 47. 0.00451

49. $11,000 51. {516} 53. {5.00}

Chapter 12, pages 629–630

1. 216 3. 1125 5. 24 7. 1260 9. 151,200

11. 66 13. 10 15. 40 17. 150 19. 2800

21. $b^5 + 10b^4 + 40b^3 + 80b^2 + 80b + 32$

23. $x^{12} - 3x^{10} + \frac{15}{4}x^8 - \frac{5}{2}x^6 + \frac{15}{16}x^4 - \frac{3}{16}x^2 + \frac{1}{64}$

25. 1, 8, 28, 56, 70, 56, 28, 8, 1 27. Sample space: {HHHH, THHH, HTHH, HHTH, HHHT, TTHH, THTH, THHT, HTTH, HTHT, HHTT, TTTH, TTHT, THTT, HTTT, TTTT} 29. $\frac{1}{20}$ 31. $\frac{11}{24}$

33. $\frac{1}{6}$ 35. $\frac{1}{2}$

Chapter 13, page 631

1. $w = 2, x = 3, y = -1$ 3. $w = 4, x = 0, y = -2, z = 3$ 5. $w = 3, x = -1, y = 4$

7. $\begin{bmatrix} 5 & -4 \\ 4 & -7 \end{bmatrix}$ 9. $[7 \quad 1 \quad 4]$ 11. $\begin{bmatrix} 34 & -24 \\ -10 & 32 \end{bmatrix}$

13. $\begin{bmatrix} 13 & -18 \\ 5 & 14 \end{bmatrix}$ 15. $\begin{bmatrix} -13 & 18 \\ -5 & -14 \end{bmatrix}$ 17. $\begin{bmatrix} 12 & -7 \\ -5 & 11 \end{bmatrix}$

19. $\begin{bmatrix} 2 & -3 \\ -8 & 17 \end{bmatrix}$ 21. $\begin{bmatrix} 5 & -1 & -10 \\ -10 & 4 & 25 \end{bmatrix}$

23. $\begin{bmatrix} 7 & -6 \\ -18 & 19 \end{bmatrix}$ 25. $A(BC) = (AB)C$

27. $(AB)C \neq (BA)C$

29. $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$ 31. $\begin{bmatrix} -\frac{5}{2} & -3 \\ -\frac{3}{2} & -2 \end{bmatrix}$

33. $\begin{bmatrix} -22 & -37 \\ 16 & 27 \end{bmatrix}$ 35. $\begin{bmatrix} -\frac{27}{2} & -\frac{37}{2} \\ 8 & 11 \end{bmatrix}$

Chapter 14, page 632

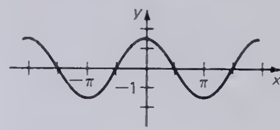
1. 594 cm 3. $\frac{5\pi^R}{6}$ 5. $\frac{7\pi^R}{3}$ 7. $-\frac{37\pi^R}{9}$

9. 315° 11. −405° 13. $\frac{1}{2}$ 15. $-\frac{1}{\sqrt{2}}$

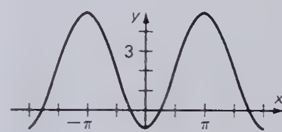
17. 0 19. $\frac{\sqrt{3}}{2}$ 21. $-\frac{\sqrt{3}}{2}$ 23. 0.5802

25. 0.6433 27. max: 3, min: −1, amp: 2

29.



31.



33. $\cos\alpha = -\frac{1}{2}$, $\sin\alpha = -\frac{\sqrt{3}}{2}$, $\tan\alpha = \sqrt{3}$, $\csc\alpha = -\frac{2}{\sqrt{3}}$, $\cot\alpha = \frac{1}{\sqrt{3}}$. 35. $m(A) = 64°30'$, $a = 63.2$, $b = 30.1$

Chapter 15, page 633

11. $\frac{\sqrt{2} - \sqrt{6}}{4}$ 13. $\frac{\sqrt{6} + \sqrt{2}}{4}$ 15. $\frac{\sqrt{2} + \sqrt{6}}{4}$

17. $\frac{\sqrt{3} + 1}{1 - \sqrt{3}}$ 23. $-\frac{24}{25}$ 25. $-\frac{7}{25}$ 27. $\frac{24}{7}$

29. $\frac{\sqrt{2 - \sqrt{3}}}{2}$ 31. $\frac{\sqrt{2 + \sqrt{2}}}{2}$ 33. $2 + \sqrt{3}$

35. $\sqrt{41}$

Chapter 16, page 634

1. 0° 3. 45° 5. $\frac{\pi}{4}$ 7. 52°; 0.91

9. 14°; 0.24 11. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ 13. $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

15. $\left\{x: x = \frac{\pi}{2} + k\pi\right\} \cup \left\{x: x = \frac{\pi}{3} + 2k\pi\right\} \cup \left\{x: x = \frac{2\pi}{3} + 2k\pi\right\}$ 17. $\left\{x: x = \frac{2\pi}{3} + 2k\pi\right\} \cup \left\{x: x = \frac{4\pi}{3} + 2k\pi\right\} \cup \{x: x = \pi + 2k\pi\}$

19. (2, −135°) 21. (25, 73°40′)

23. $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ 25. $0 - 80i$

27. $200 + 200i\sqrt{3}$ 29. $\{2(\cos 120° + i\sin 120°), 2(\cos 240° + i\sin 240°), 2(\cos 360° + i\sin 360°)\}$

31. $\{2(\cos 22°30' + i\sin 22°30'), 2(\cos 112°30' + i\sin 112°30'), 2(\cos 202°30' + i\sin 202°30'), 2(\cos 292°30' + i\sin 292°30')\}$

GHIJ-RM-89876

$x, y) : y = 2(x - 5), x \in \mathcal{R}$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$